and $Cg = ng - nC = -\dfrac{a^2 \sin\theta}{b} + b\sin\theta = -\dfrac{a^2 e^2 \sin\theta}{b}$,

$$\therefore \qquad W\omega = -\frac{a^2 e^2 \sin^3\theta}{b}.$$

Hence the centre of curvature is $\left(\dfrac{a^2 e^2 \cos^3\theta}{a}, \quad -\dfrac{a^2 e^2 \sin^3\theta}{b}\right)$.

The "Pellian Equation" and Some Series for π.

By A. C. Aitken.

§ 1. The craze for extensive π-calculation which was so strange a feature of the last century was probably brought to an end not so much by the famous 707 decimals of W. Shanks in 1873 as by the demonstrations of Hermite and Lindemann, about the same time, regarding the transcendental nature of both e and π. Sporadic minor outbreaks of the disease still occur, of course, —Ramanujan in his earlier days was not entirely immune—and the series of the present note may seem symptomatic. It is hoped, however, that they will not be devoid of interest from the point of view of elementary trigonometry.

§ 2. It is mentioned in the standard texts that a useful series for evaluating π is Gregory's inverse tangent series,

$$\arctan x = x - x^3/_3 + x^5/_5 - x^7/_7 + \ldots,$$

but that for $x = 1$ it converges to $\pi/4$ with extreme slowness. Machin's formula,

$$\arctan 1 = 4 \arctan \frac{1}{5} - \arctan \frac{1}{239},$$

and variants such as Rutherford's give a greatly enhanced convergency.

The reason why the corresponding series for $\pi/6$, $\pi/8$, $\pi/10$, $\pi/12$ have not found favour is an obvious one; the tangents of these angles are irrational, tan $\pi/10$ even involving a double surd.

Consider however, for example, the sequence of rational approxi-

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mations to $1/\sqrt{3}$ derived by finding the integer solutions of the "Pellian" equation[1] (more properly attributable to Fermat)

$$3y^2 + 1 = z^2,$$

by the method (Lagrange's) of converting $\sqrt{3}$ into a recurring continued fraction. We obtain the sequence

$$\frac{0}{1}, \quad \frac{1}{2}, \quad \frac{4}{7}, \quad \frac{15}{26}, \quad \frac{56}{97}, \quad \frac{209}{362}, \ldots.$$

where the numerators, say p_r, and the denominators, q_r, satisfy a vast number of relations, of which the following concern us here:

$$p_{r+1} + p_{r-1} = 4p_r, \qquad q_{r+1} + q_{r-1} = 4q_r, \tag{a}$$

$$p_r q_{r-1} - p_{r-1} q_r = 1, \tag{b}$$

$$p_r^2 - p_{r+1} p_{r-1} = 1, \tag{c}$$

$$q_r = 2p_r - p_{r-1}, \qquad q_{r-1} = p_r - 2p_{r-1}. \tag{d}$$

[Incidentally the convergency of the sequence is very good. Each convergent commits about $1/14$ the error of the preceding one, and $209/362$ is already correct to five decimal places.]

Hence we have

$$\arctan 1/\sqrt{3} = \arctan\frac{1}{2} + \left(\arctan\frac{4}{7} - \arctan\frac{1}{2}\right) + \left(\arctan\frac{15}{26} - \arctan\frac{4}{7}\right) + \ldots$$

$$= \Sigma\left(\arctan\frac{p_r}{q_r} - \arctan\frac{p_{r-1}}{q_{r-}}\right)$$

$$= \Sigma \arctan\frac{p_r q_{r-1} - p_{r-1} q_r}{p_r p_{r-1} + q_r q_{r-1}}$$

$$= \Sigma \arctan\frac{1}{2(p_r - p_{r-1})^2}, \qquad \text{by } (b) \text{ and } (d).$$

Thus π is given by a rapidly converging series of inverse tangents,

$$\frac{\pi}{6} = \arctan\frac{1}{2} + \arctan\frac{1}{2\,.\,3^2} + \arctan\frac{1}{2\,.\,11^2} + \arctan\frac{1}{2\,.\,41^2} + \ldots.$$

§3. Again the numerators p_r themselves yield a sequence

$$0, \quad \frac{1}{4}, \quad \frac{4}{15}, \quad \frac{15}{56}, \quad \frac{56}{209}, \ldots.$$

tending to $2 - \sqrt{3}$, or $\tan \pi/12$. Hence

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[1] Chrystal: *Algebra*, Part II, p. 478.

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$$\arctan(2-\sqrt{3}) = \arctan\frac{1}{4} + \left(\arctan\frac{4}{15} - \arctan\frac{1}{4}\right) + \left(\arctan\frac{15}{56} - \arctan\frac{4}{15}\right) \cdot$$

$$= \Sigma\left(\arctan\frac{p_r}{p_{r+1}} - \arctan\frac{p_{r-1}}{p_r}\right)$$

$$= \Sigma \arctan\frac{p_r^2 - p_{r+1}p_{r-1}}{p_r(p_{r+1}+p_{r-1})}$$

$$= \Sigma \arctan\frac{1}{4p_r^2}, \qquad \text{by } (a) \text{ and } (c).$$

Thus we have another series, superior to the first,

$$\frac{\pi}{12} = \arctan\frac{1}{4} + \arctan\frac{1}{4\,.\,4^2} + \arctan\frac{1}{4\,.\,15^2} + \arctan\frac{1}{4\,.\,56^2} + \ldots.$$

§ 4. A similar inquiry into the properties of the sequence for $1/\sqrt{2}$,

$$\frac{0}{1}, \frac{1}{1}, \frac{2}{3}, \frac{5}{7}, \frac{12}{17}, \frac{29}{41}, \frac{70}{99}, \frac{169}{239}, \frac{408}{577}, \ldots.$$

derived from integer solutions of the Pellian equations

$$2y^2 \pm 1 = z^2,$$

yields an enormous variety of such series, the two simplest being

$$\frac{\pi}{8} = \arctan\frac{1}{3} + \arctan\frac{1}{17} + \arctan\frac{1}{99} + \arctan\frac{1}{577} + \ldots.$$

$$= \arctan\frac{1}{2} - \arctan\frac{1}{12} + \arctan\frac{1}{70} - \arctan\frac{1}{408} + \ldots.$$

When we attempt the same for $\pi/10$, using the convergents to $1/\sqrt{5}$, we are frustrated by the double irrationality of $\tan \pi/10$ already mentioned.

All of these curious series owe their existence to the fact that the addition theorem for inverse tangents of rational fractions is intimately connected with the recurrence relations between the numerators and denominators of successive convergents to a continued fraction.

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