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# SOME SPECIAL CLASSES OF CARTAN MATRICES

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Let $A = (A_{ij})_{1 \leqq i,j \leqq l}$ be a *Cartan matrix*, i.e., $A_{ii} = 2$ for all $i$ and $A_{ij}$ is an integer $\leqq 0$ for $i \neq j$, with $A_{ij} = 0$ if $A_{ji} = 0$. The size $l$ of $A$ is called its *rank*, for Lie-theoretic reasons, and may be larger than its matrix rank. We associate to $A$ its *Dynkin diagram*, with vertices $1, 2, \dots, l$, with $A_{ij}A_{ji}$ lines joining $i$ to $j$, and with an arrow pointing from $i$ to $j$ if $A_{ij}/A_{ji} < 1$, i.e., pointing toward the shorter root (see below). The Cartan matrix $A$ is *indecomposable* if its diagram is connected, and *symmetrizable* if there exist positive rational numbers $q_1, \dots, q_l$ with

$$q_i A_{ij} = q_j A_{ji} \quad \text{for all } i \text{ and } j.$$

Symmetrizability is automatic if the diagram contains no cycle. We assume throughout this paper that $A$ is symmetrizable, and so "Cartan matrix" always means "symmetrizable Cartan matrix". If $A$ is indecomposable, then the symmetrizing numbers $q_1, \dots, q_l$ are unique up to a proportionality constant, since if $q_i$ is known, and $i$ is connected to $j$, then $q_j$ is known. We normalize to have $q_i = 1/k_i$, where $k_1, \dots, k_l$ are positive integers without common factor.

Let

$$R = R_{\mathbf{Z}} = \coprod_{i=1}^{l} \mathbf{Z} \cdot \alpha_i$$

be the free $Z$-module on $\alpha_1, \dots, \alpha_l$ (the simple roots); similarly

$$R_{\mathbf{Q}} = \coprod_{i=1}^{l} \mathbf{Q}\alpha_i.$$

Let ( , ) denote the symmetric bilinear form on $R_{\mathbf{Q}}$ defined by

$$(\alpha_i, \alpha_j) = (1/k_i)A_{ij} = (\alpha_j, \alpha_i).$$

The *length* (squared) of $\alpha$ is $|\alpha|^2 = (\alpha, \alpha)$; in particular $(\alpha_i, \alpha_i) = 2/k_i$. We put

$$\alpha_i^\vee = 2\,\alpha_i/(\alpha_i, \alpha_i) = k_i\alpha_i \quad (1 \leqq i \leqq l).$$

Thus $(\alpha_i^\vee, \alpha_j) = A_{ij}$. The *Weyl group* $W$ of $A$ is the group of linear automorphisms of $R$ (or $R_{\mathbf{Q}}$) generated by the reflections $s_1, \dots, s_l$, where

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$$s_i(\alpha) = \alpha - (\alpha_i^\vee, \alpha)\alpha_i,$$

i.e., $s_i(\alpha_j) = \alpha_j - A_{ij}\alpha_i$. It is a Coxeter group, generated by involutions $s_1, \dots, s_l$, with relations generated by $(s_i s_j)^{m_{ij}} = 1$, where $m_{ij} = 2, 3, 4, 6$, or $\infty$ as $A_{ij}A_{ji}$ is 0, 1, 2, 3, or $\geqq 4$ (and $m_{ij} = \infty$ means "no relation"). We have $(w(\alpha), w(\beta)) = (\alpha, \beta)$, for $w \in W$.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the Kac-Moody algebra attached to $A$, i.e., the Lie algebra (say over $\mathbf{C}$) with $3l$ generators $E_i$, $F_i$, $H_i$ $(1 \leqq i \leqq l)$ and relations

i) $[H_i, E_j] = A_{ij}E_j \quad [H_i, F_j] = -A_{ij}F_j$
 $[E_i, F_j] = \delta_{ij}H_j \quad [H_i, H_j] = 0,$
ii) $(\mathrm{ad}\ E_i)^{1-A_{ij}}(E_j) = 0 = (\mathrm{ad}\ F_i)^{1-A_{ij}}(F_j)$ (for $i \neq j$).

(In the general case, one must divide by a certain radical. But Gabber and Kac [**4**] proved that the radical is 0 in the symmetrizable case, which we are assuming.) Then

$$\mathfrak{g} = \mathfrak{h} \oplus \coprod_\alpha \mathfrak{g}^\alpha,$$

where

$$\mathfrak{h} = \coprod_{i=1}^{l} \mathbf{C} \cdot H_i$$

is abelian of dimension $l$, and $\mathfrak{g}^\alpha$ is defined as follows, for

$$\alpha = \sum_{i=1}^{l} c_i\alpha_i \neq 0 \quad \text{in } R.$$

If all $c_i \geqq 0$, then $\mathfrak{g}^\alpha$ is the space generated by all multiple commutators

$$[X_1, \dots, X_n] = [X_1, [X_2, \dots, X_n]],$$

with $n = c_1 + \dots + c_l$, in which $E_i$ appears $c_i$ times among $X_1, \dots, X_n$; if all $c_i \leqq 0$, then it is generated by all such expressions in which $F_i$ appears $-c_i$ times, and it is 0 if $\alpha$ is mixed (some $c_i > 0$ and some $c_j < 0$). We call $\alpha$ a *root* and write $\alpha \in \Delta$, if $\alpha \neq 0$ and $\mathfrak{g}^\alpha \neq 0$, i.e.,

$$m(\alpha) = \dim \mathfrak{g}^\alpha > 0.$$

If $\alpha \in \Delta$ and $w \in W$, then $w(\alpha) \in \Delta$; in fact

$$m(\alpha) = m(w(\alpha)).$$

We call $\alpha$ a *real* root and write $\alpha \in \Delta_{\mathbf{R}}$ if $\alpha = w(\alpha_i)$ for some $i = 1, \dots, l$; then $m(\alpha) = 1$, since $\mathfrak{g}^{\alpha_i} = \mathbf{C} \cdot E_i$ has dimension 1. The set of *imaginary* roots is $\Delta_I = \Delta - \Delta_{\mathbf{R}}$. If

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$$\alpha = \sum_i^l c_i \alpha_i \in R,$$

we write $\alpha > 0$ or $\alpha \in R^+$ if all $c_i \geqq 0$ and some $c_i > 0$. Then $\Delta$ is the disjoint union of $\Delta^+ = \Delta \cap R^+$ and $-\Delta^+$, i.e., a root is unmixed, and the set $\Delta_I^+ = \Delta^+ \cap \Delta_I$ of positive imaginary roots is invariant under the action of $W$. Furthermore, a root

$$\alpha = \sum_i c_i \alpha_i$$

is *connected*, i.e., the subdiagram of the Dynkin diagram of $A$, obtained by retaining only those vertices $i$ for which $c_i \neq 0$, is connected. Thus, if $\alpha \in \Delta_I^+$, then $w(\alpha)$ is positive and connected for all $w \in W$, and Kac [**5**] has shown that this necessary condition is also sufficient, if $A$ is indecomposable, so the set of imaginary roots is in a sense known, although nothing seems to be known in any generality about the multiplicities $m(\alpha)$.

Let $A$ be a Cartan matrix. It is *finite* if the form ( , ) is positive definite, i.e., $\mathfrak{g}$ is of finite dimension, i.e., $W$ is finite, i.e., $\Delta_I$ is empty; cf. [**9**] for this and for the other definitions and assertions in this paragraph. The Cartan matrix $A$ is *Euclidean* if it is indecomposable and if the form is positive semidefinite, i.e., if $A$ is singular and every principal submatrix $A^{(i)}$, obtained by striking out the $i^{\text{th}}$ row and column, if finite (possibly decomposable). If $A$ is Euclidean, then it has a *principal null root*

$$\gamma = \sum_i c_i \alpha_i,$$

where the $c_i$ are positive integers without common factor, which generates the radical of the form and such that the imaginary roots are precisely the integer multiples of $\gamma$; in this case the dimensions of the imaginary root spaces are known ([**8**]). The Cartan matrix $A$ is *hyperbolic* if it is indecomposable, not finite and not Euclidean, with every indecomposable constituent of every principal submatrix finite or Euclidean. Then $A$ is nonsingular and the form ( , ) has signature $(l - 1, 1)$, and if any principal submatrix has a Euclidean constituent, then this constituent is the entire submatrix. Moody has shown that the imaginary roots $\alpha$ of a hyperbolic Cartan matrix are characterized by the condition $(\alpha, \alpha) \leqq 0$; in view of the later and more general result of Kac mentioned above, one can rephrase this result as follows: if $A$ is hyperbolic and $\alpha \in R$, $\alpha \neq 0$, satisfies $(\alpha, \alpha) \leqq 0$, then $\alpha$ is unmixed (say positive) and connected, and since $w(\alpha)$ has the same properties for any $w \in W$, we have $\alpha \in \Delta_I$. In Section 2 of this paper we give some further results on hyperbolic Cartan matrices, including the fact that they are characterized by the property that the dual basis to $\alpha_1^\vee, \ldots, \alpha_l^\vee$ consists of negative vectors.

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Thus the hyperbolic Cartan matrices are the best understood ones, after the finite and Euclidean ones, although even for them nothing seems to be known about the multiplicities. Unfortunately, there are very few hyperbolic matrices, since their rank is at most 10, as we show in Section 3, and as was proved long ago by Chein [**2**].

It follows that there are only finitely many hyperbolic Cartan matrices of rank $\geqq 3$, since then $A_{ij}A_{ji} \leqq 4$ whenever $i \neq j$. Thus it seems desirable to find a weaker concept with a wider class of examples, and we can do this by retaining the requirement of connectivity but giving up that of unmixedness. Accordingly, we shall call an indecomposable Cartan matrix $A$ *good* if every element of $R^+$ which is *small* (i.e., $(\alpha, \alpha_i) \leqq 0$ for all $i$; cf. Section 1) is connected and hence an imaginary root. Equivalently, $A$ is good if $\Delta_I^+$ is a semigroup under addition. If any principal submatrix of $A$ is finite or Euclidean, then $A$ is good, so this class is very wide. A convenient family of examples is given by the *superaffine* (our term) Cartan matrices introduced by Feingold and Frenkel [**3**], obtained by extending the extended Dynkin diagram of a finite simple Lie algebra once more, in the simplest possible way, i.e., by connecting a new vertex $-1$ to 0 by a simple bond. All of these matrices are good, but only twenty-six of them are hyperbolic. In Section 4, we show that the Weyl group of a superaffine matrix of rank 4 (or rather a subgroup of index 2) can be realized in a natural way as a modular group over an imaginary quadratic number field.

**1. Generalities.** Let $\Gamma^+$ be the set of all $\alpha \in R^+$ with $w(\alpha) > 0$ for all $w \in W$; thus $\Gamma^+ \supset \Delta_I^+$. If $\alpha \in R^+$, we say $\alpha$ is *small* if $w(\alpha) \geqq \alpha$ for all $w \in W$. The set $\Gamma_s^+$ of small elements of $R^+$ is contained in $\Gamma^+$; both $\Gamma^+$ and $\Gamma_s^+$ are semigroups under addition. If

$$\alpha = \sum_{i=1}^{l} c_i\alpha_i \in R^+,$$

we call

$$h(\alpha) = \sum_{i=1}^{l} c_i$$

its *height*.

PROPOSITION 1. *Each $W$-orbit in $\Gamma^+$ contains a (unique) small element. If $\alpha \in R^+$, and $s_i(\alpha) \geqq \alpha$ for $1 \leqq i \leqq l$ (i.e., $(\alpha_i^\vee, \alpha) \leqq 0$ for all $i$), then $\alpha$ is small.*

*Proof.* Any $W$-orbit contains an element $\alpha$ of minimal height; then $s_i(\alpha) \geqq \alpha$ for all $i$, so it is enough to prove the second statement. If $w \in W$, let $w = s_{i_1}s_{i_2} \ldots s_{i_k}$ be an expression of minimal length $k$. Then the set $\Phi_w$ of

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all $\beta \in \Delta^+$ with $w^{-1}(\beta) < 0$ has exactly $k$ elements, namely

$$\beta_1 = \alpha_{i_1}, \quad \beta_2 = s_{i_1}\alpha_{i_2}, \dots,$$
$$\beta_k = s_{i_1} \dots s_{i_{k-1}}\alpha_{i_k}$$

(cf. [**6**], for example), and we find by induction that

$$w(\alpha) = \alpha + \sum_{j=1}^{k} a_{i_j}\beta_j \geqq \alpha,$$

where $a_i = -(\alpha_i^\vee, \alpha) \geqq 0$.

Thus we have

$$(1) \qquad \Gamma^+ = \bigcup_{\alpha \in \Gamma_s^+} W \cdot \alpha,$$

a disjoint union; the stability group of $\alpha$ in $W$ is the group generated by all $s_i$ with $(\alpha_i^\vee, \alpha) = 0$. We have

$$(2) \qquad (\alpha, \beta) \leqq 0 \text{ if } \alpha, \beta \in \Gamma^+.$$

(Since $(w(\alpha), w(\beta)) = (\alpha, \beta)$, we can assume that $\alpha$ is small, in which case (2) is obvious.) Note that $\Gamma^+$ is empty if $A$ is finite, so we may as well assume that $A$ is not finite in this section.

PROPOSITION 2. *$\Gamma^+$ is the semigroup generated by $\Delta_I^+$.*

*Proof.* It is enough to show that any small element $\alpha > 0$ is a sum of positive imaginary roots. Write

$$\alpha = \beta_1 + \dots + \beta_k,$$

where $\beta \in \Delta^+$ and $k$ is minimal. If the $\beta_i$ are all imaginary, then we are done, so assume that $\beta_k$ (say) is real, and let $w$ denote the $\beta_k$-reflection, i.e.,

$$w(\gamma) = \gamma - (\beta_k^\vee, \gamma)\beta_k.$$

If $i < k$, then $(\beta_i, \beta_k^\vee) \geqq 0$, since otherwise

$$w(\beta_i) \geqq \beta_i + \beta_k$$

and so $\beta_i + \beta_k$ is a root and we can lower $k$. (Here we use the properties of root strings; cf. [**9**].) Thus

$$(\alpha, \beta_k^\vee) \geqq (\beta_k, \beta_k^\vee) = 2,$$

so $w(\alpha)$ is lower than $\alpha$, a contradiction.

We call an indecomposable Cartan matrix $A$ *good* if every element of $\Gamma_s^+$ is connected, and hence an imaginary root, by the result of Kac mentioned in the introduction. Equivalently, $A$ is good if and only if

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$\Gamma^+ = \Delta_I^+$, by (1), i.e., $\Delta_I^+$ is a semigroup under addition. If $A$ is finite, Euclidean, or hyperbolic, then $A$ is good. (If $A$ is finite, then $\Gamma^+$ is empty, and if $A$ is Euclidean, then $\Gamma^+$ consists of the multiples $n\gamma$, $n \geqq 1$, of the principal null root $\gamma$, whose support is all of $S = \{1, 2, \ldots, l\}$; one might prefer to exclude these cases from the class.)

If $A$ is bad, then there is a disconnected $\alpha = \alpha' + \alpha''$ in $\Gamma_s^+$; here we suppose that $\alpha'$ and $\alpha'' \in R^+$ have disjoint supports $S'$ and $S''$, with

$$(\alpha_i, \alpha_j) = 0 \quad \text{for } i \in S' \text{ and } j \in S''.$$

If $(\alpha', \alpha_i) > 0$, then $i \in S'$, so

$$(\alpha'', \alpha_i) = 0 \quad \text{and} \quad (\alpha, \alpha_i) = (\alpha', \alpha_i) > 0,$$

contrary to $\alpha$ being small. Thus $\alpha'$ and $\alpha''$ are both small, and in particular there are imaginary roots with support contained in $S'$ and in $S''$. This shows that no principal submatrix $A^{(i)}$ is finite, since if it were finite, then either $S'$ or $S''$ is contained in the complement $S^{(i)}$ of $i$ in $S$, and cannot support an imaginary root. Also, $A^{(i)}$ cannot be Euclidean, since then we would have (say) $S' \subset S^{(i)}$, so $\alpha'$ is a multiple of the principal null root $\gamma_i$ of $A^{(i)}$ and hence has support equal to all of $S^{(i)}$, and then $S'' = \{i\}$ cannot support an imaginary root. Thus;

PROPOSITION 3. *Let $A$ be an indecomposable Cartan matrix. If any principal submatrix of $A$ is finite or Euclidean, then $A$ is good.*

*Remark.* Let $A$ be good and let

$$\alpha = \sum_i c_i\alpha_i \in \Gamma_s^+ \quad \text{with } (\alpha, \alpha) = 0.$$

Then $(\alpha, \alpha_i) = 0$ for $i \in S'$, the support of $\alpha$. Then the submatrix $A'$ of $A$ with support $S'$ (obtained by striking out the $j^{\text{th}}$ row and column of $A$ for all $j \notin S'$) is Euclidean (cf. [**9**], Proposition 4) and $\alpha$ is a multiple of the principal null root $\gamma'$ of $A'$. Any such small null root $\alpha$ will be discovered by a glance at the diagram of $A$; the union of the $W$-orbits of such $\alpha$ is then the set of all null positive roots, i.e., of all $\alpha \in \Delta_I^+$ with $(\alpha, \alpha) = 0$.

**2. Nonsingular and hyperbolic Cartan matrices.** If $A$ is nonsingular, i.e., if the form $(\,,\,)$ is nondegenerate, then we can express the properties discussed above most conveniently in terms of the antidual basis $\omega_1, \ldots, \omega_l$ to $\alpha_1^\vee, \ldots, \alpha_l^\vee$ of $R_{\mathbf{Q}}$:

$$(3) \qquad (\omega_i, \alpha_j^\vee) = -\delta_{ij}.$$

We write $\alpha \in R$ as

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$$(4) \qquad \alpha = \sum_{i=1}^{l} c_i \alpha_i = \sum_{i=1}^{l} c_i^\vee \alpha_i^\vee = \sum_{i=1}^{l} a_i \omega_i.$$

Thus $c_i \in \mathbf{Z}$, $c_i^\vee = c_i/k_i$, and

$$a_i = -(\alpha, \alpha_i^\vee) = -\sum_j A_{ij} c_j \in \mathbf{Z},$$

and $\alpha \in \Gamma_s^+$ if and only if all $c_i \geqq 0$ (and not all 0) and all $a_i \geqq 0$.

We take now $A$ to be hyperbolic (hence nonsingular). Let

$$R_{\mathbf{Q}}^{(i)} = \sum_{j \neq i} \mathbf{Q} \cdot \alpha_j$$

denote the root space of the principal submatrix $A^{(i)}$, for $1 \leqq i \leqq l$. The form ( , ) is nondegenerate of signature $(l - 1, 1)$ on $R_{\mathbf{Q}}$, and $\geqq 0$ on each $R_{\mathbf{Q}}^{(i)}$; more precisely, it is $> 0$ on $R_{\mathbf{Q}}^{(i)}$ if $A^{(i)}$ is finite, and $\geqq 0$, with a 1-dimensional radical (generated by the principal null root $\gamma_i$) if $A^{(i)}$ is Euclidean. If $A^{(i)}$ is finite, then

$$R_{\mathbf{Q}} = \mathbf{Q} \cdot \omega_i \oplus R_{\mathbf{Q}}^{(i)}$$

(orthogonal direct sum), so $(\omega_i, \omega_i) < 0$. If $A^{(i)}$ is Euclidean, then the principal null root $\gamma_i$ must be a multiple of $\omega_i$, so $\omega_i \in R_{\mathbf{Q}}^{(i)}$ and $(\omega_i, \omega_i) = 0$. Thus $(\omega_i, \omega_i) \leqq 0$ in either case, so by [**9**], each $\omega_i$ is unmixed. We have

$$\omega_i = -\sum_j (\omega_i, \omega_j) \alpha_j^\vee,$$

and so for fixed $i$, the $(\omega_i, \omega_j)$ are either all $\leqq 0$ or all $\geqq 0$. If $A^{(i)}$ is finite, then $(\omega_i, \omega_i) < 0$ and so $(\omega_i, \omega_j) \leqq 0$ for all $j$, i.e., $\omega_i > 0$. If $A^{(i)}$ is Euclidean, then we have

$$-(\omega_i, \alpha_i) = \sum_{j \neq i} (\omega_i, \omega_j) A_{ji} > 0,$$

with each $A_{ji} \leqq 0$, so at least one and hence all $(\omega_i, \omega_j)$ are $< 0$, and we have again $\omega_i > 0$. Thus each $\omega_i$ is positive, if $A$ is hyperbolic.

Conversely, let $A$ be indecomposable and nonsingular, with $\omega_i > 0$ for each $i$. Put as before $S = \{1, \ldots, l\}$ and $S^{(i)} = S - \{i\}$; if $C$ is a component of $S^{(i)}$, then $C \cup \{i\}$ is connected. We are given that $(\omega_i, \omega_j) \leqq 0$ for all $i$ and $j$. Suppose that $(\omega_i, \omega_j) = 0$ for some $i \neq j$. Then

$$\omega_i = -\sum_{k \neq j} (\omega_i, \omega_k) \alpha_k^\vee,$$

and so

$$0 = (\omega_i, \alpha_j) = -\sum_{k \neq j} (\omega_i, \omega_k) A_{kj},$$

with each term $\leqq 0$ and hence $= 0$. Thus $(\omega_i, \omega_k) = 0$ if $A_{kj} < 0$ and hence (by induction) if $k$ is connected to $j$ in $S^{(i)}$. Thus $(\omega_i, \omega_k) = 0$ for all $k$ in the component $C$ in $S^{(i)}$ containing $j$. Now choose $k$ with $(\omega_i, \omega_k) < 0$; then $k$ and $j$ are in different components of $S^{(i)}$, so we can connect $k$ to $i$ in $S^{(j)}$ and $j$ to $i$ in $S^{(k)}$. Centering the argument above about $j$, then about $k$, instead of $i$, we get that $(\omega_j, \omega_k) = 0$, since $(\omega_j, \omega_i) = 0$, and then $(\omega_k, \omega_i) = 0$ since $(\omega_k, \omega_j) = 0$, a contradiction. Thus in fact

$$(\omega_i, \omega_j) < 0 \text{ for all } i \neq j.$$

Now let $\alpha \in R^{(i)}$ with $(\alpha, \alpha) \leqq 0$; we will show that $\alpha$ is then a multiple of $\omega_i$ and so $(\alpha, \alpha) = 0$. We can assume that $\alpha > 0$, since if $\alpha = \alpha^+ - \alpha^-$ (with $\alpha^{\pm}$ positive and with disjoint supports), then

$$(\alpha, \alpha) = (\alpha^+, \alpha^+) + (\alpha^-, \alpha^-) - 2(\alpha^+, \alpha^-) \leqq 0,$$

and so

$$(\alpha^+, \alpha^+) + (\alpha^-, \alpha^-) \leqq 2(\alpha^+, \alpha^-) \leqq 0,$$

so at least one of the terms on the left is $\leqq 0$. Assume then that $\alpha > 0$, and that $\alpha$ has minimal height among such elements. Then $\alpha$ is small, for if $(\alpha, \alpha_j) > 0$ for some $j$, then $j \neq i$, and

$$s_j(\alpha) = \alpha - (\alpha, \alpha_j^\vee)\alpha_j \in R^{(i)},$$

so the positive part of $s_j(\alpha)$ satisfies our condition and has smaller height. Thus

$$\alpha = \sum_j a_j \omega_j \in R^{(i)} \quad \text{with } a_j \geqq 0.$$

Since $\alpha_i$ appears in $\omega_j$ for $j \neq i$, we must have $a_j = 0$. Hence $\alpha$ is a multiple of $\omega_i$, as claimed.

Thus either $( \, , \, ) > 0$ on $R_{\mathbf{Q}}^{(i)}$, so $A^{(i)}$ is finite, or $( \, , \, ) \geqq 0$ on $R_{\mathbf{Q}}^{(i)}$, with a 1-dimensional radical generated by $\omega_i$. According to Moody [**9**, Proposition 4], $A^{(i)}$ is Euclidean in this second case, provided it is indecomposable. Now the support of $\omega_i$ is $S^{(i)}$, as shown above, so if $A^{(i)}$ is decomposable, then

$$\omega_i = \alpha' + \alpha'',$$

where $\alpha'$, $\alpha''$ are positive with disconnected supports and hence with $(\alpha', \alpha'') = 0$. Then

$$0 = (\omega_i, \omega_i) = (\alpha', \alpha') + (\alpha'', \alpha''),$$

with each term $\leqq 0$ and hence $= 0$. But then $\alpha'$ and $\alpha''$ are proportional to $\omega_i$, which is clearly not the case. Thus we have proved:

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THEOREM 1. *Let $A$ be nonsingular and indecomposable. Then $A$ is hyperbolic if and only if $\omega_i > 0$ for all $i$. If so, then $(\omega_i, \omega_j) < 0$ for all $i \neq j$, $(\omega_i, \omega_i) < 0$ if $A^{(i)}$ is finite, and $(\omega_i, \omega_i) = 0$ if $A^{(i)}$ is Euclidean.*

*Remarks.* If $A$ is hyperbolic, then $\Gamma_s^+$ consists of all

$$\alpha = \sum_i a_i\omega_i \quad \text{with } a_i \in \mathbf{Z} \geqq 0$$

which are integral (have integral $\alpha_i$-coefficients $c_i$). Another way to express the result of the theorem is to say that $A$ is hyperbolic if and only if the inverse matrix $((\omega_i, \omega_j))$ to $((\alpha_i^\vee, \alpha_j^\vee))$ has all entries $\leqq 0$. If $A$ is nonsingular, indecomposable, and neither finite nor hyperbolic, then some of the $\omega_i$ are mixed, but there will be imaginary roots [**9**] and hence $\Gamma_s^+$ is not empty, so there will exist linear combinations

$$\sum_i a_i\omega_i > 0 \quad \text{with each } a_i \geqq 0.$$

Finally, we try to characterize the real roots $\alpha$ of a hyperbolic Cartan matrix $A$ by their length $(\alpha, \alpha)$; obviously, if $\alpha$ is a real root, then

$$(\alpha, \alpha) = (\alpha_i, \alpha_i) = 2/k_i \quad \text{for some } i.$$

Taking $k$ to be the maximum of $k_1, \ldots, k_l$, the smallest length of a real root is $2/k$.

PROPOSITION 4. *If $\alpha \in R$ and $(\alpha, \alpha) = 2/k$, then $\alpha$ is a real root. In particular, if $A$ is a symmetric hyperbolic Cartan matrix, then*

$$\Delta_{\mathbf{R}} = \{\alpha \in R:(\alpha, \alpha) = 2\}.$$

*Proof.* Suppose that $\alpha$ is mixed: $\alpha = \alpha^+ - \alpha^-$, where $\alpha^+$ and $\alpha^-$ are positive, with disjoint supports. Then

$$2/k = (\alpha^+, \alpha^+) + (\alpha^-, \alpha^-) - 2(\alpha^+, \alpha^-),$$

where each term on the right is $\geqq 0$. If $\alpha^+ = c_i\alpha_i$ has a one-point support, then $c_i = 1$ and the other two terms are 0. Then $A^{(i)}$ is Euclidean, and $\alpha^-$ is a positive multiple of the principal null root $\gamma_i$. But then $(\alpha^+, \alpha^-) < 0$, since $(\alpha_i, \gamma_i) < 0$, a contradiction. Hence $\alpha^+$, and similarly $\alpha^-$, has at least two points in its support. Then $\alpha^+$ and $\alpha^-$ lie in finite root spaces and so have length $\geqq 2/k$, a contradiction. Hence $\alpha$ is unmixed, say

$$\alpha = \sum_i c_i\alpha_i > 0.$$

We assume that $c_i > 0$ for more than one $i$, since otherwise $\alpha = \alpha_i \in \Delta_{\mathbf{R}}$. Since

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$$\sum_i c_i(\alpha, \alpha_i) = (\alpha, \alpha) > 0,$$

we must have $(\alpha, \alpha_i) > 0$ for some $i$. Then $s_i(\alpha)$ is still unmixed, by the above, and hence still positive, but lower. Continuing in this manner, we reduce the support eventually to a single point, and hence have a real root.

**3. Superaffine Cartan matrices.** These Cartan matrices were introduced in [**3**]. Changing the notation, we let $A = (A_{ij})_{1 \leqq i,j \leqq l}$ denote a finite indecomposable Cartan matrix, associated to a finite simple Lie algebra $\mathfrak{g}$ and a choice of Cartan subalgebra $\mathfrak{h}$ and simple roots $\alpha_1, \dots, \alpha_l$; $(\ ,\ )$ denotes the usual inner product, normalized so that the longer roots have (squared) length $(\alpha, \alpha) = 2$. Thus

$$A_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i).$$

Let $\theta$ be the highest root and put $\beta_o = -\theta$, and $\beta_i = \alpha_i$ for $1 \leqq i \leqq l$. Then the associated affine Cartan matrix is

$$\widetilde{A} = (A_{ij})_{0 \leqq i,j \leqq l},$$

where

$$A_{ij} = 2(\beta_i, \beta_j)/(\beta_i, \beta_i).$$

The matrix $\widetilde{A}$ is also called Euclidean of type I, and its imaginary root spaces all have dimension $l$; cf. [**8**].

Write

$$\theta = \sum_{i=1}^{l} n_i\alpha_i = \sum_{i=1}^{l} n_i^\vee \alpha_i^\vee,$$

where $n_i \in \mathbf{Z}_{>0}$ and $n_i^\vee = n_i/k_i$. We extend the form to

$$\widetilde{R} = \coprod_{i=0}^{l} \mathbf{Z} \cdot \alpha_i$$

(or to $\widetilde{R}_{\mathbf{Q}}$) by

$$(\alpha_i, \alpha_j) = (1/k_i)A_{ij},$$

where $k_0 = 1$; note that $\alpha_0$ is a formal symbol, i.e., $\widetilde{R} \simeq \mathbf{Z}^{l+1}$, and we must not identify $\alpha_0$ with $-\theta$. Let

$$\gamma = \alpha_0 + \theta = \sum_{i=0}^{l} n_i\alpha_i = \sum_{i=0}^{l} n_i^\vee \alpha_i^\vee$$

be the principal null root of $\widetilde{A}$; it generates the radical of $(\ ,\ )$. Another way to say this: we have a natural map $\phi: \widetilde{R} \to R$, projecting $\alpha_0$ to $-\theta$,

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with kernel $\mathbf{Z} \cdot \gamma$ and with

$$(\alpha, \alpha') = (\phi\alpha, \phi\alpha').$$

The Dynkin diagram of $\widetilde{A}$ is the extended Dynkin diagram of $A$; it is given in all cases in the appendix of Bourbaki [**1**], together with many other useful facts, including the highest root $\theta$. We extend once more, in the simplest way, with a simple bond joining a new index $-1$ to 0, to get the associated superaffine Cartan matrix $\hat{A} = (A_{ij})_{-1 \leqq i,j \leqq l}$:

$$\hat{A} = \begin{pmatrix} 2 & -1 \quad 0 \quad \dots \quad 0 \\ -1 & \\ 0 & \\ \cdot & \widetilde{A} \\ \cdot & \\ \cdot & \\ 0 & \end{pmatrix}$$

We put $k_{-1} = 1$ and extend the form ( , ) to the formal sum

$$\hat{R} = \sum_{i=-1}^{l} \mathbf{Z} \cdot \alpha_i$$

by $(\alpha_i^\vee, \alpha_j) = A_{ij}$, where as always $\alpha_i^\vee = k_i\alpha_i$.

The matrix $\hat{A}$ is good, by Proposition 3, because $\hat{A}^{(-1)} = \widetilde{A}$ is Euclidean, or because $\hat{A}^{(0)} = A_1 + A$ is finite. It is nonsingular; in fact

$$\det \hat{A} = -\det A.$$

Here, let $C_i$ be the $i^{\text{th}}$ column of $\hat{A}$, and replace $C_0$ by

$$\sum_{i=0}^{l} n_i C_i = \begin{pmatrix} -1 \\ 0 \\ \cdot \\ \cdot \\ \cdot \\ 0 \end{pmatrix}.$$

*Remark*. Since $\hat{A}$ is nonsingular, the formal elements $\alpha_{-1}, \dots, \alpha_l$ define independent functionals on $\hat{\mathfrak{h}}$, the span of $H_0, \dots, H_l$, by

$$\alpha_i(H_j) = A_{ji},$$

and we may identify them. This gives a natural definition of the extended algebra $\widetilde{\mathfrak{h}}^e = \hat{\mathfrak{h}}$; cf. [**6**]. The Weyl group $\widetilde{W}$ of $\widetilde{A}$ can be regarded as the group of automorphisms of $\widetilde{\mathfrak{h}}^e = \hat{\mathfrak{h}}$ generated by $s_0, \dots, s_l$, cf. [**6**], so we have

$$\hat{W} \supset \widetilde{W} \supset W,$$

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where $\hat{W}$ resp. $\tilde{W}$ resp. $W$ is the group generated by the reflections $s_i$ with $i \geqq -1$ resp. $i \geqq 0$ resp. $i \geqq 1$.

The antidual basis $\omega_{-1}, \ldots, \omega_l$ to $\alpha^\vee_{-1}, \ldots, \alpha^\vee_l$ is as follows. Clearly

(5) $\omega_{-1} = \gamma,\ \omega_0 = \alpha_{-1} + 2\gamma;$

note that $\omega_0$ is a root and $\omega_{-1}$ is a null root. Let $\eta_1, \ldots, \eta_l$ be the dual basis (with plus sign) to $\alpha^\vee_1, \ldots, \alpha^\vee_l$ in $R_{\mathbf{Q}}$:

$$(\eta_i, \alpha^\vee_j) = \delta_{ij} \quad (1 \leqq i, j \leqq l).$$

By the classical theory, we have $(\eta_i, \eta_j) > 0$ for all $i, j \geqq 1$; the values are listed in the appendix of [**1**]. Now $(\eta_i, \gamma) = 0$, so

$$(\eta_i, \alpha_0) = -(\eta_i, \theta) = -n^\vee_i = n^\vee_i(\omega_0, \alpha_0),$$

and $(\eta_i, \alpha_{-1}) = 0$, so we have

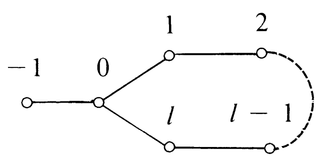
(6) $\omega_i = n^\vee_i \omega_0 - \eta_i \quad (1 \leqq i \leqq l).$

We do not necessarily have $\omega_i > 0$, as we show next; $\hat{A}$ is in general not hyperbolic.

Suppose for example that $A = A_l$, i.e., $\mathfrak{g} = \mathfrak{sl}_{l+1}$. If $l = 1$ we have

$$\hat{A} = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -2 \\ 0 & -2 & 2 \end{pmatrix},$$

studied in detail by Feingold and Frenkel, and certainly hyperbolic. For $l \geqq 2$ the diagram of $\hat{A}$ is




A glance at this diagram shows that $\hat{A}^{(i)}_l$ is not finite or Euclidean for $l$ large and (say) $i$ near $l/2$. Analytically, we can check that some $\omega_i$ is not positive, and so $\hat{A}_l$ is not hyperbolic, for $l > 7$, as follows. We have all $k_i = 1$ and $\theta = \alpha_1 + \ldots + \alpha_l$, so all $n_i = 1$. Then $\omega_i = \omega_0 - \eta_i$ and so

$$(\omega_i, \omega_j) = -2 + (\eta_i, \eta_j) \quad \text{for } i, j \geqq 1.$$

Now

$$(\eta_i, \eta_j) = i(1 - j/(l + 1)) \quad \text{for } 1 \leqq i \leqq j$$

(cf. [**1**]). If $l$ is odd, take $i = (l + 1)/2$, getting

$$(\omega_i, \omega_i) = -2 + (l + 1)/4 = (l - 7)/4,$$

so $\omega_i$ is not positive if $l > 7$. Similarly, if $l$ is even, take $i = l/2$, getting $\omega_i$

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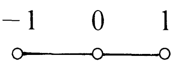
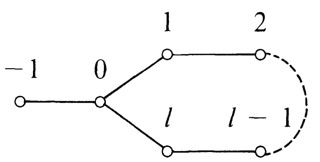
$\not> 0$ for $l > 7$. If $l \leqq 7$, then $\hat{A}_l$ is hyperbolic; to check this, we need only verify that $\hat{A}_l^{(i)}$ is finite or affine (as it turns out) for $1 \leqq i \leqq (l + 1)/2$ (using the symmetry $i \leftrightarrow (l + 1 - i)$ ). We find that $\hat{A}_l^{(1)} = A_{l+1}$ and $\hat{A}_l^{(2)} = D_{l+1}$ are finite for all $l$, and $\hat{A}_l^{(3)} = E_{l+1}$ for $l = 5, 6, 7$, and $\hat{A}_7^{(4)} = \tilde{E}_7$. The cases $A = B_l$ $(l \geqq 2)$, $C_l (l \geqq 3)$, $D_l (l \geqq 4)$, $E_6$, $E_7$, $E_8$, $F_4$, $G_2$ can be treated by the same methods; the result is:

THEOREM 2. *There are exactly twenty-six superaffine Cartan matrices $\hat{A}$ which are hyperbolic. They are listed below, together with their Dynkin diagrams (which show that the twenty-six cases are indeed nonisomorphic).*

i) $\hat{A}_l$, $1 \leqq l \leqq 7$.

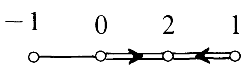



$(l = 1)$

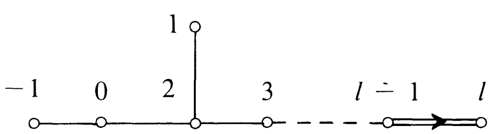



$(l \geqq 2)$

ii) $\hat{B}_l$, $2 \leqq l \leqq 8$.

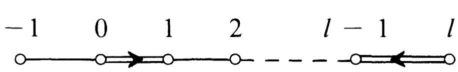



$(l = 2)$

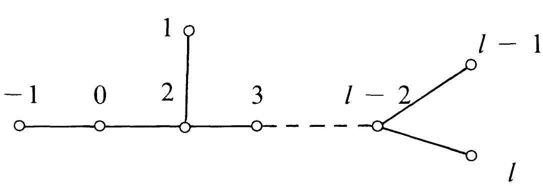



$(l \geqq 3)$

iii) $\hat{C}_l$:$3 \leqq l \leqq 4$.




iv) $\hat{D}_l$:$4 \leqq l \leqq 8$.

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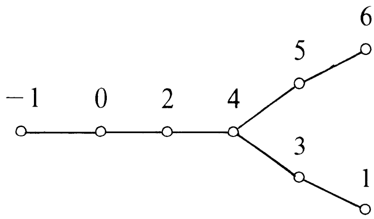
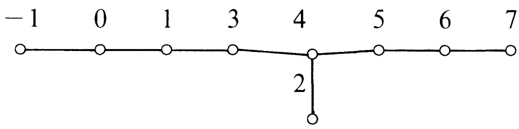
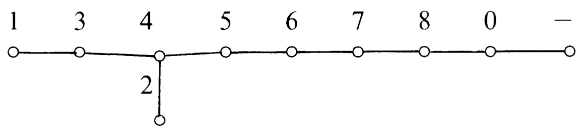
v) $\hat{E}_6$.



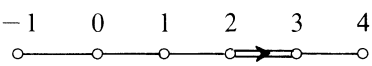

$\hat{E}_7$.



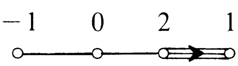

$\hat{E}_8$.




$\hat{F}_4$.

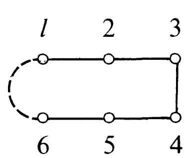



$\hat{G}_2$.




(*The indexing is that of Bourbaki.*) *The factors* $\hat{A}^{(i)}$ *are finite, except for* $\hat{A}^{(-1)} = \tilde{A}$ *and* $\hat{A}_7^{(4)} = \tilde{E}_7$, $\hat{B}_8^{(8)} = \tilde{E}_8$, $\hat{C}_4^{(4)}$ *is the dual of* $\tilde{F}_4$, *and* $\hat{D}_8^{(8)} = \hat{D}_8^{(7)} = \tilde{E}_8$.

The same kind of reasoning (using the diagram) shows that any hyperbolic Cartan matrix $A$ has rank $l \leqq 10$; cf. [**2**]. Deleting any vertex from the diagram of $A$ gives a finite or Euclidean diagram, i.e., an entry in a known list ([**7**], for example, has a list of all Euclidean diagrams, called affine in that paper); deleting any two vertices gives a finite diagram. The only finite or Euclidean diagram which contains a cycle is that for $\tilde{A}_l$, $l \geqq 2$. Suppose that one of the principal submatrices, say $A^{(1)}$, has a cycle in its diagram. We can assume then that $A^{(1)}$ has the diagram

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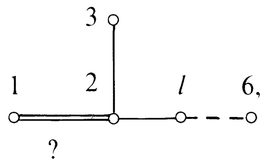
and that 1 is connected to 2. Suppose now that $l \geqq 6$, and that 1 is connected to $j \neq 2$ as well. If $j = l$ resp. $l - 1$, then the diagram contains a 3-cycle resp. a 4-cycle, contrary to $l \geqq 6$; hence $j \leqq l - 2$, and the diagram contains a $j$-cycle, also impossible. Thus 1 is connected only to 2. The bond is simple, since otherwise we would get, after dropping 4 and 5:

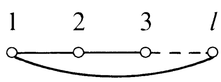



not a finite diagram for $l \geqq 6$. Thus:

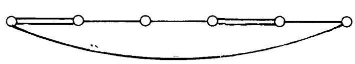
PROPOSITION 5. *If $A$ is hyperbolic of rank $l \geqq 6$, and its diagram contains a cycle of length $< l$, then*

$$A = \hat{A}_{l-2} \quad \textit{and} \quad l \leqq 9.$$

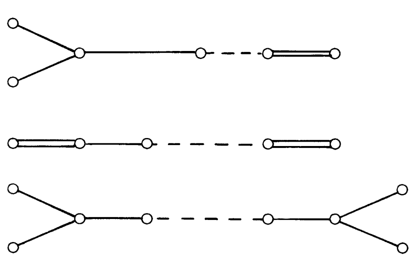
Suppose now that $A$ is hyperbolic of rank $l \geqq 6$ and is not superaffine (as we have just classified the hyperbolic superaffine Cartan matrices). Thus the diagram contains no cycle of length $< l$. Suppose that it contains a cycle of length $l$, say with diagram




possibly with multiple bonds. If there is a multiple bond, say from 1 to 2, then deleting 4 and 5 gives a finite diagram of rank $\geqq 4$ with a multiple bond in the middle, necessarily $F_4$:



Otherwise we have $\tilde{A}_{l-1}$, which is not hyperbolic. Thus, assuming now that $A$ is of rank $\geqq 7$ (and hyperbolic but not superaffine), there will be no cycle in the diagram. Then the diagram has an end, say 1, so the diagram for $A^{(1)}$ is connected, either finite or Euclidean, and 1 is connected to only one other vertex. If $A^{(1)}$ is Euclidean and not $\tilde{E}_6$, $\tilde{E}_7$, or $\tilde{E}_8$, then its diagram is of one of the following forms (cf. [7] ):



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where the arrow on any double bond can point in either direction. The vertex 1 cannot be connected to a vertex at either end of a double bond, since dropping two vertices at the other end must yield a finite diagram. By the same reasoning, the middle case does not occur. In the first case, we must get a finite diagram upon dropping the two vertices on the right, so 1 is connected by a simple bond to one of the vertices on the left, and we have $\hat{B}_{l-2}$ or its dual (with the arrow on the double bond reversed), with $l \leqq 10$. The third case implies that $A = \hat{D}_{l-2}$ is superaffine. By the same reasoning, we find that $A^{(1)} = \tilde{E}_6$ or $\tilde{E}_7$ is not possible and that $A^{(1)} = \tilde{E}_8$ implies that $A$ is the dual of $\hat{B}_8$. Finally, if $A^{(1)}$ is finite (and indecomposable), we find by the same sort of argument that $A$ is again the dual of $\hat{B}_{l-2}$ with $l \leqq 10$. Thus:

THEOREM 3. *Let $A$ be a hyperbolic Cartan matrix of rank $l$. Then $l \leqq 10$, and $A$ is superaffine or dual to superaffine if $l \geqq 7$.*

**4. The Weyl group in some superaffine cases (rank $\leqq$ 4).** Write $\alpha \in \hat{R}_{\mathbf{Q}}$ in the form

(7) $$\alpha = a\gamma + c(\alpha_{-1} + \gamma) + \beta \quad (a, c \in \mathbf{Q}, \beta \in R_{\mathbf{Q}}).$$

Then $\alpha$ is integral, i.e., $\alpha \in \hat{R}$, if and only if $a$ and $c$ are in $\mathbf{Z}$ and $\beta \in R$. Here

$$\gamma = \omega_{-1} \text{ and } \alpha_{-1} + \gamma = s_{-1}(\gamma)$$

are null roots orthogonal to $R_{\mathbf{Q}}$, so we have

(8) $$(\alpha, \alpha) = -2ac + (\beta, \beta), \quad \text{or}$$

$$(\alpha_1, \alpha_2) = -a_1c_2 - a_2c_1 + (\beta_1, \beta_2).$$

Note that $s_{-1}$ interchanges $a$ and $c$, and leaves $\beta$ fixed, while $s_1, \dots, s_l$ act on $\beta$ alone (as the finite Weyl group $W$). As for $s_0$, we have

$$(\alpha, \overset{\vee}{\alpha}_0) = (\alpha, \alpha_0) = -c + (\beta, \alpha_0) = -c - (\beta, \theta),$$

and so

(9) $$s_0(\alpha) = \alpha + (c + (\beta, \theta))\alpha_0$$
$$= \alpha + (c + (\beta, \theta))(\gamma - \theta).$$

Suppose now that $l \leqq 2$. Let $W^+$ be the even part of the Weyl group $W$ of $A$ and let $K$ be the field of $|W^+|$-th roots of 1; thus $[K:\mathbf{Q}] = l$, specifically $K = \mathbf{Q}$ resp. $\mathbf{Q}(i)$ resp. $\mathbf{Q}(\rho)$, where $i = \sqrt[4]{1}$ and $\rho = \sqrt[3]{1}$, as $A = A_1$ resp. $B_2$ resp. $A_2$ or $G_2$. By looking at a picture of the roots (e.g., on p. V-5 of **[10]**), we see that there is a unique isometric isomorphism

(10) $$\phi: R_{\mathbf{Q}} \xrightarrow{\sim} K$$

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with $\phi(\theta) = 1$. (The metric on $K$ is $(b, b) = 2b\bar{b}$, i.e., $(b_1, b_2) = \mathrm{tr}(b_1\bar{b}_2)$, and we know that $(\theta, \theta) = 2$.) If $\mathfrak{O}$ is the ring of integers of $K$, one finds by an easy calculation that

$$(11) \qquad \phi(R) = \begin{cases} \mathfrak{O} & (A = A_1, A_2) \\ (1/(1 + i))\mathfrak{O} & (A = B_2) \\ (1/1(- \rho))\mathfrak{O} & (A = G_2). \end{cases}$$

Now let $H(K)$ be the space of $2 \times 2$ Hermitian matrices over $K$, so $\alpha \in H(K)$ means

$$\alpha = \begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \overline{{}^t\alpha} = \alpha^*,$$

where $a, b \in K$; we give $H(K)$ the inner product

$$(\alpha, \alpha) = -2 \det (\alpha).$$

By (7) and (8), we have an isometric isomorphism

$$(12) \qquad \hat{\phi}:\hat{R}_{\mathbf{Q}} \rightrightarrows H(K),$$

carrying $\alpha$, as in (7), to $\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix}$, where $b = \phi(\beta)$. The Weyl group $\hat{W}$ acts faithfully on this space (with its form), and so does the group $G = PGL(2, \mathfrak{O})$, by

$$(13) \qquad g \cdot \alpha = g\alpha g^*.$$

Each group preserves the set $R$ of integral vectors; checking this for $G$ uses the fact that $\phi(R)$ is contained in the inverse different of $K$, by (11). Let $\hat{W}^+$ denote the subgroup of $\hat{W}$ consisting of all elements of determinant 1, and let $G^+ = PSL(2, \mathfrak{O})$; then

$$(\hat{W}:\hat{W}^+) = 2 = (G:G^+).$$

Now

$$s_{-1}\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \begin{pmatrix} c & \bar{b} \\ b & a \end{pmatrix},$$

so

$$s_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G$$

if $l = 1$, but $s_{-1} \notin G$ if $l = 2$. The result is:

PROPOSITION 6. *Let $l \leqq 2$. Then we have natural isomorphisms according to cases as follows*:
i) $\hat{W} \simeq G$ *if* $A = A_1$;
ii) $\hat{W}^+ \simeq G^+$, *if* $A = A_2$;
iii) $\hat{W}^+ \simeq G$, *if* $A = B_2$ *or* $G_2$.

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*Proof.* We have

$$(\beta, \theta) = b + \bar{b}, \quad \text{and}$$

$$\gamma - \theta = \begin{pmatrix} 1 & -1 \\ -1 & 0 \end{pmatrix},$$

so by (9) we have

$$(14) \quad s_0\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \begin{pmatrix} a + b + \bar{b} + c & -\bar{b} - c \\ -b-c & c \end{pmatrix}.$$

If $l = 1$, then we have

$$s_{-1} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, s_0 = \begin{pmatrix} 1 & 1 \\ 0 & -1 \end{pmatrix}, s_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},$$

and these three elements generate $G$, so we have $\hat{W} = G = PGL(2, \mathbf{Z})$ as in [**3**]. Since $\hat{W}$ is a Coxeter group, this gives a Lie-theoretic proof that the relations among these generators of $G$ are what they are.

The group $\hat{W}^+$ is generated by $t_i = s_i s_{-1}$, for $i \geqq 0$. Using (14), we find

$$(15) \quad t_0 = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \in G^+.$$

Thus, for $l = 2$, we have $t_0 \in G^+$, but $s_{-1} \notin G$. We check according to cases whether $t_1$ and $t_2$ are in $G$ or $G^+$.

For $A = A_2$, the map $\phi:R_{\mathbf{Q}} \xrightarrow{\sim} K$ sends $\alpha_1$ to $-\rho$ and $\alpha_2$ to $-\bar{\rho}$. Now $s_1$ sends $\alpha_1$ to $-\alpha_1$ and $\alpha_2$ to $\alpha_1 + \alpha_2 = \theta$, so on $K$ it acts as $\rho \mapsto -\rho$ and $-\bar{\rho} \mapsto 1$, i.e., $b \mapsto -\bar{b}\,\bar{\rho}$. Thus

$$t_1\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \begin{pmatrix} c & -\bar{\rho}\,\bar{b} \\ -\rho b & a \end{pmatrix},$$

so

$$t_1 = \begin{pmatrix} 0 & \rho \\ -\bar{\rho} & 0 \end{pmatrix} \in G^+$$

and similarly

$$t_2 = \begin{pmatrix} 0 & \bar{\rho} \\ -\rho & 0 \end{pmatrix} \in G^+.$$

Thus $G^+$ contains the three generators $t_0$, $t_1$, $t_2$ of $\hat{W}^+$, and these three elements generate $G^+$, by a simple computation, so we have $\hat{W}^+ = G^+$.

For $A = B_2$, the map $\phi$ sends $\theta = \alpha_1 + 2\alpha_2 = s_2(\alpha_1)$ to 1 and $\alpha_1$ to $i$; since $s_1$ fixes $\theta$ and sends $\alpha_1$ to $-\alpha_1$, it acts as $b \mapsto \bar{b}$ on $K$. Thus

$$t_1\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} = \begin{pmatrix} c & \bar{b} \\ b & a \end{pmatrix},$$

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so

$$t_1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} \in G^+.$$

Similarly, $s_2$ interchanges $\alpha_1$ and $\theta$, hence 1 and $i$ in $K$, so $s_2$ sends $b$ to $i\bar{b}$. Then

$$t_2 = \begin{pmatrix} 0 & i \\ 1 & 0 \end{pmatrix} \in G.$$

One finds in this case that these three elements generate $G$, so we have $\hat{W}^+ = G$.

Finally, for $A = G_2$, our map sends $\theta = 3\alpha_1 + 2\alpha_2$ to 1 and $\alpha_1$ to $1/(2\rho + 1)$, so $s_1$ fixes 1 and sends $2\rho + 1$ to its negative, i.e., $\rho$ to $-(\rho + 1) = \bar{\rho}$, and so

$$t_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in G.$$

The map sends $\alpha_2$ to $-\bar{\rho}$, and $s_2$ maps $\theta$ onto $\theta - \alpha_2$, so $s_2$ sends $\rho$ to $-\bar{\rho}$ and 1 to $1 + \bar{\rho} = -\rho$, i.e., $b$ to $-\rho\bar{b}$, and we have

$$t_2 = \begin{pmatrix} 0 & \bar{\rho} \\ -\rho & 0 \end{pmatrix} \in G^+.$$

Again, these three elements generate $G$, and we have $\hat{W}^+ = G$.

Finally, in view of Proposition 4, we can make a few comments on the real roots in these cases. In the case $l = 1$, the set $\hat{\Delta}_{\mathbf{R}}$ consists of the $\hat{W}$-orbits of

$$\alpha_{-1} = t_o(\alpha_o) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

and of

$$\alpha_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

These two orbits are disjoint and fill up the set of all integral

$$\alpha = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

of determinant $-1$ (i.e., with $(\alpha, \alpha) = 2$), by Proposition 4 or by matrix calculation, as in **[3]**.

If $l = 2$, then $\hat{\Delta}_{\mathbf{R}}$ is covered by the orbits under $\hat{W}$ of $\alpha_{-1}$, $\alpha_1$, $\alpha_2$, and also by the orbits under $\hat{W}^+$ of these three elements, since each is fixed by an element of determinant $-1$ (e.g. $s_1$, $s_{-1}$, $s_{-1}$ respectively). If $A = A_2$, then

$$\alpha_2 = s_1 s_2(\alpha_1) \text{ and } \alpha_1 = s_0 s_1(\alpha_0),$$

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so $\hat{\Delta}_{\mathbf{R}} = \hat{W}^{+} \cdot \alpha_{-1}$ is a single orbit and equal to the set of all $\alpha \in \hat{R}$ with $(\alpha, \alpha) = 2$, i.e., of determinant $-1$, by Proposition 4. If $A = B_2$ resp. $G_2$, then the short roots are a single orbit and are equal to the set of all $\alpha \in \hat{R}$ with $(\alpha, \alpha) = 2/k$, where $k = 2$ resp. 3, by Proposition 4. The long roots form two orbits resp. one orbit, as one checks. They are the same as the set of all $\alpha \in \hat{R}$ with $(\alpha, \alpha) = 2$; this follows from the following rather artificial companion to Proposition 4 (the proof is much the same and will be left to the reader):

PROPOSITION 7. *Let $A$ be hyperbolic, with $k_i = 1$ for $2 \leqq i \leqq l$, with $k_1 = 2$ or 3, and with $A^{(1)}$ finite. Then any $\alpha \in R$ with $(\alpha, \alpha) = 2$ is a real root.*

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