A note on approximation by rational functions

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1. In the present paper two problems on approximation by rational functions will be treated. The one concerns rational functions whose poles are of any order but lie at two preassigned points. The other problem relates to rational functions which have simple poles only.

It is known[1] that the sequence

$$\{(a-t)^n(\bar{a}-t)^{-n-1}\} \qquad [\Im(a)>0;\ n=0,\pm 1,\pm 2,\ldots]$$

is an orthogonal closed system of the space $L_p(-\infty,\infty)$ for $1<p<\infty$. But a better result, known for $p=2$ only,[2] will be proved in the present paper (§ 2); the partial sums of the orthogonal series associated with a function $f(t)$ of $L_p(-\infty,\infty)$ $[1<p<\infty]$ converge to $f(t)$ in the mean of index p.

With regard to the second problem, A. Erdélyi has proved:[3] *In order that the sequence* $\{(t-\beta_n)^{-1}\}$ $[n=1,2,\ldots;\ \Im(\beta_n)<0;\ \beta_j\neq\beta_k$ *for* $j\neq k]$ *should be closed with respect to the space* $\mathfrak{H}'_2$,[4] *it is necessary and sufficient that the series* $\Sigma(1+|\beta_n|^2)^{-1}\Im(\beta_n)$ diverges.

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[1] E. Hille, *Compositio Mathem.*, 6 (1939), p. 99, $p=2$. H. Kober, *Quart. J. o Math.* (Oxford), 14 (1943), 49-54, referred to as K III; p. 51.

[2] H. Kober, *Bull. American Math. Soc.*, 49 (1943), 437-443, referred to as K II. Explicit formulæ are given also for $p=1$ and $p=\infty$. The author's paper *Bull. American Math. Soc.*, 48 (1942), 421-426, is referred to as K I.

[3] *J. London Math. Soc.*, 18 (1943), 72-77. The above formulation is slightly different from the original one. A. Erdélyi gives also an explicit formula for the approximating functions, orthonormalising the sequence.

[4] $\mathfrak{H}_p$ is the set of functions $F(z)$ $[z=x+iy]$ which, for $y>0$, are regular and such that

$$\int_{-\infty}^{\infty}|F(x+iy)|^p\,dx\leq M^p \quad [0<p<\infty] \text{ or } |F(z)|<M \quad [p=\infty],$$

respectively, where $M=M(F,p)$ is independent of y; see E. Hille and J. D. Tamarkin, *Fundamenta Mathem.*, 25 (1935), 329-352, referred to as H.-T. Every $F(z)$ of $\mathfrak{H}_p$ tends to a limit-function $F(x)$ as $y\to 0$. $\mathfrak{H}'_p$ is the set of these limit-functions; with "norm" $|F(x)|_p$ [see (1.1)], $\mathfrak{H}_p$ and $\mathfrak{H}'_p$ are complete normed linear spaces for $1\leq p\leq\infty$, see K I, lemma 3, p. 442.

Erdélyi's proof is based on a theorem due to O. Szász. To cope with the general case, concerning the space $\mathfrak{H}'_p$ for $1 < p < \infty$, another method is to be applied, using the Blaschke function associated with elements of $\mathfrak{H}_p$; to extend the results to the case $p = \infty$, properties of the Riesz class H_p are required.[1] A number of results of a similar type will be proved, concerning the spaces H_p, $L_p(-\pi, \pi)$ and $L_p(-\infty, \infty)$.

The symbols $|f(t)|_p$ and $|f|_p$ will be used to denote

$$(1.1) \quad \left(\int_{-}^{\infty} |f(t)|^p \, dt\right)^{1/p} [0 < p < \infty] \text{ or } \underset{-\infty \leq t \leq \infty}{ess.u.b.} |f(t)| \quad [p = \infty].$$

Similarly

$$(1.2) \quad \| f(e^{i\delta}) \|_p = \left(\int_{-\pi}^{\pi} |f(e^{i\delta})|^p \, d\delta\right)^{1/p} \text{ or } = \underset{-\pi < \delta < \pi}{ess.u.b.} |f(e^{i\delta})|.$$

Any element $g(\delta)$ of $L_p(-\pi, \pi)$ is written in the form $f(e^{i\delta})$. Positive finite constants, depending on p only, are denoted by K_p.

I am greatly indebted to the referee for suggesting a number of alterations in the present paper.

2. THEOREM 1. *If* $\Im(a) > 0$, $1 < p < \infty$, $F(t) \in L_p(-\infty, \infty)$, *and if*

$$(2.1) \quad c_k = \frac{i(a - \bar{a})}{2\pi} \int_{-\infty}^{\infty} \frac{(t - \bar{a})^k}{(a - t)^{k+1}} F(t) \, dt \quad [k = 0, \pm 1, \pm 2, \ldots],$$

then

$$(2.2) \quad \int_{-\infty}^{\infty} \left| F(x) - \sum_{k=-n}^{n} c_k \frac{(a - x)^k}{(x - \bar{a})^{k+1}} \right|^p dx \to 0 \qquad \text{as } n \to \infty.$$

THEOREM 1'. *If* $\Im(a) > 0$, $1 < p < \infty$, *and if, for some sequence* $\{c_k\}$ $[k = 0, \pm 1, \pm 2, \ldots]$, (2.2) *holds, then* $F(x)$ *belongs to* $L_p(-\infty, \infty)$, *and the* c_k's *satisfy* (2.1).

The proof of Theorem 1, using Hilbert's operator

$$(2.3) \quad \mathfrak{H} g = \mathfrak{H}[g(t); x] = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{g(t) \, dt}{t - x} = \lim_{\epsilon \to 0} \int_{\epsilon}^{\infty} \frac{g(x+t) - g(x-t)}{\pi t} dt,$$

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[1] F. Riesz, *Math. Zeitschr*, 18 (1923), 87-95. H_p is the set of functions $f(z)$ which, for $|z| < 1$, are regular and satisfy the inequality

$$\int_{-\pi}^{\pi} |f(re^{i\delta})|^p \, d\delta \leq M^p \quad [0 < p < \infty] \text{ or } |f(z)| < M \quad [p = \infty],$$

respectively, where $0 \leq r < 1$ and $M = M(f, p)$; H'_p is the set of the limit functions $f(e^{i\delta})$ $[z \to e^{i\delta}]$.

employs an argument advanced by M. Riesz in his classical theorem on Fourier series.[1] It will suffice to take $a = i$, $\bar{a} = -i$, the general case being deduced from this one by the substitution $t = \Re(a) + t' \Im(a)$. Let

$$(2.4) \qquad s_n F = s_n [F; x] = \sum_{k=-n}^{k=n} c_k \frac{(i-x)^k}{(i+x)^{k+1}}; \; q = \frac{(i-x)}{(i+x)} \frac{(i+t)}{(i-t)}.$$

Using (2.1), we have

$$s_n F = \frac{1}{2\pi i} \int_{-\infty}^{\infty} (q^{-n} - q^{n+1}) \frac{F(t)\, dt}{t-x}.$$

Hence

$$(2.5) \qquad |s_n F| \leqq \tfrac{1}{2} \left| \mathfrak{H} \left\{ \left(\frac{i-t}{i+t} \right)^n F(t) \right\} \right| + \tfrac{1}{2} \left| \mathfrak{H} \left\{ \left(\frac{i+t}{i-t} \right)^{n+1} F(t) \right\} \right|,$$

since $|i-x|\,|i+x|^{-1} = 1$. As $|\mathfrak{H} g|_p \leqq K_p |g|_p \; [1 < p < \infty]$ and $|i-t|\,|i+t|^{-1} = 1$, it follows from (2.5) that

$$(2.6) \qquad |s_n [F(t); x]|_p \leqq K_p |F(t)|_p.$$

Let now $F_m(x)$ be any rational function of the form

$$\sum_{k=-m}^{m} A_k (i-x)^k (i+x)^{-k-1}.$$

Using orthogonality, from (2.1) we deduce that $c_k [F_m] = 0$ or $= A_k$ for $|k| > m$ or $|k| \leqq m$, respectively. Hence $s_n F_m = F_m$ whenever $n \geqq m$, and so

$$|s_n F - F|_p \leqq |s_n [F - F_m]|_p + |F_m - F|_p \leqq (K_p + 1) |F - F_m|_p,$$

applying (2.6) to the function $F - F_m$. Observing that, given $\epsilon > 0$, F_m can be chosen so that $|F - F_m|_p \leqq \epsilon (1 + K_p)^{-1}$, we have $|s_n F - F|_p \leqq \epsilon$ for $n \geqq m$. Thus we have proved the theorem.

Theorem 1′ is proved by a familiar argument, using orthogonality.

Corollary. In order that there should exist numbers $c_0, c_1, c_2, \ldots$ *such that*

$$(2.7) \qquad \int_{-\infty}^{\infty} \left| F(x) - \sum_{k=0}^{n} c_k \frac{(a-x)^k}{(x-\bar{a})^{k+1}} \right|^p dx \to 0 \qquad \left[\begin{array}{c} 1 < p < \infty; \\ a \text{ given}, \Im(a) > 0 \end{array} \right]$$

as $n \to \infty$, *it is necessary and sufficient that* $F(x)$ *belongs to* $\mathfrak{H}'_p$. *If this condition is fulfilled, then the* c_k'*s satisfy* (2.1), *and*

[1] *Math. Zeitschr*, 27 (1928), 218-244, §§ 13-14. There also the theory of Hilbert's operator is given, §§ 17-20.

$$(2.8) \qquad \int_{-\infty}^{\infty} \left| F(z) - \sum_{k=0}^{n} c_k \frac{(a-z)^k}{(z-\bar{a})^{k+1}} \right|^p dx \to 0 \qquad [z = x + iy]$$

as $n \to \infty$, *uniformly for* $y \geqq 0$.

That $F(x) \in \mathfrak{H}'_p$ implies $c_k = 0$ for $k < 0$, and the necessity of the condition, are deduced by means of known results;[1] (2.8) follows from (2.7) by a maximum-modulus theorem,[2] due to E. Hille and J. D. Tamarkin.

For comparison, we add:[3]

A necessary and sufficient condition that $F(x)$ *can be approximated, in the mean of index* p, *by rational functions vanishing at infinity and regular in* $y \geqq 0$ *is that* $F(x)$ *belongs to* $\mathfrak{H}'_p$ $[1 < p < \infty]$.

3. In this section and in § 4 the approximation to elements of H_p by rational functions with simple poles is treated. On the unit circle $\Gamma\,[|z| = 1]$ the rational function $(z-a)^{-1}$ takes the form $(e^{i\delta} - a)^{-1}$, where $-\pi \leqq \delta \leqq \pi$. By $\{a_n\}$ or $\{b_n\}$ $[n = 1, 2, \ldots .]$ we denote sequences of points lying in the interior or exterior of Γ, respectively, all the points being different from one another; without loss of generality, the origin and the point at infinity are excluded. The set of functions $f(e^{i\delta})$, with norm $\|f(e^{i\delta})\|_\infty$, which are continuous, and of period 2π, thus forming a sub-space of $L_\infty(-\pi, \pi)$, is denoted by C, and the intersection of the spaces H'_∞ and C is denoted by $(H'_\infty;\ C)$.

Theorem 2. *The divergence of the product* $\Pi|b_n|$ *is a necessary and sufficient condition for the sequence* $\{(e^{i\delta} - b_n)^{-1}\}$ *to be closed with respect to* H'_p *when* $1 \leqq p < \infty$; *with respect to* $(H'_\infty;\ C)$ *when* $p = \infty$.

By a well-known maximum-modulus theorem, the statement concerning the sufficiency of the condition is equivalent to this one:

If $\Pi\,|b_n|$ *diverges, and if* $f(e^{i\delta}) \in H'_p$ $[1 \leqq p < \infty]$ *or* $f(e^{i\delta}) \in (H'_\infty; C)$, *then there are numbers* $B_{n,k}$ *such that, uniformly for* $0 \leqq r \leqq 1$,

$$(3.1) \qquad \|f(re^{i\delta}) - \sum_{k=1}^{n} B_{n,k}\,(r\,e^{i\delta} - b_k)^{-1}\|_p \to 0 \qquad \text{as } n \to \infty.$$

[1] K II, Lemma, p. 442; Lemma 3, p. 440.

[2] H.-T., 2.1 (iii), p. 339.

[3] This follows from Theorem 2 (a) and from Lemma 3, K II, pp. 438 and 440. Compare K III, Theorem 2, p. 54, and K I, Lemma 4, p. 423.

This result is, for $p = \infty$; deduced from theorems due to J. L. Walsh;[1] we have to observe that $f(e^{i\delta}) \epsilon (H'_\infty;\ C)$ implies that $f(z)$ is regular for $|z| < 1$ and continuous for $|z| \leqq 1$, and *vice versa*. Now we need

Lemma 1. *If $0 < p < q \leqq \infty$, and if the sequence $\{(e^{i\delta} - b_n)^{-1}\}$ is closed with respect to H'_q if $q < \infty$ but with respect to $(H'_\infty;\ C)$ if $q = \infty$, then it is also closed with respect to H'_p.*

For there exist polynomials $P_n(z)$ $[n = 1, 2, \ldots.]$ such that[2]

$$(3.2) \qquad \| f(e^{i\delta}) - P_n(e^{i\delta}) \|_p \to 0 \qquad \text{as } n \to \infty \qquad (f(e^{i\delta}) \epsilon H'_p).$$

Also for each n there is a sequence $\{s_{n,m}\}$ of finite linear forms in the $(e^{i\delta} - b_k)^{-1}$ such that

$$(3.3) \qquad \| P_n(e^{i\delta}) - s_{n,m} \|_q \to 0 \qquad \text{as } m \to \infty.$$

Since $p < q$, (3.3) remains true if q is replaced by p, by Hölder's inequality if $q < \infty$. This, with (3.2), gives the result.

By the lemma, the sufficiency of the condition in the theorem, even for $0 < p < \infty$, follows from the case $p = \infty$. Again by the lemma, the necessity of the condition when $1 < p \leqq \infty$ follows from the case $p = 1$, treated in the next section.

4. *The necessity of the condition for $p = 1$.* We need[3]

Lemma 2. *A set of functions in a normed linear space (B) is closed with respect to (B) if and only if every linear functional in (B) which vanishes at all elements of the set vanishes identically in (B).*

Therefore we have to show: If $\Pi |b_n|$ is finite, we can construct a linear functional Φ in H'_1 which does not vanish identically, while $\Phi[(e^{i\delta} - b_n)^{-1}] = 0$ for $n = 1, 2, \ldots.$

Since $\Pi |b_n| < \infty$, we have $\Pi |b_n|^{-1} > 0$. Hence the Blaschke function[4]

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[1] Interpolation and approximation in the complex domain, *American Math. Soc. Coll. Publications*, vol. 20, New York, 1935, see chapter 9, in particular 9.6. For $p = \infty$ the sufficiency of the condition in theorem 2 and, incidentally, (3.1), can alternatively be proved by an argument similar to that of § 6; but using the generalization of a theorem by Littlewood, treated in the Appendix to the present paper; see *J. London Math. Soc.*, 1 (1926), 229-231.

[2] K II, Lemma 1, p. 439.

[3] S. Banach, *Théorie des opérations linéaires*, Warsaw, 1932, p. 58.

[4] F. Riesz, *loc. cit.*, p. 89.

$$(4.1) \qquad G(z^{-1}) = z \prod_{n=1,2,\ldots} |b_n|^{-1} \frac{1 - z b_n}{1 - z/\bar{b}_n} \qquad [|z| < 1]$$

belongs to H_∞. It does not vanish identically for $|z| < 1$, since $|G(e^{-i\delta})| = 1$ for almost all δ in $(-\pi, \pi)$, $G(e^{-i\delta})$ being the limit-function of $G(z^{-1})$ as $z \to e^{i\delta}$. By the Cauchy integral theorem and the Lebesgue theorem on bounded convergence, we have

$$(4.2) \qquad G(z^{-1}) = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{G(e^{-i\psi}) e^{i\psi} d\psi}{e^{i\psi} - z} \qquad [|z| < 1].$$

By (4.1), we have $G(z^{-1}) = 0$ for $z = 0$, therefore, by (4.2), we have $\int_{-\pi}^{\pi} G(e^{-i\psi}) d\psi = 0$, and we can deduce from (4.2) that

$$(4.3) \qquad G(w) = -\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{G(e^{i\delta}) e^{i\delta} d\delta}{e^{i\delta} - w} \qquad [|w| > 1].$$

Let $|w_0| > 1$ and $G(w_0) \neq 0$, and let $h(e^{i\delta}) = -(2\pi)^{-1} G(e^{i\delta}) e^{i\delta}$. Then the functional

$$\Phi[f] = \int_{-\pi}^{\pi} f(e^{i\delta}) h(e^{i\delta}) d\delta \qquad [f \in H'_1]$$

possesses all the properties required. For (i) it is linear, since $|\Phi[f]| \leqq (2\pi)^{-1} \|f(e^{i\delta})\|_1$, as $|h(e^{i\delta})| \equiv (2\pi)^{-1}$; (ii) it vanishes for $f = (e^{i\delta} - b_n)^{-1}$ $[n = 1, 2, \ldots]$ since, by (4.3) and (4.1),

$$\Phi[(e^{i\delta} - b_n)^{-1}] = G(b_n) = 0;$$

(iii) it does not vanish for $f = (e^{i\delta} - w_0)^{-1} \in H'_1$, since $\Phi[(e^{i\delta} - w_0)^{-1}] = G(w_0) \neq 0$. This completes the proof of the theorem.

THEOREM 2′. *The sequence* $(e^{i\delta} - a_1)^{-1}$, $(e^{i\delta} - b_1)^{-1}$, $(e^{i\delta} - a_2)^{-1}$, $(e^{i\delta} - b_2)^{-1}$, ... *is closed with respect to* $L_p(-\pi, \pi)$ $[1 \leqq p < \infty]$ *or to* C *if, and only if, both products* $\Pi |a_n|$ *and* $\Pi |b_n|$ *diverge.*

Given an element $f(e^{i\delta})$ of $L_p(-\pi, \pi)$ $[1 \leqq p < \infty]$ or of C $[p = \infty$; see § 3], we can find a sequence of functions

$$g_m(e^{i\delta}) = \sum_{k=-m}^{m} a_{k,m} e^{ik\delta} = \left(\sum_{k=-m}^{-1} + \sum_{k=0}^{m}\right) a_{k,m} e^{ik\delta} = g_m^I + g_m^{II}$$

$[m = 1, 2, \ldots]$ such that $\|f(e^{i\delta}) - g_m(e^{i\delta})\|_p \to 0$ as $m \to \infty$. For each m, we can now apply theorem 2 to $g_m^{II}(e^{i\delta})$ and, *mutatis mutandis*, to $g_m^I(e^{i\delta})$. Thus we deduce the sufficiency of the conditions. The proof of their necessity is left to the reader.

5. Approximation to elements of $\mathfrak{H}_p$.

Let $F(z) \epsilon \mathfrak{H}_p$ and $|F(x) - R_n(x)|_p \to 0$ as $n \to \infty$, where the rational functions $R_n(z)$ are regular for $y \geqq 0$. We note that this implies that, uniformly for $y \geqq 0$,

$$\int_{-\infty}^{\infty} |F(x+iy) - R_n(x+iy)|^p \, dx \to 0 \qquad \text{as } n \to \infty$$

and that, for any $\epsilon > 0$, $|F(z) - R_n(z)| \to 0$ uniformly with respect to z for $y \geqq \epsilon$ (compare 2.8).[1] Again we refer to the theorem stated at the end of § 2.

The symbol $\mathfrak{C}$ is used to denote the set of functions $F(x)$ which are continuous in any finite interval and whose limits, as $x \to \pm \infty$, both exist and are finite and equal; the "norm" is $|F(x)|_\infty$. Throughout the following investigation we suppose that $\mathfrak{J}(\alpha_n) > 0$, $\mathfrak{J}(\beta_n) < 0$, and that $\alpha_j \neq \alpha_k$, $\beta_j \neq \beta_k$ for $j \neq k$.

THEOREM 3 (*a*). *The sequence* $\{(x-\beta_n)^{-1}\}$ $[n = 1, 2, \ldots]$ *is closed with respect to* $\mathfrak{H}'_p$ $[1 < p < \infty]$ *if and only if the series* $S(\beta)$ *diverges, where*

$$S(\beta) = \sum_{n=1,2,\ldots} \frac{\mathfrak{J}(\beta_n)}{1 + |\beta_n|^2}.$$

THEOREM 3 (*b*). *The sequence* 1; $(x-\beta_1)^{-1}$; $(x-\beta_2)^{-1}$, ... *is closed with respect to* $(\mathfrak{H}'_\infty; \mathfrak{C})$ *if and only if the series* $S(\beta)$ *diverges.*

The last statement is reduced to the corresponding one of theorem 2 by the conformal transformation $w = (1+iz) \cdot (1-iz)^{-1}$ which transforms $y \geqq 0$ into $|w| \leqq 1$. Write

$$e^{i\delta} = \frac{i-x}{i+x}; \; x = \tan \tfrac{1}{2}\delta; \; F(x) = f(e^{i\delta}); \; F(z) = f(w); \; b_n = \frac{i-\beta_n}{i+\beta_n}.$$

Then $F(x) \epsilon (\mathfrak{H}'_\infty; \mathfrak{C})$ implies $f(e^{i\delta}) \epsilon (H'_\infty; C)$, and *vice versa*. Moreover,

$$|b_n| > 1; \; \frac{1}{e^{i\delta} - b_n} = B_n + \frac{B'_n}{x - \beta_n}; \; B_n = \frac{i\beta_n - 1}{2}, \; B'_n = \frac{(1 - i\beta_n)^2}{2i},$$

and it is not difficult to see that the series $S(\beta)$ and $\Sigma\,(1 - |b_n|^{-2})$, consequently also $S(\beta)$ and the product $\Pi\,|b_n|$, are equiconvergent.[2] It is now easy to complete the proof.

6. Proof of theorem 3 (*a*) [the case $1 < p < \infty$]. Suppose that $S(\beta)$ diverges. By lemma 2, we have to show that every linear

[1] See footnote 2 on page 126, and K II, p. 441, footnote 15.

[2] Compare H.-T., Theorem 2.2, p. 340.

functional $\Phi[F]$ in $\mathfrak{H}'_p$ which vanishes for $F_n(x) = (x - \beta_n)^{-1}$ $[n = 1, 2, \ldots.]$ vanishes for any $F(x) \in \mathfrak{H}'_p$. We can represent any linear functional in $\mathfrak{H}'_p$ in the form[1]

$$(6.1) \qquad \Phi[F] = \int_{-\infty}^{\infty} F(t)\, H(t)\, dt \qquad \begin{bmatrix} H(t) \in L_q(-\infty, \infty) \\ 1/p + 1/q = 1 \end{bmatrix}.$$

Let

$$(6.2) \qquad G(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{H(-t)\, dt}{t - z} = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{H(t)\, dt}{t + z} \qquad [y > 0].$$

Since $\Phi[F_n] = 0$, we have $G(-\beta_n) = 0$ for $n = 1, 2, \ldots.$ But $G(z)$ belongs to $\mathfrak{H}_q$, as is shown at the end of this section. Hence $G(z)$ vanishes identically for $y > 0$, since otherwise the existence of the Blaschke function associated with $G(z)$ would require[2] that $S(-\beta) < \infty$. Therefore

$$(6.3) \qquad \int_{-\infty}^{\infty} \frac{H(t)\, dt}{(t + z)^n} = 0 \qquad \begin{bmatrix} n = 1, 2, \ldots. \\ y > 0 \end{bmatrix},$$

the repeated differentiation under the integral being permitted. Now $\{(t - i)^k (t + i)^{-k-1}\}$ $[k = 0, 1, 2, \ldots.]$ is a closed system of the space $\mathfrak{H}'_p$;[3] so is, therefore, $\{(t + i)^{-n}\}$ $[n = 1, 2, \ldots.]$. By (6.3), $\Phi[(t + i)^{-n}]$ vanishes for $n = 1, 2, \ldots.$ Hence, by lemma 2, $\Phi[F]$ vanishes identically in $\mathfrak{H}'_p$. Thus the condition is sufficient.

To prove its necessity we proceed as in § 4. Since $S(-\beta) < \infty$ implies that the Blaschke function[4]

$$(6.4) \qquad k_0(z) = \prod_{n=1,2,\ldots} \frac{z + \beta_n}{z + \bar{\beta}_n} \frac{i + \bar{\beta}_n}{i + \beta_n} \left| \frac{i + \beta_n}{i - \beta_n} \right| \qquad [y > 0]$$

is regular and bounded for $y > 0$, and that $|k_0(x)| = 1$ for almost all x in $(-\infty, \infty)$, we have $k_0(z_0) \neq 0$ for some z_0 $[y_0 > 0]$. The function $k(z) = (i + z)^{-1} k_0(z)$ belongs to $\mathfrak{H}_q$ $[1/q + 1/p = 1]$, therefore we have

$$(6.5) \qquad k(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{k(t)\, dt}{t - z} = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{k(-t)\, dt}{t + z} \qquad [y > 0].$$

Writing $H(t) = -(2\pi i)^{-1} k(-t)$ and defining Φ by (6.1), we have by (6.5) and (6.4)

$$\Phi[(t - \beta_n)^{-1}] = k(-\beta_n) = 0; \quad \Phi[(t + z_0)^{-1}] = k(z_0) \neq 0,$$

which completes the proof.

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[1] S. Banach, *loc. cit.*, pp. 61-65 ; $\mathfrak{H}'_p$ is a sub-space of $L_p\ (-\infty, \infty)$.

[2] Compare H.-T., Theorem 2.2, p. 340.

[3] K II, Theorem 2 (*a*), p. 438.

[4] H.-T., Theorem 2.2, p. 340. There the formula is misprinted.

To show that the function $G(z)$, defined by (6.2), belongs to $\mathfrak{H}_q$, we represent $H(-t)$ in the form

(6.6) $$H(-t) = h_1(t) + h_2(t); \; h_1(z) \in \mathfrak{H}_q, \; h_2(-z) \in \mathfrak{H}_q.$$ [1]

By a fundamental property of the class $\mathfrak{H}_q$, we have[2]

(6.7) $$\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{h_1(t)\,dt}{t-z} = h_1(z), \qquad \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{h_2(t)\,dt}{t-z} = 0 \qquad [y > 0].$$

Using (6.2) we have, therefore, $G(z) = h_1(z) \in \mathfrak{H}_q$.

7. The spaces $L_p(-\infty, \infty)$ and $\mathfrak{C}$.

THEOREM 3′ (a). *The system* $(x-\alpha_1)^{-1}$, $(x-\beta_1)^{-1}$, $(x-\alpha_2)^{-1}$, $(x-\beta_2)^{-1}$ *is closed with respect to* $L_p(-\infty, \infty)$ $[1 < p < \infty]$ *if, and only if, both series* $S(\alpha)$ *and* $S(\beta)$ *diverge.*

(b). *The system* $1, (x-\alpha_1)^{-1}, (x-\beta_1)^{-1}, \ldots.$ *is closed with respect to* $\mathfrak{C}$ *if, and only if, both* $S(\alpha)$ *and* $S(\beta)$ *diverge.*

In case (a) the sufficiency of the condition follows from theorem 3(a), since any function of $L_p(-\infty, \infty)$ may be decomposed as in (6.6). In case (b) it follows from the two facts: (i) $\{h_n(x)\} = \{(x-i)^n(x+i)^{-n}\}$ $[n = 0, \pm 1, \pm 2, \ldots.]$ is a closed system of $\mathfrak{C}$,[3] (ii) for each n, $h_n(x)$ can be approximated uniformly in $(-\infty, \infty)$ by a finite linear form of $1, (x-\gamma_1)^{-1}, (x-\gamma_2)^{-1}, \ldots,$ where $\gamma_r = \beta_r$ or α_r according as $n \geqq 0$ or $n < 0$. The truth of (i) is known; (ii) follows from theorem 3(b), as $h_n(x)$ $[n \geqq 0]$ and $h_n(-x)$ $[n < 0]$ belong to $(\mathfrak{H}'_\infty; \mathfrak{C})$.

To prove the necessity of the condition for both cases (a) and (b), we suppose that $S(-\beta) < \infty$ and take $k_1(z) = (z+i)^{-2}k_0(z)$, defining $k_0(z)$ by (6.4). Then $k_1(z) \in \mathfrak{H}_p$ for any $p \geqq 1$, and we can replace k by k_1 in (6.5). Furthermore, we have by Cauchy's theorem

(7.1) $$\int_{-\infty}^{\infty} k_1(t)\,dt = 0; \qquad \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{k_1(-t)\,dt}{t+z} = 0 \qquad [y < 0].$$

If we take $H(t) = -(2\pi i)^{-1}k_1(-t)$, the integral in (6.1) defines a linear functional $\Phi[F]$ in $L_p(-\infty, \infty)$ $[1 \leqq p < \infty]$ and also in $\mathfrak{C}$,

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[1] K I, p. 422, lines 8-9 from the bottom.

[2] H.-T., Theorem 2.1 (ii), p. 338. Using the H.-T. notation, we have
$I(h_2(t); z) = -I(h_2(-t); -z) = -I(h_2(-t); -\bar{z}) + P(h_2(-t); -\bar{z}) = -h_2(\bar{z}) + h_2(\bar{z}) = 0.$
As the referee has pointed out to me, the second equation of (6.7) can be deduced from the first one also this way: Replace $h_1(t)$ by $(t-\xi)(t+\xi)^{-1}h_2(-t) = g_\xi(t)$ $[I(\xi) > 0]$, observing that $g_\xi(z)$ belongs to $\mathfrak{H}_p$, and take finally $\xi = z$.

[3] J. L. Walsh, *loc. cit.*, Chapter 2, Theorem 16.

and we have $\Phi[1]=0$, $\Phi[(t-\alpha_n)^{-1}]=0$ by (7.1), and $\Phi[(t-\beta_n)^{-1}]=0$, $\Phi[(t+z_0)^{-1}]\neq 0$ for some z_0 $[y_0>0]$ as in § 6. Hence the system is not closed in the space concerned which completes the proof.

8. The spaces $\mathfrak{H}_1'$ and L_1. We remark that theorem 3 (a) holds for $p=1$. In this case, however, any linear form of the $(x-\beta_n)^{-1}$ which approximates to $F(x)\in\mathfrak{H}_1'$ is required to be $O(|x|^{-2})$ as $|x|\to\infty$. Similarly theorem 3′ (a) holds for $p=1$, the linear forms of the $(x-\alpha_1)^{-1}$, $(x-\beta_1)^{-1}$, $(x-\alpha_2)^{-1}$, $(x-\beta_2)^{-1}$, ... being required to belong to $L_1(-\infty,\infty)$.

APPENDIX: On a result due to J. E. Littlewood.[1]

9. THEOREM. *If $g(\delta)$ is of bounded variation over $(-\pi,\pi)$, with total variation V_g, and if*

$$G(z)=\frac{1}{2\pi}\int_{-\pi}^{\pi}\frac{e^{i\delta}\,dg(\delta)}{e^{i\delta}-z}\qquad [\,|z|<1], \tag{9.1}$$

then for $0<\lambda<1$, *and uniformly in* $0<r<1$,

$$\left(\int_{-\pi}^{\pi}|G(re^{i\delta})|^{\lambda}\right)^{1/\lambda}\leqq A_\lambda\int_{-\pi}^{\pi}|dg(\delta)|=A_\lambda\,V_g, \tag{9.2}$$

where A_λ depends on λ only.

This result, in a slightly different formulation, has been proved by Littlewood for the special case when $g(\delta)$ is an integral. The sufficiency of the condition in theorem 2 concerning the case $p=\infty$ follows from it at once by the argument of § 6. For every linear functional in $(H'_\infty;\ C)$ is of the form

$$\Phi[f]=\int_{-\pi}^{\pi}f(e^{i\delta})\,dg(\delta)\qquad [V_g<\infty], \tag{9.3}$$

and the function

$$\Psi(z)=\frac{1}{z}\int_{-\pi}^{\pi}\frac{dg(\delta)}{e^{i\delta}-z^{-1}}=\int_{-\pi}^{\pi}\frac{e^{i\delta}\,dg(-\delta)}{e^{i\delta}-z}\qquad [\,|z|<1], \tag{9.4}$$

belonging to H_λ in consequence of the above theorem, vanishes for $z=b_n^{-1}$ $[n=1,2,\ldots.]$ when we suppose that $\Phi[(e^{i\delta}-b_n)^{-1}]=0$, it vanishes, therefore, identically, as $\Pi|b_n|=\infty$. Differentiating (9.4) n times and taking $z=0$, we arrive at $\Phi[e^{in\delta}]=0$ $[n=0,1,2,\ldots.]$; and observing that the sequence $\{e^{in\delta}\}$ $[n=0,1,\ldots.]$ is closed with respect to $(H'_\infty;\ C)$, we can easily complete the proof.

[1] *Cf.* footnote 1 on page 127.

10. The proof of the theorem follows the lines of Littlewood's proof. Without loss of generality we suppose that $g(\delta)$ is real and non-decreasing and that $0 < V_g < \infty$. Putting

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-ik\delta}\,dg(\delta), \qquad k = 0, \pm 1, \pm 2, \ldots,$$

we have $c_0 > 0$, $G(z) = \sum_{n=0}^{\infty} c_n z^n\ [|z| < 1]$. Now the Poisson integral $(2\pi)^{-1}\int_{-\pi}^{\pi}(1 - |z|^2)\,|e^{i\delta} - z|^{-2}\,dg$ is evidently positive and is equal to

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left\{\frac{e^{i\delta}}{e^{i\delta} - z} + \frac{\bar{z}\,e^{i\delta}}{1 - \bar{z}e^{i\delta}}\right\} dg(\delta) = G(z) + \overline{G(z) - c_0} = 2\mathfrak{R}(G(z)) - c_0.$$

Thus the function $w = \eta(z) = G(z) - \frac{1}{2}c_0$ maps $|z| < 1$ on the half-plane $\mathfrak{R}(w) > 0$ or on part of it, while $\eta_0(z) = \frac{1}{2}c_0(1 + z)(1 - z)^{-1}$ maps $|z| < 1$ on the whole half-plane. Moreover, we have $\eta(0) = \eta_0(0)$. Hence, by Littlewood's argument,

$$\|\eta(re^{i\delta})\|_\lambda \leqq \|\eta_0(re^{i\delta})\|_\lambda \leqq K_\lambda c_0 = K_\lambda V_g \qquad \begin{bmatrix} 0 \leqq r < 1, \\ 0 < \lambda < 1 \end{bmatrix},$$

which gives ultimately the result required.

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