ON THE GOTLIEB-CSIMA TIME-TABLING ALGORITHM

M. A. H. DEMPSTER

1. Introduction. This paper concerns an algorithm, proposed by C. C. Gotlieb **(4)** and modified by J. Csima **(1; 2)**, for a recent combinatorial problem whose application includes the construction of school time-tables. Theoretically, the problem is related to systems of distinct subset representatives, the construction of Latin arrays, the colouring of graphs, and flows in networks **(1; 2; 3)**. It was conjectured by Gotlieb and Csima that if solutions to a given time-table problem existed, i.e. if time-tables incorporating certain pre-assigned meetings existed, their algorithm would find one. In the contrary case, it would indicate which pre-assignments were incompatible with the remainder. A computer-generated counterexample to this conjecture has recently been reported by J. Lions **(6)**, and it is the purpose of this paper to analyse the situation in detail. It is known that in the absence of pre-assignments, a time-table can always be found **(1)**.

Suppose a non-negative integral $n \times n$ matrix $R = (\rho_{ij})$ is given whose row and column sums are all m. Then one may form an initial 0–1 availability array $A^0 = (\alpha^0{}_{ijk})$ for the time-table problem with *requirements matrix* R, $T(R)$, as follows. Consider as a 3-dimensional array, a stack of m $n \times n$ 0–1 matrices $A^0{}_k$ whose non-zero entries correspond to those of R. In school time-tabling, $\rho_{i'j'} = 4$, for example, expresses a requirement for teacher i' to meet class j' for four periods in the m-period day. The unit entries $\alpha^0{}_{i'j'k}$, $k = 1, \ldots, m$, of A^0 represent possible meetings and express the obvious fact that before scheduling is begun, teacher i' and class j' are available to meet in *any* four of the m periods of the day.

Define the *union* of two 0–1 arrays $X = (\xi_{ijk})$ and $Y = (\eta_{ijk})$ of comparable dimensions to be the 0–1 array

$$X \cup Y = (\xi_{ijk} + \eta_{ijk} - \xi_{ijk} \cdot \eta_{ijk})$$

and their *intersection* to be the 0–1 array

$$X \cap Y = (\xi_{ijk} \cdot \eta_{ijk}).$$

Define these concepts similarly for 0–1 matrices and vectors of comparable dimensions. For example, if $X = (1010101)$ and $Y = (0111011)$, then $X \cup Y = (1111111)$ and $X \cap Y = (0010001)$. A 0–1 array (or matrix or vector) X may be said to be *contained* in a 0–1 array (or matrix or vector) Y of comparable dimensions, denoted $X \subset Y$, if $X \cap Y = X$. An entry ξ_{ijk} of

Received July 15, 1966.

https://doi.org/10.4153/CJM-1968-013-7 Published online by Cambridge University Press

an array X will be referred to as an *element* of X and denoted $\xi_{ijk} \in X$. The array $0 = (\zeta_{ijk})$ with all zero elements will be called the *zero array*. Any 0–1 array $A = (\alpha_{ijk}) \neq 0$ such that $A \subset A^0$ will be said to be an *availability array* for $T(R)$. A non-zero element $\alpha_{ijk} \in A$ will be referred to as an *availability* of A and will be denoted by (i, j, k) when the array A of which it is an element is clear. A *solution* to $T(R)$, i.e. a time-table, will be taken to be an availability array $S = (\sigma_{ijk})$ which may be considered as a stack of m permutation matrices whose matrix sum $\sum_{k=1}^{m} S_k$ is R. Each availability of S, of course, represents a meeting of the time-table. Such a unit element $\sigma_{ijk} \in S$ will be referred to as a *solution element* of S and will be denoted by $[i, j, k]$ when the *specific* solution S to $T(R)$ of which it is an element is clear.

The algorithm proposed by Gotlieb begins with the initial availability array A^0 and a set $\mathscr{S}$ of specified final solution elements (special or pre-assignments), and moves toward a solution to $T(R)$ by changing certain availabilities to zero. The idea was to specify certain 2-dimensional conditions (herein called the strong (planar) Hall conditions; cf. (**1**; **2**; **4**)) which would be powerful enough to eliminate those availabilities which were not contained in any solution to $T(R)$ incorporating the elements of $\mathscr{S}$. Hence it was originally conjectured by Gotlieb that if it were possible to reduce A^0 by this means to an availability array A^1 containing the elements of $\mathscr{S}$ and satisfying these conditions, a solution to $T(R)$ incorporatng the specified solution elements existed. Moreover, *any* availability in the array A^1 could be introduced as a specified solution element in A^1 and it would always be possible to reduce A^1 to a new availability array A^2 satisfying the strong Hall conditions. Choosing any availability of A^2 as a specified solution element and applying the reduction procedure, it would thus be possible to iterate toward a solution. In the light of computer experiments reported by Csima and Gotlieb (**2**), it was seen that the strong Hall conditions are insufficient to eliminate all availabilities not contained in any solution to $T(R)$ incorporating the elements of $\mathscr{S}$. Thus the conjecture was modified to state the existence of *some* availability of A^1 which would lead to A^2 and eventually, through some availability of A^2, A^3, etc., to a solution. It is this conjecture to which counterexamples have recently been discovered. That is, it is possible to generate by the above procedure an array which satisfies the strong Hall conditions, but in which no solution to $T(R)$ is contained.

In this paper, 3-dimensional necessary and sufficient conditions for an availability to be an element of a solution to $T(R)$ are given. Moreover, the situation is characterized in which an availability of an array satisfying the strong Hall conditions is not an element of any solution It is an easy consequence of a theorem of G. Birkhoff (see **8**, Theorem 5.3, p. 56) that any element of $\mathscr{S}$ may be included in some solution to $T(R)$ contained in A^0. The present results lead to a different algorithm enlarging on this fact (**3**). They are also related to the solution of other multidimensional assignment problems (to be treated in another paper).

The time-table problem $T(R)$ described above has been called a "tight" problem by Csima. It can be shown that any actual school time-table problem for a day (typically represented by a non-square requirements matrix whose row and column sums may differ) can be canonically embedded in a problem of this form and, moreover, that a solution for the latter yields a solution for the former **(1)**. Necessary and sufficient conditions for the existence of a time-table incorporating a set of pre-assigned meetings which do not require this embedding may be formulated in terms of constrained network flows **(3)**. Although actual time-tabling problems typically involve complexities, it would appear that many of these may be treated using the concept of a requirements matrix and a set of specified final solution elements to be introduced sequentially **(7)**.

The next section of the paper discusses some necessary preliminaries to the analysis. In § 3, a precise statement of the Gotlieb–Csima conjecture is given and a form of their "long-range feasibility" algorithm described. Section 4 contains the main results of the paper. In § 5, some implications of these results are drawn.

2. Preliminaries. A *planar section*, or simply a *plane*, of an availability array $A = (\alpha_{ijk})$ will be taken to be any 0–1 matrix formed by holding one of the three indices i, j, and k fixed. For example, for $k = k'$, $A_{k'} = (\alpha_{ijk'})$ is an $n \times n$ 0–1 matrix which will be called a *horizontal* plane, while for $i = i'$ and $j = j'$, $A_{i'} = (\alpha_{i'jk})$ and $A_{j'} = (\alpha_{ij'k})$ are both $m \times n$ 0–1 matrices which will be called *vertical* planes (see Figure 1). A *line* of the array A will

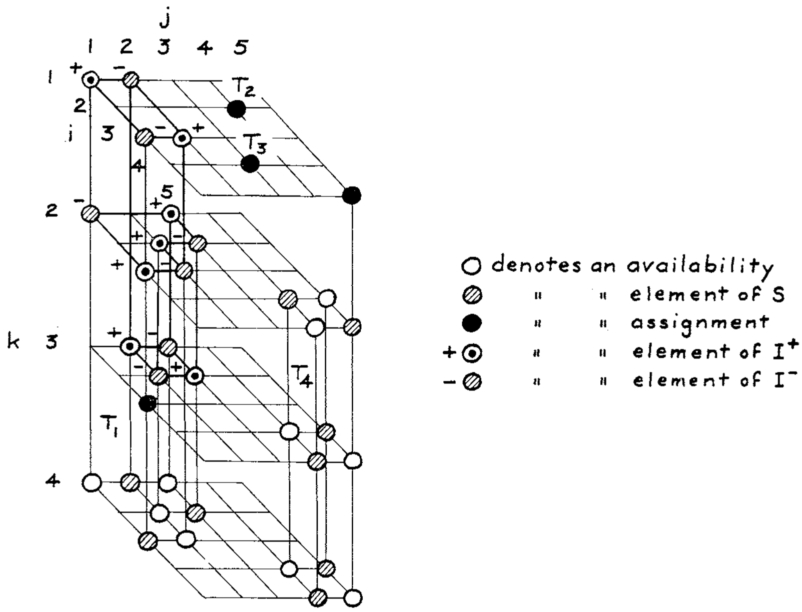

FIGURE 1.

https://doi.org/10.4153/CJM-1968-013-7 Published online by Cambridge University Press

be taken to be any 0–1 vector formed by holding two of the three indices i, j, and k fixed. For example, for $j = j'$ and $k = k'$, $A_{j'k'} = (\alpha_{ij'k'})$ is the jth column vector of the horizontal plane $A_{k'}$. The *modulus* $||X||$ of an array $X = (\xi_{ijk})$ is defined to be $\sum_{i,j,k} |\xi_{ijk}|$. For this purpose, lines and planes will be identified with the arrays defined by adding appropriate zero elements.

Each plane of an availability array A leads to the following *planar problem.* Does there exist a set of availabilities contained in the plane whose sum along any line equals the requirements for that line in the problem $T(R)$? For example, for $A_{i'} = (\alpha_{i'jk})$, does there exist $P_{i'} = (\pi_{i'jk}) \subset A_{i'}$ such that

$$\sum_{j=1}^{n} \pi_{i'jk} = 1 \quad (k = 1, \ldots, m) \quad \text{and} \quad \sum_{k=1}^{m} \pi_{i'jk} = \rho_{i'j} \quad (j = 1, \ldots, n)?$$

Such a set will be called a *planar solution* and will be identified with the 0–1 matrix defined by adding appropriate zero elements. It is clear that while the existence of a solution $S \subset A$ to $T(R)$ implies the existence of a planar solution for each fixed i, j, or k (e.g. for $i = i'$, $S_{i'} \subset A_{i'}$), there will in general, for $A \neq A^0$, be many planar solutions which may not be found in this way.

Let $\mathscr{I}$ be the set of all subsets of $N = \{1, \ldots, n\}$. Then an availability array A is said to satisfy the (*planar*) *Hall conditions* if:

(i) for fixed i, $||\bigcup_{j\in J} A_{ij}|| \geqslant \sum_{j\in J} \rho_{ij}$ for all $J \in \mathscr{I}$,

(ii) for fixed j, $||\bigcup_{i\in I} A_{ij}|| \geqslant \sum_{i\in I} \rho_{ij}$ for all $I \in \mathscr{I}$,

and

(iii) for fixed k, $||\bigcup_{j\in J} A_{jk}|| \geqslant \sum_{j\in J} 1$ for all $J \in \mathscr{I}$.

These conditions are a translation into the present situation of the classical necessary and sufficient conditions for the existence of a system of distinct subset representatives due to P. Hall (see, e.g., **7**, Theorem 1.1, p. 48). In this setting, such a system is a planar solution, and thus the Hall conditions imply the existence of a planar solution contained in A for each fixed i, j, or k. An availability array which does not satisfy these conditions will be said to be *infeasible.*

A partition $\mathscr{P}_2$ of the set N will be said to be *finer* than another partition $\mathscr{P}_1$ of N if all the sets of $\mathscr{P}_2$ are subsets of sets of $\mathscr{P}_1$ and at least two sets of $\mathscr{P}_2$ are proper subsets of a set of $\mathscr{P}_1$. The availability array A will be said to satisfy the *strong* (*planar*) *Hall conditions* if it satisfies the Hall conditions, and if moreover:

(i) for fixed i, there exists a finest partition $\mathscr{J}_i$ of N such that for all $J \in \mathscr{J}_i$, $||\bigcup_{j\in J} A_{ij}|| = \sum_{j\in J} \rho_{ij}$ and $A_{ij} \cap A_{ij'} = 0$ for $j \in J$ and $j' \in N \sim J$;

(ii) for fixed j, there exists a finest partition $\mathscr{I}_j$ of N such that for all $I \in \mathscr{I}_j$, $||\bigcup_{i\in I} A_{ij}|| = \sum_{i\in I} \rho_{ij}$ and $A_{ij} \cap A_{i'j} = 0$ for $i \in I$ and $i' \in N \sim I$;

and

(iii) for fixed k, there exists a finest partition $\mathscr{J}_k$ of N such that for all $J \in \mathscr{J}_k$, $||\bigcup_{j\in J} A_{jk}|| = \sum_{j\in J} 1$ and $A_{jk} \cap A_{j'k} = 0$ for $j \in J$ and $j' \in N \sim J$.

The availabilities of any plane of an availability array A satisfying the strong Hall conditions are clearly partitioned into sets which will be called *planar tight sets* by one of the index partitions $\mathscr{J}_i, \mathscr{J}_j$, and $\mathscr{J}_k$ $(i, j = 1, \ldots, n; k = 1, \ldots, m)$ (cf. **1**; **2**; **4**). It follows from the definition of the strong Hall conditions that such a plane may be put in block diagonal form by suitable permutations of its rows and columns. The blocks correspond to planar tight sets and are by no means necessarily solid with availabilities. They are always square in horizontal planes, but are generally rectangular in vertical planes. The availabilities of the full array A may be partitioned into *tight sets* T_α, where α is an availability, as follows. For any such $\alpha \in A$, the right set T_α may be generated by taking as its elements, α, and any availability which is in a planar tight set with an availability already in T_α. It is clear that every availability is in such a set. Moreover, no availability is in two such sets, for if it were it would be in two planar tight sets in some plane, contradicting the fact that the planar tight sets in a plane are a partition of the availabilities in that plane. Tight set partitions may have a relatively complicated structure for an arbitrary availability array A (see Figure 1). The tight set partition for the initial availability array A^0, however, consists of a single set containing all the availabilities, since A^0 satisfies the strong Hall conditions with the trivial index partitions $\{N\}$.

As a consequence of the minimality of the index partitions of an availability array A satisfying the strong Hall conditions, notice, for example in the plane A_i, that

$$||\textstyle\bigcup_{j\in H} A_{ij}|| > \sum_{j\in H} \rho_{ij} \tag{2.1}$$

if, and only if, the index set H is properly contained in a set J of the index partition $\mathscr{J}_i$ for which $\sum_{j\in J} \rho_{ij} > 0$. The sets of lines indexed by such sets H are called "slack sets" by Gotlieb and Csima. Ways are known of reducing a given availability array satisfying the Hall conditions to a *unique* availability array contained in it which satisfies the strong Hall conditions and has maximum modulus, i.e. has the largest number of availabilities left in it (see, e.g., **1**, Chapter 3, where the term "restriction" is used). These are all equivalent to finding smallest sets of lines in a plane which satisfy the Hall conditions with equality, changing enough availabilities to zero to give the zero intersections of the strong Hall conditions in that plane, and then iterating the procedure on all affected planes until enough availabilities have been removed to yield an array satisfying the strong Hall conditions. Any such process applied to an arbitrary availability array A will be referred to as a *reduction process* and the (unique) resulting array, if it exists, will be called the *reduced* (*availability*) *array*, and denoted by $r(A)$.

An availability $\alpha = (i, j, k)$ of an arbitrary availability array A will be said to be *assigned* if there are no other availabilities in the ith row and jth column of the kth horizontal plane A_k (and, if $\rho_{ij} = 1$, in the line A_{ij}). *Assignment* of α is the operation of changing all the other availabilities in these

https://doi.org/10.4153/CJM-1968-013-7 Published online by Cambridge University Press

lines to zeros. The new array obtained from A by the assignment of α will be denoted by A_α. The availability α is an element of any solution contained in A_α. Of course, A_α may be infeasible, i.e. may not satisfy the Hall conditions, even though the array A is not.

Finally, two availabilities of an availability array A will be said to be *compatible* if they are elements of a common solution to $T(R)$ contained in A. Otherwise, they will be said to be *incompatible.*

3. The Gotlieb–Csima algorithm and conjecture. It is clear that a solution is the only type of availability array satisfying the strong Hall conditions in which all sets of index partitions referring to non-zero lines may be taken to be the elements of N, i.e. that only a solution satisfies the Hall conditions with all inequalities replaced by equalities. The Hall conditions are thus necessary for the existence of a solution to $T(R)$ contained in a given availability array and hence an infeasible array contains no solution. However, specific R and A which constitute a simple counterexample to the sufficiency of the Hall conditions have been given by Csima and Gotlieb **(2)**. It might be conjectured that if any solution to $T(R)$ is contained in an availability array A satisfying the strong Hall conditions, there would exist a solution S_α through *every* availability α of A. This conjecture is equivalent to the original conjecture of Gotlieb, however, and, as mentioned above, Gotlieb and Csima have reported a computer-generated counterexample **(2)**. (Several others will be exhibited in the next section of this paper.) Their modified conjecture may be stated as follows.

CONJECTURE. *Let S be a solution to $T(R)$ contained in an availability array A satisfying the strong Hall conditions and let α be an availability of A. Then either A_α is infeasible, or there exists a solution S_α through α contained in $r(A_\alpha)$.*

If it were true, this statement, in conjunction with Birkhoff's theorem and the necessity of the Hall conditions, would justify a slightly modified form of the Gotlieb–Csima algorithm. That is, if solutions to $T(R)$ incorporating a set of specified final solution elements $\mathscr{S}$ existed, the algorithm would find one. In the contrary case, it would indicate which elements of $\mathscr{S}$ were incompatible with the remainder. However, the computer-generated array reported by Lions **(6)** constitutes a counterexample to the conjecture, and another will be exhibited in § 4. Nevertheless, it is instructive to see how the truth of the conjecture would justify the algorithm. The following description assumes that the set $\mathscr{S} = \{\sigma^t\}$ is equipped with a (linear) priority ordering (cf. **4**, p. 76).

Form the initial availability array A^0 from the requirements matrix R. Since any element of $\mathscr{S}$ may be included in a solution contained in A^0, in particular, the element first in the priority ordering σ^1 forms part of such a solution. The availability array $A^0{}_{\sigma^1}$ resulting from the assignment of σ^1 satisfies the Hall conditions since it contains a solution. It may therefore be reduced by any reduction process to the unique reduced array A^1 $(=r(A^0{}_{\sigma^1}))$ which satisfies the strong Hall conditions and contains the solution. If some elements of $\mathscr{S}$ are not contained in A^1, these are incompatible with σ^1. Otherwise, either the second element of the priority ordering σ^2 is incompatible with σ^1, or σ^2 may be assigned and the reduction process applied again to yield a new reduced availability array A^2 which contains a solution through σ^1 and σ^2. In the first case, if the conjecture as stated above were true, $A^1{}_{\sigma^2}$ would be infeasible, and in the course of reducing $A^1{}_{\sigma^2}$ the reduction process would discover a plane in which the Hall conditions were violated. In fact, although σ^1 and σ^2 are incompatible, $A^1{}_{\sigma^2}$ may not be infeasible and the incompatibility may go undetected, at least temporarily. If σ^2 is found to be incompatible with σ^1, the next element of $\mathscr{S}$ in priority which is an availability of A^1, σ^t, should be assigned to form $A^1{}_{\sigma^t}$ and the reduction process applied to this array. This procedure may be iterated to produce a sequence of reduced arrays $A^0 \supset A^1 \supset A^2 \supset \ldots \supset A^T$, the last of which, if the conjecture were true, would contain as assignments as many elements of $\mathscr{S}$ as are compatible with respect to the priority ordering. From this point onwards, the process would be continued by attempting to assign any availability of the current array in order to produce a sequence of reduced arrays $A^T \supset A^{T+1} \supset \ldots$ converging to a solution S to $T(R)$. If the conjecture were true, each of these availability arrays would contain a solution to $T(R)$ with the required properties, the next array for the sequence could always be found, and the final solution S would have the required properties. In fact, the incompatibility of some σ^t with σ^s $(1 \leqslant s \leqslant t)$, or of some arbitrarily assigned availability with $\mathscr{S}$, may not be discovered for several iterations until an infeasible array results. At this time, it may be impossible to pinpoint the source of the infeasibility.

4. Interchanges and necessarily redundant chains. In order to specify necessary and sufficient conditions for the existence of a solution through an availability of a reduced array, consider the introduction of a new element into an existing solution S contained in an availability array A satisfying the strong Hall conditions. That is, suppose there exists a solution $S' \subset A$ and containing the new element. Without loss of generality, suppose this availability belongs to S', and that $S'_k \neq S_k$, for $k = 1, \ldots, p \leqslant m$, and $S'_k = S_k$, for $k = \mathrm{p} + 1, \ldots, m$. Then since

$$\sum_{k=1}^{m} S'_k = \sum_{k=1}^{m} S_k = R,$$

we have

$$\sum_{k=1}^{p} (S'_k - S_k) = 0.$$

Notice that the elements of the last sum are $n \times n$ matrices with entries

0, -1, and $+1$, the negative elements corresponding to the old solution S and the positive elements to the new, S'.

In general, a 0, ± 1 array, $I(A) = (\iota_{ijk})$, such that $|I| = (|\iota_{ijk}|) \subset A$ and:

(i) for fixed j, k,

$$\sum_{i=1}^{n} \iota_{ijk} = 0 \quad \text{and} \quad ||I_{jk}|| \leqslant 2,$$

(ii) for fixed i, k,

$$\sum_{j=1}^{n} \iota_{ijk} = 0 \quad \text{and} \quad ||I_{ik}|| \leqslant 2,$$

(iii) for fixed i, j,

$$\sum_{k=1}^{m} \iota_{ijk} = 0 \quad \text{and} \quad ||I_{ij}|| \leqslant 2[m/2],$$

will be called an *interchange*. The symbol I^+ will be used to denote the 0–1 array $(\max\{0, \iota_{ijk}\})$ and I^- the 0–1 array $(-\min\{\iota_{ijk}, 0\})$, so that as usual $I = I^+ - I^-$ and $|I| = I^+ + I^-$, the arithmetic operations being performed elementwise.

It is easily checked that the pair of solutions S', S specify an interchange I with $I^+ = S' - S' \cap S$ and $I^- = S - S' \cap S$. Conversely, any interchange $I'(A)$ such that $I'^- \subset S$ specifies a new solution $S' = I'^+ + (S - I'^-)$. Indeed, conditions (i) and (ii) imply, for fixed k, that there is a single one in every row and column of S'_k, i.e. that S'_k is a permutation matrix. Moreover, from condition (iii),

$$\sum_{k=1}^{m} S'_k = \sum_{k=1}^{m} I'^+ + (S - I'^-) = \sum_{k=1}^{m} S_k + \sum_{k=1}^{m} I' = R + 0 = R,$$

so that S' is a solution to $T(R)$. Hence

THEOREM 1. *Given a solution S to $T(R)$ contained in an availability array A satisfying the strong Hall conditions, and an availability $\alpha \in A$ and not to S, there exists a solution $S' \subset A$ through α if, and only if, there exists an interchange I such that $I \subset A$, $\alpha \in I^+$, and $I^- \subset S'$.*

The interchange of Theorem 1 will be said to be an *interchange with S through α*. Essentially, an interchange describes a balancing operation in three dimensions in which the availabilities of I^- are removed from the solution S, and in their place are added an equal number of availabilities (including α), those of I^+, to form the new solution S' through α.

An important result follows immediately.

THEOREM 2. *Let α be an availability of an availability array A satisfying the strong Hall conditions. Then the following statements are equivalent:*

(i) *There exists a solution to $T(R)$ through α contained in A.*

(ii) *There exists a solution S to $T(R)$ contained in A and an interchange with S through α.*

(iii) *There exists an interchange through α with every solution to $T(R)$ contained in A.*

Figure 1 shows a $5 \times 5 \times 4$ reduced availability array A for the timetable problem with requirements matrix

$$R = \begin{bmatrix} 1 & 0 & 0 & 1 & 2 \\ 0 & 0 & 1 & 2 & 1 \\ 0 & 2 & 2 & 0 & 0 \\ 3 & 0 & 0 & 0 & 1 \\ 0 & 2 & 1 & 1 & 0 \end{bmatrix}.$$

The array is partitioned into four tight sets $T_1, \ldots, T_4$, and the first four elements of a set of specified final solution elements

$$\mathscr{S} = \{[2, 4, 1], [4, 3, 1], [5, 5, 1], [3, 1, 3], (1, 1, 1)\} \subset A$$

are incorporated into a solution $S \subset A$. An interchange I with S through $(1, 1, 1)$ involving elements of T_1 is marked. Several others are possible.

It is clear that the concept of a *planar interchange* may be defined using the appropriate selection of two of the three conditions (i), (ii), and (iii) of the definition of an interchange. Then Theorem 1 may be specialized as follows.

COROLLARY 3. *Given a planar solution P to a planar problem defined for the availability array A satisfying the strong Hall conditions, and an availability α belonging to the plane under consideration and not to P, there exists a new planar solution contained in A through α if, and only if, there exists a planar interchange with P through α.*

The next result is implicit in an efficient reduction algorithm due to Lions which is based upon the Hungarian algorithm of H. W. Kuhn (see **5**).

THEOREM 4. *Given a solution S to $T(R)$ contained in an availability array A and an availability $\alpha \in A$, there exists a planar interchange with the appropriate section of S in all three planes through α if, and only if, $\alpha \in r(A)$.*

Remark. If an availability $\alpha \in A$ but not to $r(A)$ there may exist a planar interchange with the appropriate section of S in one or two planes through α.

Proof. Consider the availability $\alpha_1 = (i_1, j_1, k_1) \in A$. In the interests of economy of notation, $r(A)$ will be denoted by B.

Suppose there exists a planar interchange with the appropriate section of S in all three planes through α_1. Then, in particular, there exists a planar interchange I_{i_1} with S_{i_1} through α_1. Let H denote the set of indices j of the columns of A_{i_1} in which there are elements of I_{i_1}. In each of these columns there are elements of S_{i_1} belonging to $I^-{}_{i_1}$ and an availability of A_{i_1} belonging to $I^+{}_{i_1}$. Hence, by the definition of a planar interchange,

$$||\bigcup_{j\in J} I_{i_1 j}|| = ||\bigcup_{j\in J} I^+{}_{i_1 j}|| + ||\bigcup_{j\in J} I^-{}_{i_1 j}|| = 2||\bigcup_{j\in J} I^-{}_{i_1 j}|| = 2\sum_{j\in J} \rho_{i_1 j},$$

i.e.

$$||\bigcup_{j\in J} I_{i_1 j}|| > \sum_{j\in J} \rho_{i_1 j}$$

for any index subset $J \subset H$ (including $J = H$). A fortiori for $J \subset H$,

$$||\bigcup_{j\in J} B_{i_1 j}|| > \sum_{j\in J} \rho_{i_1 j}$$

and hence, by (2.1), $H \subset J_i \in \mathscr{J}_{i_1}$, the index partition of the strong Hall conditions for B_{i_1}. Since $j_1 \in H$, and there is an element of S_{i_1} belonging to $I^-{}_{i_1}$, it follows that $\alpha_1 \in B_{i_1}$. Similar considerations for the planes A_{j_1} and A_{k_1} involving the planar interchanges I_{j_1} and I_{k_1} imply that $\alpha_1 \in B_{j_1}$ and B_{k_1}. Hence $\alpha_1 \in B$, in particular to the tight set T_σ generated by any $\sigma \in I^-{}_{i_1}$, $I^-{}_{j_1}$ or $I^-{}_{k_1}$, as required.

Conversely, assume $\alpha_1 \in B \subset A$ and consider the plane A_{i_1}. Then there exists a set $J \in \mathscr{J}_{i_1}$, the index partition of the strong Hall conditions for B_{i_1}, such that $j_1 \in J$, $||\bigcup_{j\in J} B_{i_1 j}|| = \sum_{j\in J} \rho_{i_1 j}$, and, by the minimality of $\mathscr{J}_{i_1}$ (cf. (2.1)),

$$||\bigcup_{j\in H} B_{i_1 j}|| > \sum_{j\in H} \rho_{i_1 j} \qquad \text{for all } H \subsetneqq J. \tag{4.1}$$

If $\alpha_1 \in S$, there is nothing to prove. Hence suppose $\alpha_1 \notin S$. Then since there is an element of S_{k_1} in every row of A_{k_1}, and in particular the i_1th, there exists a solution element $\sigma_1 = [i_1, j_2, k_1] \in A_{k_1}$. Moreover, $j_2 \in J$, i.e. $\sigma_1 \in B_{k_1}$, for there are no solution elements in the k_1th row and jth column of A_{k_1} if $j \notin J$. Indeed, if there were, there would be no solution element in some column of B_{k_1}, which is impossible. Now there exists an unassigned availability $\alpha_2 = (i_1, j_2, k_2) \in B_{i_1}$, for otherwise, for the line $B_{i_1 j_2}$,

$$||B_{i_1 j_2}|| = \rho_{i_1 j_2},$$

contradicting (4.1). By similar arguments there exist a solution element $\sigma_2 = [i_1, j_3, k_2] \in B_{k_2}, j_2 \in J$, and an unassigned availability $\alpha_3 = (i_1, j_3, k_3) \in B_{i_1}$, where $k_3 \neq k_2$. If $j_3 = j_1$, α_3 may be taken to be α_1, and letting $I^+{}_{i_1} = \{\alpha_1, \alpha_2\}$ and $I^-{}_{i_1} = \{\sigma_1, \sigma_2\}$, the required planar interchange I_{i_1} has been found. If $j_3 \neq j_1$, k_3 may be chosen not equal to k_1. Indeed, otherwise

$$||\bigcup_{j=j_2,j_3} B_{i_1 j}|| = \sum_{j=j_2,j_3} \rho_{i_1 j},$$

contradicting (4.1) (see Figure 2).

This process may be continued until the solution element $\sigma_T = [i_1, j_T, k_{T-1}]$, for which $j_T = j_1$, has been found. Unless $j_t = j_1$, it is always possible to pick $k_t \neq k_1, \ldots, k_{t-1}$, for otherwise

$$\left|\left|\bigcup_{\tau=1}^{t-1} B_{i_1 j_\tau}\right|\right| = \sum_{\tau=1}^{t-1} \rho_{i_1 j_\tau},$$

https://doi.org/10.4153/CJM-1968-013-7 Published online by Cambridge University Press

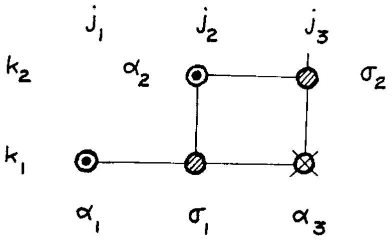

FIGURE 2.

contradicting (4.1). When σ_T has been found, $2T$ availabilities have been encountered (counting the end points), and letting $I^+{}_{i_1} = \{\alpha_1, \ldots, \alpha_T\}$ and $I^-{}_{i_1} = \{\sigma_1, \ldots, \sigma_T\}$, the required planar interchange I_{i_1} has been found (see Figure 3). Notice that the process must terminate after at most $2\sum_{j\in J} \rho_{i_1 j}$ steps. Similar considerations apply to the planes A_{j_1} and A_{k_1}.

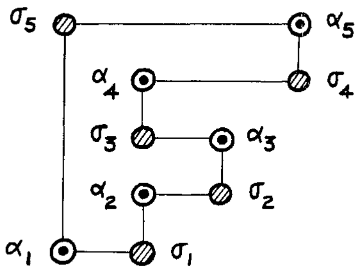

FIGURE 3.

COROLLARY 5. *Given a solution S to $T(R)$ contained in the availability array A satisfying the strong Hall conditions and an availability $\alpha \in A$, there exists a planar solution containing α in all three planes through α.*

Proof. Immediate from Corollary 3 and Theorem 4.

Thus, while the Hall conditions imply the existence of a planar solution in every plane of an availability array which satisfies them, the strong Hall conditions imply considerably more for an array which is known to contain a solution.

Unfortunately, the construction of Theorem 4 cannot be extended directly to three dimensions, since a 3-dimensional interchange has a more complicated structure than its planar counterpart. Indeed, a planar interchange I is formed from a simple cyclic chain of availabilities taken alternately from I^+ and I^-. An interchange, however, consists of (perhaps many) 2-dimensional chains fitted together so that each availability involved in the interchange is part of a chain in each of the three planes through it (see Figure 1). Formal definitions of chains and cycles applicable to three dimensions will be useful

https://doi.org/10.4153/CJM-1968-013-7 Published online by Cambridge University Press

in describing the situation when no interchange through a given availability of an array exists.

Let A be an availability array and let S be a solution to $T(R)$ contained in it. Then a *chain* will be defined to be a sequence of availabilities of A such that even numbered but not odd-numbered elements of the sequence are elements of S, and every two consecutive elements, but no three consecutive elements, are contained in a line of A. A finite chain will be called a *cycle* if the sequence of availabilities formed by appending the chain to itself is a chain. A chain will be said to be *simple* if no proper subset of its elements forms a cycle, and *planar* if its elements lie in a plane of A. A finite chain whose elements are distinct will be said to be (*necessarily*) *redundant* (see Figure 4) if:

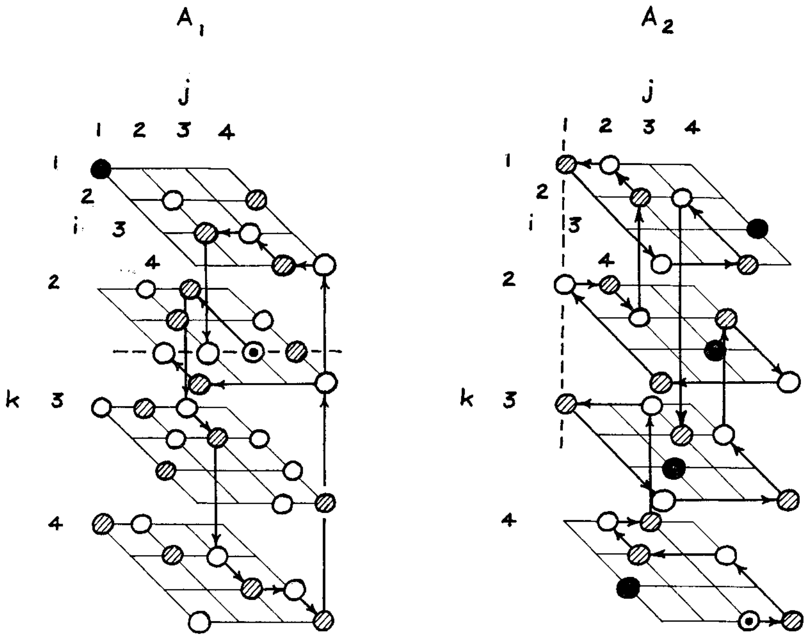

FIGURE 4.

(i) there exists a line containing either more odd-numbered than even-numbered elements of the chain, or vice versa, and

(ii) every consecutive pair of even-numbered and odd-numbered elements of the chain are either: (a) the only availabilities in the line containing them, or (b) contained in a line with other availabilities of A which are elements of a redundant chain whose elements include all previous elements of the sequence under consideration.

In case (b) of condition (ii), there are points of choice in constructing the redundant chain, but upon making a choice only a redundant chain results.

https://doi.org/10.4153/CJM-1968-013-7 Published online by Cambridge University Press

In this case, the alternative chains will be said to be *branches* of a redundant chain with *branch points*. A chain which has the availability α as an element will be described as a chain *through* α.

Suppose that an availability array A satisfies the strong Hall conditions and that there exists an interchange I with S through an availability $\alpha \in A$ and not to S (refer to Figure 1). It follows that every chain, cycle, planar interchange, and interchange through α must be constructed from the availabilities of the tight set T_α to which α belongs (cf. the proof of Theorem 4). Note also that the availabilities involved in a planar section of the interchange I form a simple planar cycle. There are in general, however, many more simple cycles through α than the three simple planar cycles which are formed by sections of I in the three planes through α. Indeed, more complicated simple cycles through α may be built up recursively, for it is easily seen that any planar section of I forms a simple planar cycle through an even-numbered and an odd-numbered element of *any* cycle which is made up of availabilities of I^+ and I^-, passes through α, and intersects the plane of section.

Now suppose there does not exist an interchange with S through an availability $\alpha \in A$. Theorem 4 guarantees the existence of a planar interchange with the appropriate section of S in all three planes through any availability belonging to T_α. Hence, since no interchange can be constructed from all the possible planar interchanges involving elements of T_α, there must exist one or more planar sections of T_α, which are necessarily involved in any interchange through α, but in which none of the possible planar interchanges match in the manner of the previous paragraph. This can only occur if there exists a redundant chain through α. Indeed, necessary involvement with α is brought about by condition (ii) for such chains, and condition (i) expresses the only ways in which planar interchanges can fail to match.

In summary

THEOREM 6. *Given a solution S to $T(R)$ contained in an availability array A satisfying the strong Hall conditions and an availability $\alpha \in A$, there exists through α either an interchange with S, or a necessarily redundant chain.*

The latter situation is illustrated in Figure 4. The figure shows two $4 \times 4 \times 4$ availability arrays, A_1 and A_2, for the time-table problem with requirements matrix

$$R = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 0 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 2 \end{bmatrix}$$

which satisfy the strong Hall conditions. Both arrays contain a solution. In A_1, a redundant chain with a branch point is marked which has elements in

https://doi.org/10.4153/CJM-1968-013-7 Published online by Cambridge University Press

a horizontal line with more odd-numbered than even-numbered elements of the chain. The redundant chain in A_2 has elements in a vertical line with more even-numbered than odd-numbered elements of the chain. Note that this chain includes all the unassigned availabilities of the array. Examples of redundant chains for small problems are quite difficult to construct, but this is not surprising in view of the fact that their instance during the progress of the time-tabling algorithm seems to be fairly rare (see **1**, p. 77). The arrays shown in Figure 4 are not reduced, since in both cases availabilities have been arbitrarily removed. However, the $9 \times 9 \times 9$ computer-generated reduced array resulting from the 36th assignment reported by Gotlieb and Csima may be shown to contain a redundant chain with a single branch point through two availabilities incompatible with the final solution obtained. In each case, subsequent to their assignment, the reduction process found the Hall conditions to be violated.

That this is not the general case is illustrated by Figure 5. It shows an

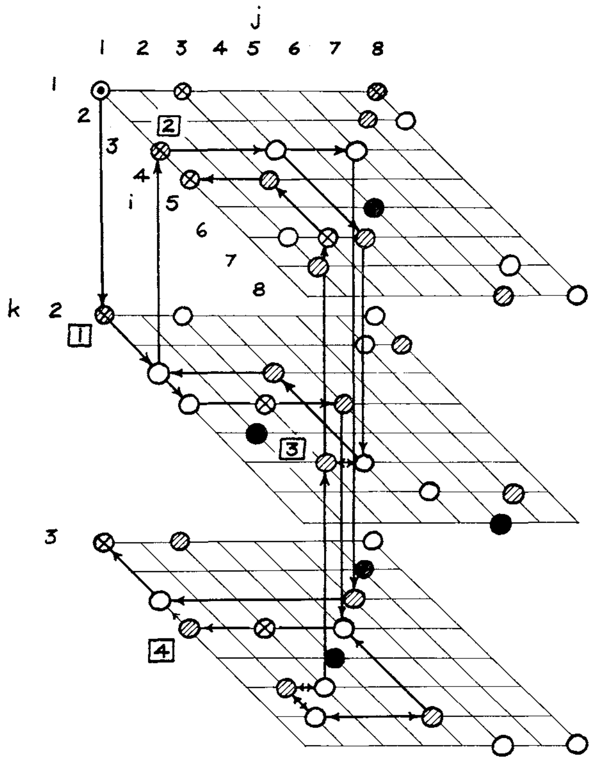

FIGURE 5.

https://doi.org/10.4153/CJM-1968-013-7 Published online by Cambridge University Press

$8 \times 8 \times 3$ reduced availability array for the time-table problem with requirements matrix

$$R = \begin{bmatrix} 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 1 \end{bmatrix}.$$

The first five elements of a set of specified final solution elements

$$\mathscr{S} = \{[5, 5, 1], [5, 2, 2], [8, 6, 2], [2, 7, 3], [5, 4, 3], (1, 1, 1)\} \subset A$$

are incorporated into a solution $S \subset A$. A redundant chain with four branch points is marked through the availability $(1, 1, 1)$. Although $(1, 1, 1)$ is thus incompatible with the other elements of $\mathscr{S}$, upon its assignment and the elimination of the crossed availabilities, the strong Hall conditions are left satisfied. Thus the incompatibility remains temporarily undiscovered by the reduction process. A similar event occurs for the $12 \times 12 \times 3$ computer-generated symmetric reduced array reported by Lions. A redundant chain with four branch points similar to that of Figure 5 may be shown to originate at the incompatible availability of that array.

The difficulties illustrated in Figure 5 are due to the existence of multi-branched redundant chains, and there *is* a circumstance in which a reduction process will always find the Hall conditions to be violated upon the assignment of an incompatible availability.

COROLLARY 7. *Given a solution S to $T(R)$ contained in an availability array A satisfying the strong Hall conditions and an availability $\alpha \in A$, if there exists a necessarily redundant chain with no branch points through α, then A_α is infeasible.*

Proof. Observe that the assignment of an odd-numbered element of a redundant chain with no branch points will cause a reduction process subsequently to assign all the odd-numbered elements of the chain except those in the distinguished line of condition (i) of the definition of a redundant chain (see A_2, Figure 4). In this line, one of the elements of the chain changed to zero by the reduction process will cause a line through this element to contain one less availability than is required in any solution to $T(R)$. The Hall conditions will be violated in the two planes through this line.

Notice that, in general, no solution passes through an odd-numbered element of a redundant chain only if the element precedes the first branch point in the defining sequences of the chain. More precisely, to be incompatible with the assigned elements of a solution S to $T(R)$, such an availability must be common to *all* the redundant chains making up a redundant chain with branch points.

5. Conclusions. The instance of the Gotlieb–Csima algorithm failing to detect incompatibilities in the course of actual time-tabling appears to be extremely rare **(7)**. Nevertheless, in the spirit of Gotlieb and Csima, it might be desirable to try to develop a restriction algorithm which would locate and eliminate availabilities not included in some solution. However, since it would have to locate redundant chains and eliminate all odd-numbered availabilities of each chain prior to its first branch point, such an algorithm is likely to be inefficient. Reduction procedures themselves have proved time-consuming, although certainly not prohibitively so.

Perhaps a more promising approach might be to begin with an initial solution S^0 to $T(R)$, and introduce the elements of $\mathscr{S}$ by constructing interchanges in order to produce a sequence of solutions $S^1, S^2, \ldots$. The interchanges with a given solution to $T(R)$ may be identified with a subset of, essentially, the cycles of the requirements matrix R. By virtue of Theorem 2, condition (iii), this subset is dependent only on the elements of $\mathscr{S}$ already incorporated in the solution, i.e. its assigned elements, and not on the solution itself. When the elements of $\mathscr{S}$ incorporated in the current solution are few, the cycles are typically 4-cycles. When the elements of $\mathscr{S}$ in the current solution are many, the possible cycles, although complicated, are few. Hence it seems reasonable to search exhaustively for interchanges. For this purpose, the problem may be translated into a graph recolouring problem in which an interchange corresponds to a true cycle of the 0–1 incidence matrix of a certain graph constructed from the requirements matrix. A theoretically more satisfying approach to the problem of incorporating the elements of $\mathscr{S}$ into a solution to $T(R)$ consists in translating the problem into one of constrained network flows. An algorithm for this form, essentially a 3-dimensional generalization of the Hungarian algorithm, promises to be efficient and, moreover, should be capable of producing time-tables optimal with respect to availability costs. Both methods are described elsewhere **(3)**.

6. Acknowledgments. The research reported in this paper was begun with the support of J. Kates and Associates, Toronto, and completed while I was the I.B.M. Research Fellow for 1965–66 in the University of Oxford. I would like to thank Drs. B. A. Griffith and J. Lions of J. Kates and Associates both for stimulating my interest in time-table problems, and for their helpful comments on earlier drafts of this paper. Conversations with them were responsible for most of the ideas presented here.

References

1. J. Csima, *Investigations on a time-table problem*, Doctoral dissertation, (issued as a report of the Institute of Computer Science), Univ. of Toronto (1965).

2. J. Csima and C. C. Gotlieb, *Tests on a computer method for constructing school time-tables*, Comm. ACM, *8* (1965), 160–163.

3. M. A. H. Dempster, *Two algorithms for the time-table problem* (to appear).

4. C. C. Gotlieb, *The construction of class-teacher time-tables*, Proc. I.F.I.P. Congress 62, Munich (Amsterdam, 1963), pp. 73–77.

5. J. Lions, *Matrix reduction using the Hungarian algorithm for the generation of school time-tables*, Comm. ACM, *9* (1966), 349–354.

6. ——— *A counter-example for Gotlieb's Method for the construction of school timetables*, Comm. ACM, *9* (1966), 697–698.

7. ——— *The Ontario school scheduling program*, Comput. J., *10* (May, 1967), 14–21.

8. H. J. Ryser, *Combinatorial mathematics*, Carus Math. Monographs No. 14 (New York, 1963).

Balliol College,
Oxford University