A special net of quadrics

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Introduction.

It is well-known that a general net of quadric surfaces cannot be obtained as the net of polar quadrics of the points of a plane in regard to a cubic surface; in order that it may be so obtained it must have various properties that a general net of quadrics does not have. The locus of the vertices of the cones which belong to the net of quadrics is a curve ϑ—the Jacobian curve of the net of quadrics, and the trisecants of ϑ generate a scroll. Any plane which contains two trisecants of ϑ is a bitangent plane of the scroll and, for a general net of quadrics, there are eighteen of these bitangent planes passing through an arbitrary point. When however the net of quadrics is a net of polar quadrics it is found that *any plane which contains two trisecants of ϑ contains two other trisecants also*; it thus contains four trisecants in all and counts six times as a bitangent plane of the scroll. The bitangent developable of the scroll, which is, for a general net of quadrics, of class eighteen, degenerates, in the special case when the net of quadrics is a net of polar quadrics, into a developable of class three counted six times; the planes of the developable are therefore the osculating planes of a twisted cubic γ. The plane which, together with a cubic surface, gives rise to the net of polar quadrics must be one of the osculating planes of γ. It is also found, further, that the osculating planes of γ are grouped into pentahedra, the vertices of all these pentahedra lying on ϑ.

When a net of quadrics consists of the polar quadrics of the points of a plane in regard to a cubic surface, this same net of quadrics can arise from different planes and different cubic surfaces. It was found by Schur[1] that the net can be obtained from any osculating

[1] *Math. Annalen*, 18 (1881); see in particular pp. 23-27. See also Töplitz: *Math. Annalen*, 11 (1877), 434-463; Töplitz proves that the ∞^1 planes which give rise to the net of quadrics all osculate the same twisted cubic. There is also a paper by A. C. Dixon: *Proc. London Math. Soc.* (2), 7 (1909), 150-156, in which he gives an algebraical proof that the faces of the pentahedra all osculate the twisted cubic obtained by Töplitz, but he seems unaware of Schur's work.

plane of γ and that, when any osculating plane of γ has been chosen, there is a singly-infinite set of cubic surfaces in regard to all of which the same net of polar quadrics is obtained; thus a given net of polar quadrics can be obtained from ∞^1 planes and ∞^2 cubic surfaces, each of the planes being associated with ∞^1 of the cubic surfaces. Each of the cubic surfaces has for its Sylvester pentahedron one of the pentahedra, already noticed, whose faces osculate γ and whose vertices lie on ϑ; given any one of the pentahedra and any osculating plane of γ there is in fact one cubic surface having this pentahedron for its Sylvester pentahedron and such that the polar quadrics, of the points of the given osculating plane of γ in regard to the cubic surface, are the members of the net from which we started. When a net of polar quadrics is given it is therefore not possible to specify a unique plane and cubic surface that give rise to it; but the twisted cubic γ *is* unique, and it is in some ways more satisfactory to define the net of quadrics by means of γ than by means of a plane and a cubic surface. An osculating plane of γ belongs to one and only one of the pentahedra, so that these pentahedra are sets of an involution of sets of five osculating planes of γ. The net of polar quadrics is indeed completely defined when a g_5^1 is given on γ; the osculating planes of γ at the points of a set of the g_5^1 form a pentahedron and, as can easily be shown and as will appear explicitly below, there is a unique net of quadrics in regard to each member of which all the pentahedra so arising from the different sets of the g_5^1 are self-conjugate. This net is a net of polar quadrics, and its Jacobian curve is the locus of the vertices of the pentahedra.

It is easy to give algebraic equations of the planes and cubic surfaces that give rise to the same net of polar quadrics. Let us write

$$P_i \equiv x_0 + \theta_i x_1 + \theta_i^2 x_2 + \theta_i^3 x_3,$$

so that the plane $P_i = 0$ osculates a twisted cubic, whatever the value of θ_i. Then it is easily verified that, if we take the polar quadrics of the points of the plane

$$x_0 + ax_1 + a^2x_2 + a^3x_3 = 0$$

with respect to any cubic surface of the pencil

$$\frac{a_1 P_1^3}{a-\theta_1} + \frac{a_2 P_2^3}{a-\theta_2} + \frac{a_3 P_3^3}{a-\theta_3} + \frac{a_4 P_4^3}{a-\theta_4} + \frac{a_5 P_5^3}{a-\theta_5} + \lambda (x_0 + ax_1 + a^2 x_2 + a^3 x_3)^3 = 0,$$

where the different surfaces of the pencil are obtained by varying λ, the net of quadrics that is obtained *does not depend on* λ. Furthermore[1]: this net of quadrics *does not depend on* a, being in fact the net determined by the three quadrics

$$a_1 P_1^2 + a_2 P_2^2 + a_3 P_3^2 + a_4 P_4^2 + a_5 P_5^2 = 0,$$
$$a_1 \theta_1 P_1^2 + a_2 \theta_2 P_2^2 + a_3 \theta_3 P_3^2 + a_4 \theta_4 P_4^2 + a_5 \theta_5 P_5^2 = 0,$$
$$a_1 \theta_1^2 P_1^2 + a_2 \theta_2^2 P_2^2 + a_3 \theta_3^2 P_3^2 + a_4 \theta_4^2 P_4^2 + a_5 \theta_5^2 P_5^2 = 0.$$

That the faces of the Sylvester pentahedra, of the cubic surfaces obtained by giving different values to λ in the equation

$$A_1 P_1^3 + A_2 P_2^3 + A_3 P_3^3 + A_4 P_4^3 + A_5 P_5^3 + \lambda A_6 P_6^3,$$

osculate the same twisted cubic, and that their points of contact with the curve are the sets of a g_5^1 thereon, is quite clear when we refer to the usual algebraical process of reducing the left-hand side of the equation of a cubic surface from the sum of six to the sum of five cubes.[2]

The choice of a twisted cubic and a g_5^1 thereon, rather than of a plane and a cubic surface, as a means of defining a net of polar quadrics, has certainly one very important advantage, since it can be extended immediately to space of higher dimensions; we have only to take in $[n]$ a rational normal curve of order n and a g_{n+2}^1 thereon. This paper is concerned with the properties of that net of quadrics in $[n]$ which arises from a rational normal curve of order n and a g_{n+2}^1 thereon in the same way as a net of polar quadrics arises from a g_5^1 on a twisted cubic. It is shown how the curve ϑ is obtained as the locus of the vertices of the $(n+2)$-hedra which are formed by the osculating primes of the rational normal curve, and how there arise certain families of secant spaces of ϑ. It is also shown how ϑ can be put into birational correspondence with a special form of plane curve of order $n+1$. The net of quadrics is introduced in §4, the equations of the

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[1] The equation of the doubly-infinite family of cubic surfaces contains the parameter λ to the first degree and the parameter a to degree 14. Töplitz states, in the footnote to p. 449 of his paper, that this second parameter enters to degree 8. I believe however that the word *achten* is a misprint for *achtzehnten*, since it seems, on reading carefully through his work, that the second parameter enters to degree 18 in his equation (9). He goes on to say that it does not seem possible to lessen this degree, but I have not been able to account for the discrepancy between his 18 and the degree 14 to which a enters in the above equation. This equation, with a entering to degree 14, is also implied in Dixon's work on pp. 154-155 of his paper.

[2] See, for example, Baker, *Principles of Geometry* 3 (Cambridge, 1933), 206-208.

quadrics being given explicitly in §5. §§6-9 are concerned with the calculation of the classes of certain developables associated with the set of ∞^1 $(n+2)$-hedra; these developables are contravariants of the net of quadrics. In §§10-13 the orders of the scrolls generated by the secant spaces of ϑ are obtained; these results afford another means of calculating the classes of the developables just mentioned, and this brief calculation is given in §14. In §§15-16 the genera of the scrolls and developables are calculated, and finally the results obtained are stated explicitly for the smaller values of n.

Before commencing the work it may not be out of place to point out the extent to which this net of quadrics is specialised.

The quadrics in $[n]$ are members of a linear system of dimension $\frac{1}{2}n(n+3)$, so that the "freedom" of nets of quadrics in $[n]$ is the same as that of planes in a space $[\frac{1}{2}n(n+3)]$; this freedom is equal to

$$3\{\tfrac{1}{2}n(n+3)-2\} = \tfrac{3}{2}(n-1)(n+4).$$

Now the special net of quadrics is determined by a rational normal curve of order n and a g^1_{n+2} thereon; since the freedom of a rational normal curve of order n is $(n-1)(n+3)$, and that of a g^1_{n+2} on a rational curve is $2(n+1)$, the freedom of the specialised net of quadrics is

$$(n+3)(n-1)+2(n+1) = n^2+4n-1.$$

But $$\tfrac{3}{2}(n-1)(n+4)-(n^2+4n-1) = \tfrac{1}{2}(n^2+n-10).$$

Hence if a general net of quadrics in $[n]$ is so specialised that it becomes one of the special nets that we are to consider, it must be subjected to $\frac{1}{2}(n^2+n-10)$ conditions. When $n=3$ this gives the known fact that a net of polar quadrics in [3] can be obtained by imposing *one* condition on a general net of quadric surfaces.

When $n=4$ we have a net of quadrics arising from a g^1_6 on a rational normal quartic curve. A detailed study of this net, together with other special nets of quadrics in [4], is now in the press.[1]

The curve ϑ and its properties.

1. The homogeneous coordinates $\{x_0, x_1, x_2, \dots x_r, \dots, x_n\}$ of a point on a rational normal curve C of order n in $[n]$ may be taken to be

$$\left\{\theta^n, -n\,\theta^{n-1}, \binom{n}{2}\theta^{n-2}, \dots, (-1)^r\binom{n}{r}\theta^{n-r}, \dots, (-1)^n\right\},$$

[1] *Acta mathematica* 66 (1936), 253-332.

where θ is a parameter giving the position of the point on C. Every prime meets C in n points, and the equation of the *osculating prime* at the point whose parameter is θ is

$$x_0 + \theta x_1 + \theta^2 x_2 + \ldots . + \theta^r x_r + \ldots . + \theta^n x_n = 0.$$

The parameters of any set of $n + 2$ points on C are the roots of some equation of degree $n + 2$ in θ; let us then consider the singly-infinite family of sets of $n + 2$ points on C given by the equation

$$\lambda (a_0\theta^{n+2} + a_1\theta^{n+1} + \ldots + a_{n+1}\theta + a_{n+2}) + \mu (b_0\theta^{n+2} + b_1\theta^{n+1} + \ldots + b_{n+1}\theta + b_{n+2}) = 0,$$

where the a's and b's are fixed coefficients and $\lambda : \mu$ is a variable parameter; each value of the ratio $\lambda : \mu$ gives one set of $n + 2$ points on C. It is supposed that the two polynomials

$$a_0 \theta^{n+2} + a_1 \theta^{n+1} + \ldots . + a_{n+1} \theta + a_{n+2},$$
$$b_0 \theta^{n+2} + b_1 \theta^{n+1} + \ldots . + b_{n+1} \theta + b_{n+2}$$

have no common factor. If θ is given the equation determines the value of $\lambda : \mu$ uniquely; in other words, a given point of C belongs to one and only one of the sets of $n + 2$ points. We therefore say that the sets of points form an *involution* of sets of $n + 2$ points on C. Call the involution J.

Suppose now that we take the osculating primes of C at the points of a set of J, and omit any two of them; the remaining n primes have a common point, and we obtain in this way, by omitting different pairs of the $n + 2$ osculating primes, $\frac{1}{2}(n + 1)(n + 2)$ points associated with each set of J. The locus of these points, as the set varies in the involution J, is a curve ϑ, some of whose properties it is proposed to obtain.

Through any point $\{\xi_0, \xi_1, \ldots, \xi_n\}$ of $[n]$ there pass n osculating primes of C, the parameters of their points of osculation being the roots of the equation

$$\xi_0 + \theta\xi_1 + \ldots . + \theta^n \xi_n = 0.$$

If then $\{\xi_0, \xi_1, \ldots, \xi_n\}$ is on ϑ we must have an identity

$$\lambda (a_0 \theta^{n+2} + a_1 \theta^{n+1} + \ldots . + a_{n+1} \theta + a_{n+2}) + \mu (b_0 \theta^{n+2} + b_1 \theta^{n+1} + \ldots . + b_{n+1} \theta + b_{n+2})$$
$$\equiv (\alpha\theta^2 + \beta\theta + \gamma)(\xi_0 + \theta\xi_1 + \ldots . + \theta^n \xi_n).$$

Equating the coefficients of the different powers of θ in this identity we obtain $n + 3$ equations, linear and homogeneous in the five quantities λ, μ, α, β, γ. If these equations are simultaneously satisfied the point ξ must be such that its cordinates satisfy

$$\left\| \begin{matrix} a_0 & a_1 & a_2 & a_3 & \dots & a_{n-1} & a_n & a_{n+1} & a_{n+2} \\ b_0 & b_1 & b_2 & b_3 & \dots & b_{n-1} & b_n & b_{n+1} & b_{n+2} \\ x_n & x_{n-1} & x_{n-2} & x_{n-3} & \dots & x_1 & x_0 & 0 & 0 \\ 0 & x_n & x_{n-1} & x_{n-2} & \dots & x_2 & x_1 & x_0 & 0 \\ 0 & 0 & x_n & x_{n-1} & \dots & x_3 & x_2 & x_1 & x_0 \end{matrix} \right\| = 0;$$

all determinants, of five rows and columns, of this matrix of five rows and $n+3$ columns are to vanish. The locus given by the simultaneous vanishing of these determinants is a curve[1], and its order, being the coefficient of t^{n-1} in the expansion of $(1-t)^{-3}$, is $\frac{1}{2}n(n+1)$.

2. We have therefore found that ϑ is a curve of order $\frac{1}{2}n(n+1)$; it has properties which curves in $[n]$ do not in general have, and one set of these properties, relating to secant spaces of the curve, can be pointed out immediately. If we take any set of J and omit three of its points, the osculating primes at the remaining $n-1$ points have a line in common; each of the three omitted primes meets this line, and the point of intersection, being common to the primes which osculate C at n points belonging to the same set of J, is on ϑ; hence the line is a trisecant of ϑ. Thus ϑ *has an infinity of trisecants.* Again, if we take any set of J and omit four of its points, the osculating primes at the remaining $n-2$ points have a plane in common; each of the four omitted primes meets this plane in a trisecant of ϑ, and the intersection of any two of these four trisecants is on ϑ. Thus *the plane meets ϑ in six points, these being the vertices of a quadrilateral.* There is a singly-infinite family of these secant planes of ϑ. Similarly we obtain a family of secant solids of ϑ; each secant solid meets ϑ in the ten vertices of a pentahedron whose five faces are all secant planes of ϑ, and the secant solid is the intersection of $n-3$ primes osculating C at points which all belong to the same set of J;

[1] Suppose we have a matrix of p rows and q columns ($p \leq q$), the elements of the matrix being homogeneous polynomials in the coordinates of a point in $[n]$, all those elements in the i^{th} row being of degree r_i. Then the locus given by the simultaneous vanishing of all the determinants of p rows and columns belonging to the matrix is of dimension $n-(q-p+1)$, and its order is the coefficient of t^{q-p+1} in the expansion of $\{(1-r_1t)(1-r_2t)\dots\dots(1-r_pt)\}^{-1}$. See Baker: *Principles of Geometry* 6 (Cambridge, 1933), 109. Here $p=5$; $r_1=r_2=0$, $r_3=r_4=r_5=1$.

and so on. A secant $[p]$ of ϑ, where $0 < p < n - 1$, meets ϑ in $\frac{1}{2}(p+1)(p+2)$ points and contains $p + 2$ secant $[p-1]$'s; the points in which the secant $[p]$ meets ϑ are the points of intersection of the sets of p of the $p + 2$ secant $[p-1]$'s. If $p < n - 2$ the secant $[p]$ is itself contained in $n-p$ secant $[p+1]$'s. Through each point of ϑ there pass $\binom{n}{p}$ secant $[p]$'s.

The secant spaces of dimension p generate a scroll R_{p+1} on which ϑ is of multiplicity $\binom{n}{p}$; in particular the trisecants of ϑ generate a ruled surface on which ϑ is of multiplicity n. If $r < p$ the scrolls R_{r+1} all lie on R_{p+1} and are multiple loci thereon, the actual degree of multiplicity of R_{r+1} on R_{p+1} being $\binom{n-r}{p-r}$. All the scrolls R_{p+1} lie on the primal R_{n-1} which is generated by the secant spaces of dimension $n - 2$.

3. The rational normal curve C can be put in (1, 1) correspondence with a conic γ; the involution J on C then gives rise to a corresponding involution j on γ. Moreover, any point of ϑ is the intersection of the primes which osculate C at n points belonging to a set of J; there are two other points belonging to this set of J, and we can suppose that the point of ϑ from which we started corresponds to that point in the plane of γ which is the intersection of the tangents of γ at the two points which correspond to these two remaining points of the set. Conversely: through any point of the plane of γ there pass two of its tangents, and the points of contact of these tangents will not, in general, belong to the same set of the involution j. If they do the points of C to which they correspond belong to the same set of the involution J, and the osculating primes of C at the remaining n points of the set meet in a point of ϑ. This point of ϑ may then be regarded as corresponding to the point in the plane of γ. The curve ϑ is therefore in (1, 1) correspondence with the plane curve ζ which is the locus of intersections of those pairs of tangents of γ whose points of contact belong to the same set of the involution j. An intersection of ζ and γ must be a double point of a set of j, and also every point of γ which is a double point of a set of j is on ζ; thus the curve ζ meets γ in the points of the Jacobian set of j, and only in these points. The number of points in the Jacobian set of j, being the degree of the discriminant of a polynomial of order $n + 2$, is $2(n + 1)$; hence ζ must be a curve of order $n + 1$. Through

any point of ζ there pass two tangents of γ, their points of contact belonging to the same set of j; the two points of contact cannot belong simultaneously to any other set of j, and there are no further tangents of γ passing through the point; hence ζ has no multiple points, and is therefore of genus $\frac{1}{2}n(n-1)$. Since the curves ζ and ϑ are in (1, 1) correspondence ϑ must also be of genus $\frac{1}{2}n(n-1)$.

The tangents of γ at the points of a set of j form an $(n+2)$-gram whose $\frac{1}{2}(n+1)(n+2)$ vertices are all on ζ; the curve ζ is therefore circumscribed to a singly-infinite set of $(n+2)$-grams, which are themselves circumscribed to γ. The $\frac{1}{2}(n+1)(n+2)$ vertices of an $(n+2)$-gram correspond, in the (1, 1) correspondence between ζ and ϑ, to the $\frac{1}{2}(n+1)(n+2)$ vertices of an $(n+2)$-hedron, the $n+2$ faces of this being the osculating primes of C at the points of a set of J. The sides of the $(n+2)$-gram correspond to the faces of the $(n+2)$-hedron, and the vertices of the $(n+2)$-gram correspond to those of the $(n+2)$-hedron, in such a way that those vertices which do not lie on a particular side of the $(n+2)$-gram correspond to those vertices which do lie in that face of the $(n+2)$-hedron which corresponds to the particular side. If a side of an $(n+2)$-gram is omitted the vertices of the remaining $(n+1)$-gram are $\frac{1}{2}n(n+1)$ in number; they correspond to the vertices of the $(n+2)$-hedron which lie in that face corresponding to the omitted side of the $(n+2)$-gram; these are the $\frac{1}{2}n(n+1)$ intersections of the face of the $(n+2)$-hedron with ϑ.

The locus of the vertices of $(n+2)$-grams which are circumscribed to γ, and whose sets of $n+2$ points of contact with γ belong to an involution of sets of $n+2$ points, has been seen to be a curve of order $n+1$; similarly the locus of the vertices of $(n+1)$-grams which are circumscribed to γ, and whose sets of $n+1$ points of contact with γ belong to an involution of sets of $n+1$ points, is a curve of order n. Take now any two of the $(n+2)$-grams which are inscribed in ζ, and omit one side of each; we obtain two $(n+1)$-grams whose vertices lie on ζ. Moreover the two sets of $n+1$ points on γ which are the points of contact of the sides of these two $(n+1)$-grams determine a unique involution of sets of $n+1$ points on γ; the locus of vertices of the $(n+1)$-grams arising from this involution is a curve of order n. Since each $(n+1)$-gram has $\frac{1}{2}n(n+1)$ vertices, the vertices of the two $(n+1)$-grams by which the involution was determined account for all the $n(n+1)$ intersections of ζ with this curve of order n; wherefore if any two $(n+1)$-grams are obtained from any two inscribed $(n+2)$-grams of ζ by omitting one side of each, their

$n(n+1)$ vertices form the complete intersection of ζ with a curve of order n. If, for the moment, we keep one of these $(n+1)$-grams fixed and allow the other to vary continuously (as we may do since it is determined by the tangent of γ which is omitted from an $(n+2)$-gram), we see that if any side is omitted from an inscribed $(n+2)$-gram of ζ, *the* $\frac{1}{2}n(n+1)$ *vertices of the resulting* $(n+1)$-*gram are the points of contact of* ζ *with a curve of order* n; this curve of order n therefore touches ζ wherever it meets it. There is thus obtained a singly-infinite set of *contact-curves* of order n of ζ; any two curves of the set being such that their two sets of contacts together constitute a set of points which is the complete intersection of ζ with a curve of order n. The set of $\frac{1}{2}n(n+1)$ contacts of ζ with one of these contact-curves corresponds, in the (1, 1) correspondence between ϑ and ζ, to the set of $\frac{1}{2}n(n+1)$ intersections of ϑ with an osculating prime of C.

The net of quadrics.

4. The equation of an osculating prime of C is

$$P \equiv x_0 + \theta x_1 + \theta^2 x_2 + \ldots + \theta^n x_n = 0.$$

Suppose now that $\theta_1, \theta_2, \ldots, \theta_{n+2}$ are the roots of

$$a_0 \theta^{n+2} + a_1 \theta^{n+1} + \ldots + a_{n+2} = 0,$$

and so are the parameters of the points of a set of J, while $\theta_{n+3}, \theta_{n+4}, \ldots, \theta_{2n+4}$ are the roots of

$$b_0 \theta^{n+2} + b_1 \theta^{n+1} + \ldots + b_{n+2} = 0,$$

and so are the parameters of the points of a second set of J. Write

$$P_i \equiv x_0 + \theta_i x_1 + \theta_i^2 x_2 + \ldots + \theta_i^n x_n \qquad (i = 1, 2, 3, \ldots, 2n+4).$$

Then the $2n+4$ expressions P_i^2, since they depend linearly on only $2n+1$ quadratic functions of the coordinates x, must be connected by three linearly independent linear relations; suppose these are

$$\sum_{i=1}^{2n+4} A_i P_i^2 \equiv 0, \quad \sum_{i=1}^{2n+4} B_i P_i^2 \equiv 0, \quad \sum_{i=1}^{2n+4} C_i P_i^2 \equiv 0.$$

There thus arise the three linearly independent quadric primals with equations

$$\sum_{i=1}^{n+2} A_i P_i^2 = 0, \quad \sum_{i=1}^{n+2} B_i P_i^2 = 0, \quad \sum_{i=1}^{n+2} C_i P_i^2 = 0,$$

these being, respectively, the same equations as

$$\sum_{i=n+3}^{2n+4} A_i P_i^2 = 0, \quad \sum_{i=n+3}^{2n+4} B_i P_i^2 = 0, \quad \sum_{i=n+3}^{2n+4} C_i P_i^2 = 0.$$

These three quadrics determine a net of quadric primals. Consider now the first form for the three equations of the quadrics; the left hand side of each of the three equations is the sum of constant multiples of the $n+2$ squares $P_1^2, P_2^2, \ldots, P_{n+2}^2$. It follows at once that every point of the $[n-s]$ common to any s of the $n+2$ primes $P_1=0, P_2=0, \ldots, P_{n+2}=0$ is conjugate to every point of the $[s-2]$ common to the remaining $n+2-s$ primes in regard to every quadric of the net; we may say that the $n+2$ primes form a self-conjugate $(n+2)$-hedron in regard to all the quadrics of the net. In particular, putting $s=2$, or n, every vertex of the $(n+2)$-hedron, being common to n of its bounding primes, is conjugate to every point of the $[n-2]$, which is the intersection of the two remaining bounding primes, in regard to every quadric of the net. But if the polar primes of a point in regard to all the quadrics of a net in $[n]$ have an $[n-2]$ in common the point[1] must be the vertex of a cone belonging to the net of quadrics and so lie on the Jacobian curve of the net; hence all the vertices of the $(n+2)$-hedron bounded by the primes $P_1=0, P_2=0, \ldots, P_{n+2}=0$ are vertices of cones belonging to the net of quadrics. This is verified at once from the equations of the quadrics; for the three equations can be combined linearly in such a way that the coefficients of any two of the $n+2$ squares vanish; the particular quadric obtained is then linearly dependent on the squares of n primes, and is therefore a cone whose vertex is the common point of these n primes. Hence if we take that set of J the parameters of whose points are $\theta_1, \theta_2, \ldots, \theta_{n+2}$, the $\frac{1}{2}(n+1)(n+2)$ points of ϑ arising from this set are on the Jacobian curve of the net of quadrics. It follows similarly, by considering the second form for the equations of the quadrics of the net, that if we take the set of J the parameters of whose points are $\theta_{n+3}, \theta_{n+4}, \ldots, \theta_{2n+4}$, the $\frac{1}{2}(n+1)(n+2)$ points of ϑ arising from this set are also on the Jacobian curve of the net of quadrics.

We have obtained the net of quadrics by considering two sets of the involution J; we can similarly obtain a net of quadrics by considering any other pair of sets of J, and it follows similarly that its Jacobian curve meets ϑ in the $(n+1)(n+2)$ vertices of the two associated $(n+2)$-hedra. The fact is, however, that all the nets of

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[1] If $\Gamma_0=0, \Gamma_1=0, \Gamma_2=0$ are the equations of three independent quadrics of the net, the coordinates of such a point must cause all the three-rowed determinants of the matrix $\left\| \frac{\partial \Gamma_i}{\partial x_j} \right\|$, of three rows and $n+1$ columns, to vanish.

quadrics so obtained are one and the same net of quadrics, and it then follows that the Jacobian curve of this net of quadrics is ϑ itself.

5. Suppose that $\phi_1, \phi_2, \ldots, \phi_{n+2}$ are the roots of

$$\lambda (a_0 \theta^{n+2} + a_1 \theta^{n+1} + \ldots + a_{n+2}) + \mu (b_0 \theta^{n+2} + b_1 \theta^{n+1} + \ldots + b_{n+2}) = 0,$$

and write

$$Q_i \equiv x_0 + \phi_i x_1 + \phi_i^2 x_2 + \ldots + \phi_i^n x_n, \qquad (i = 1, 2, \ldots, n+2).$$

Then we have three identities

$$\sum_{i=1}^{n+2} (A_i P_i^2 + A'_i Q_i^2) \equiv 0, \quad \sum_{i=1}^{n+2} (B_i P_i^2 + B'_i Q_i^2) \equiv 0, \quad \sum_{i=1}^{n+2} (C_i P_i^2 + C'_i Q_i^2) \equiv 0,$$

where the $n+2$ expressions P_i are those previously defined. If it can now be shown that the mutual ratios of the $3n+6$ quantities

$$A_1, A_2, \ldots, A_{n+2}, B_1, B_2, \ldots, B_{n+2}, C_1, C_2, \ldots, C_{n+2},$$

occurring in these identities are the same as the ratios of the corresponding quantities, denoted by the same symbols, occurring in the previous identities, the truth of the statement that the net of quadrics arising from a pair of sets of J is the same, no matter which pair of sets is taken, will follow without further argument.

The identities connecting the $2n+4$ squares P_i^2 and Q_i^2 can be written down explicitly. Let us write

$$f(\theta) \equiv (\theta - \theta_1)(\theta - \theta_2) \ldots (\theta - \theta_{n+2}),$$
$$g(\theta) \equiv (\theta - \phi_1)(\theta - \phi_2) \ldots (\theta - \phi_{n+2}).$$

Then it can immediately be verified that the three identities are[1]

$$\sum_{i=1}^{n+2} \left\{ \frac{P_i^2}{f'(\theta_i)\, g(\theta_i)} + \frac{Q_i^2}{f(\phi_i)\, g'(\phi_i)} \right\} \equiv 0,$$
$$\sum_{i=1}^{n+2} \left\{ \frac{\theta_i P_i^2}{f'(\theta_i)\, g(\theta_i)} + \frac{\phi_i Q_i^2}{f(\phi_i)\, g'(\phi_i)} \right\} \equiv 0,$$
$$\sum_{i=1}^{n+2} \left\{ \frac{\theta_i^2 P_i^2}{f'(\theta_i)\, g(\theta_i)} + \frac{\phi_i^2 Q_i^2}{f(\phi_i)\, g'(\phi_i)} \right\} \equiv 0.$$

Thus we have

$$A_i = \frac{1}{f'(\theta_i)\, g(\theta_i)}, \quad B_i = \frac{\theta_i}{f'(\theta_i)\, g(\theta_i)}, \quad C_i = \frac{\theta_i^2}{f'(\theta_i)\, g(\theta_i)}.$$

Now

$$(\lambda a_0 + \mu b_0)\, g(\theta_i) = \lambda (a_0 \theta_i^{n+2} + a_1 \theta_i^{n+1} + \ldots + a_{n+2}) + \mu (b_0 \theta_i^{n+2} + b_1 \theta_i^{n+1} + \ldots + b_{n+2})$$
$$= \mu (b_0 \theta_i^{n+2} + b_1 \theta_i^{n+2} + \ldots + b_{n+2}).$$

[1] When $n = 3$ the forms of the equations of the three quadrics arising from these identities are equivalent to the less symmetrical set of three equations given by Dixon, *loc. cit.*, p. 154.

Thus, except for a factor of proportionality $(\lambda a_0 + \mu b_0)/\mu$, which is the same for all of them, the quantities A_i, B_i, C_i are independent of the choice of the set $(\phi_1, \phi_2, \ldots, \phi_{n+2})$ among the sets of J. Thus we always obtain the same net of quadrics.

The developables D.

6. Suppose we take the space $[s]$ which is common to any $n-s$ faces of one of the self-conjugate $(n+2)$-hedra, and the space $[n-s-2]$ which is common to the remaining $s+2$ faces; these two spaces determine a unique prime which contains them both. There is associated in this way with each of the $(n+2)$-hedra a finite number of primes, and when all the primes associated with the different $(n+2)$-hedra are taken we obtain a singly-infinite family of primes generating a developable; we may denote this developable by either D_{s+1} or D_{n-s-1}; it is a combinantal contravariant of the net of quadrics. The number of generating primes of D_{s+1} associated with any one of the $(n+2)$-hedra is $\binom{n+2}{n-s}$, except in the particular case when n is even and $s = \frac{1}{2}n - 1$, in which case the number is only one half of this. We propose to obtain the *class* d_{s+1} of the developable D_{s+1}, this being the number of its generating primes which pass through an arbitrary point of $[n]$.

The $n+2$ expressions

$$Q_i \equiv x_0 + \phi_i x_1 + \phi_i^2 x_2 + \ldots + \phi_i^n x_n \qquad (i = 1, 2, \ldots, n+2)$$

satisfy the identity

$$\sum_{i=1}^{n+2} \frac{Q_i}{g'(\phi_i)} \equiv 0;$$

this being the only linear identity which they do satisfy. Thus the equation of the prime which joins the space $[s]$ common to the primes

$$Q_1 = 0,\ Q_2 = 0, \ldots, Q_{n-s} = 0,$$

to the space $[n-s-2]$ common to the primes

$$Q_{n-s+1} = 0,\ Q_{n-s+2} = 0, \ldots, Q_{n+2} = 0,$$

can be written in either of the two forms

$$\sum_{i=1}^{n-s} \frac{Q_i}{g'(\phi_i)} = 0, \qquad \text{or} \qquad \sum_{i=n-s+1}^{n+2} \frac{Q_i}{g'(\phi_i)} = 0.$$

The question now to be considered is—how many of these primes pass through an arbitrary point of $[n]$?

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7. We will first, before obtaining d_{s+1}, obtain d_1 and d_2, for the sake of simplicity and illustration; indeed, to simplify the matter still further, we will first obtain d_1 for the net of quadric surfaces in [3] with an infinity of self-conjugate pentahedra. In [3] D_1 is the only developable which exists.

Suppose then that we take an involution of sets of five points on a twisted cubic: the parameters $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ of the points of a set are the roots of the quintic

$$\lambda(a_0\theta^5+a_1\theta^4+a_2\theta^3+a_3\theta^2+a_4\theta+a_5)+\mu(b_0\theta^5+b_1\theta^4+b_2\theta^3+b_3\theta^2+b_4\theta+b_5)=0,$$

different values of the ratio $\lambda:\mu$ giving different sets of the involution. If

$$Q_i \equiv x_0+\phi_i x_1+\phi_i^2 x_2+\phi_i^3 x_3, \qquad (i=1,2,3,4,5)$$
$$g(\theta)\equiv(\theta-\phi_1)(\theta-\phi_2)(\theta-\phi_3)(\theta-\phi_4)(\theta-\phi_5),$$

then

$$\sum_{i=1}^{5}\frac{Q_i}{g'(\phi_i)}\equiv 0,$$

and the equation of the plane which joins the line of intersection of the osculating planes of the twisted cubic at the points whose parameters are ϕ_1 and ϕ_2 to the point of intersection of the osculating planes at the three points whose parameters are ϕ_3, ϕ_4, ϕ_5 is

$$\frac{x_0+\phi_1 x_1+\phi_1^2 x_2+\phi_1^3 x_3}{(\phi_1-\phi_2)(\phi_1-\phi_3)(\phi_1-\phi_4)(\phi_1-\phi_5)}+\frac{x_0+\phi_2 x_1+\phi_2^2 x_2+\phi_2^3 x_3}{(\phi_2-\phi_3)(\phi_2-\phi_4)(\phi_2-\phi_5)(\phi_2-\phi_1)}=0.$$

When this is cleared of fractions it can be written

$$\begin{aligned}&x_0\{\phi_1^2+\phi_1\phi_2+\phi_2^2-(\phi_1+\phi_2)(\phi_3+\phi_4+\phi_5)+(\phi_4\phi_5+\phi_5\phi_3+\phi_3\phi_4)\}\\&+x_1\{\phi_1\phi_2(\phi_1+\phi_2)-\phi_1\phi_2(\phi_3+\phi_4+\phi_5)+\phi_3\phi_4\phi_5\}\\&+x_2\{\phi_1^2\phi_2^2-\phi_1\phi_2(\phi_4\phi_5+\phi_5\phi_3+\phi_3\phi_4)+(\phi_1+\phi_2)\phi_3\phi_4\phi_5\}\\&+x_3\{\phi_1^2\phi_2^2(\phi_3+\phi_4+\phi_5)-\phi_1\phi_2(\phi_1+\phi_2)(\phi_4\phi_5+\phi_5\phi_3+\phi_3\phi_4)+(\phi_1^2+\phi_1\phi_2+\phi_2^2)\phi_3\phi_4\phi_5\}=0.\end{aligned}$$

The left-hand side of this equation is a symmetric function of ϕ_1 and ϕ_2, and also a symmetric function of ϕ_3, ϕ_4, ϕ_5; but it is not a symmetric function of the five parameters ϕ; suppose then that we call this equation

$$\{12\}=0.$$

The expression $\{12\}$ is of the second order in each of ϕ_1 and ϕ_2 and of the first order in each of ϕ_3, ϕ_4, ϕ_5.

We can form a symmetric function of $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$ having $\{12\}$ as a factor; such a function is

$$\{12\}\{13\}\{14\}\{15\}\{23\}\{24\}\{25\}\{34\}\{35\}\{45\}.$$

When equated to zero this represents ten planes associated with one of the pentahedra; we may call them the diagonal planes of the pentahedron. Moreover, being a symmetric function, it is a rational function of the coefficients of the quintic equation whose roots are $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5$. Since ϕ_1 occurs squared in each of the four factors $\{12\}, \{13\}, \{14\}, \{15\}$, and to the first degree only in the remaining six factors, the highest power of ϕ_1 which occurs in the symmetric function is $\phi_1{}^{14}$. If then the symmetric function is multiplied by $(\lambda a_0 + \mu b_0)^{14}$ the resulting product is a homogeneous binary form of degree 14 in λ and μ; when equated to zero it gives 14 values for the ratio $\lambda : \mu$, and hence there are 14 pentahedra for which a plane of D_1 passes through (x_0, x_1, x_2, x_3). Hence, for the net of quadrics in [3], $d_1 = 14$.

The value of d_1 for the net of quadrics in $[n]$ is found by an exactly similar argument. The equation of a generating prime of D_1 is now

$$\frac{x_0 + \phi_1 x_1 + \phi_1^2 x_2 + \ldots . + \phi_1^n x_n}{(\phi_1 - \phi_2)(\phi_1 - \phi_3) \ldots (\phi_1 - \phi_{n+1})(\phi_1 - \phi_{n+2})} + \frac{x_0 + \phi_2 x_1 + \phi_2^2 x_2 + \ldots . + \phi_2^n x_n}{(\phi_2 - \phi_3)(\phi_2 - \phi_4) \ldots (\phi_2 - \phi_{n+2})(\phi_2 - \phi_1)} = 0.$$

Multiplying by the product of the two denominators, and then removing the factor $(\phi_1 - \phi_2)^2$, this equation becomes

$$\{12\} = 0,$$

where $\{12\}$ is a symmetric function of the pair of variables ϕ_1, ϕ_2 and also of the set of n variables $\phi_3, \phi_4, \ldots . , \phi_{n+2}$; the two variables ϕ_1 and ϕ_2 occur in $\{12\}$ to the $(n-1)^{\text{th}}$ but no higher power, while the remaining variables occur only to the first power. Then the product

$$\{12\}\{13\}\{14\} \ldots \{1\,\overline{n+1}\}\{1\,\overline{n+2}\}\{23\}\{24\} \ldots \{2\,\overline{n+2}\}\{34\} \ldots \{\overline{n+1}\;\overline{n+2}\}$$

in a symmetric function of all the $n+2$ variables $\phi_1, \phi_2, \ldots . , \phi_{n+2}$. The number of factors $\{r\,s\}$ in which the digit 1 occurs is $n+1$, while the number in which the digit 1 does not occur is $\frac{1}{2}n(n+1)$; hence the degree to which ϕ_1 occurs in the whole product is

$$(n+1)(n-1) + \tfrac{1}{2}n(n+1) = \tfrac{1}{2}(n+1)(3n-2).$$

Whence the value of d_1 is $\frac{1}{2}(n+1)(3n-2)$.

The property of having a singly-infinite family of secant $[n-2]$'s in (1, 1) correspondence with its points is also possessed by the Jacobian curve of a *general* net of quadrics in $[n]$, and the formula $\frac{1}{2}(n+1)(3n-2)$ for the class of the developable D_1 has been obtained previously in a different way[1]. But the other developables D_2, D_3, ... do not exist for a general net of quadrics.

8. We next obtain the class of the developable D_2. A generating prime of D_2 joins a trisecant of ϑ to a secant $[n-3]$ of ϑ, and its equation is

$$\frac{x_0+\phi_1 x_1+\ldots+\phi_1^n x_n}{(\phi_1-\phi_2)(\phi_1-\phi_3)\ldots(\phi_1-\phi_{n+2})}+\frac{x_0+\phi_2 x_1+\ldots+\phi_2^n x_n}{(\phi_2-\phi_3)\ldots(\phi_2-\phi_{n+2})(\phi_2-\phi_1)}$$
$$+\frac{x_0+\phi_3 x_1+\ldots+\phi_3^n x_n}{(\phi_3-\phi_4)\ldots(\phi_3-\phi_1)(\phi_3-\phi_2)}=0.$$

Multiplying by the product of the three denominators, and then dividing by $(\phi_2-\phi_3)^2(\phi_3-\phi_1)^2(\phi_1-\phi_2)^2$, we obtain an equation

$$\{123\}=0.$$

The expression $\{123\}$ is a symmetric function of ϕ_1, ϕ_2, ϕ_3; the highest power of each of these which occurs in the symmetric function is $n-2$; also $\{123\}$ is a symmetric function of $\phi_4, \phi_5, \ldots, \phi_{n+2}$, these occurring squared. We then form the symmetric function

$$\{123\}\{124\}\{125\}\ldots\{12\,\overline{n+2}\}\{134\}\ldots\{n\,\overline{n+1}\,\overline{n+2}\}.$$

The number of symbols $\{r\,s\,t\}$ occurring in this function, which is a symmetric function of the whole set of $n+2$ quantities $\phi_1, \phi_2, \ldots, \phi_{n+2}$, is $\binom{n+2}{3}$; the number of symbols which contain the digit 1 is $\binom{n+1}{2}$ and the number which do not contain this digit is $\binom{n+2}{3}-\binom{n+1}{2}=\binom{n+1}{3}$. The highest power to which ϕ_1 occurs in this symmetric function is therefore

$$\binom{n+1}{2}(n-2)+\binom{n+1}{3}2=\tfrac{1}{2}(n+1)n(n-2)+\tfrac{1}{3}(n+1)n(n-1)$$
$$=\tfrac{1}{6}n(n+1)(5n-8).$$

The class of D_2 is therefore given by the equation $d_2=\frac{1}{6}n(n+1)(5n-8)$.

There is, however, one exception to this statement, and this is an additional reason why it is perhaps worth while to obtain d_2 separately before proceeding to obtain the general formula for d_{s+1}. In the case when $n=4$, and the net of quadrics is in four-dimensional space, there are six quantities $\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6$; and the expression $\{123\}$ is exactly the same as $\{456\}$. Thus, in this particular case, we

[1] *Proc. Edinburgh Math. Soc.* (2), 3 (1933), 263.

do not need to take the product of the twenty expressions $\{r\,s\,t\}$ in order to form a symmetric function; these expressions are equal in pairs, and we need only take the product of ten of them. We can take the ten symbols containing any given digit; for example the product

$$\{123\}\{124\}\{125\}\{126\}\{134\}\{135\}\{136\}\{145\}\{146\}\{156\}$$

is a symmetric function. The class of D_2 is therefore 20, half the value given by the above formula for d_2, in the particular case when $n = 4$.

9. We may now obtain the value of d_{s+1}; after these preparatory illustrations it will be sufficient to give only the main steps of the argument. The equation of a generating prime of D_{s+1} is

$$\sum_{i=1}^{s+2} \frac{x_0 + \phi_i x_1 + \phi_i^2 x_2 + \dots + \phi_i^n x_n}{g'(\phi_i)} = 0.$$

Multiplying by the continued product of the $s + 2$ denominators, and then dividing by

$$\{(\phi_1-\phi_2)(\phi_1-\phi_3)\dots(\phi_1-\phi_{s+2})(\phi_2-\phi_3)\dots(\phi_2-\phi_{s+2})(\phi_3-\phi_4)\dots(\phi_{s+1}-\phi_{s+2})\}^2,$$

we denote the quotient by the symbol

$$\{123\,\dots\,\overline{s+2}\}.$$

This is a symmetric function of the set of $s+2$ variables $\phi_1, \phi_2, \dots, \phi_{s+2}$, each of these occurring to a power $n - s - 1$; it is also a symmetric function of the set of $n - s$ variables $\phi_{s+3}, \phi_{s+4}, \dots, \phi_{n+2}$, each of these occurring to a power $s + 1$. We now form a symmetric function of the set of $n + 2$ variables $\phi_1, \phi_2, \dots, \phi_{n+2}$ by multiplying together the $\binom{n+2}{s+2}$ expressions, like $\{123\,\dots\,\overline{s+2}\}$, obtained by taking all possible combinations of $s + 2$ of the first $n + 2$ integers; the number of these expressions which contain the digit 1 is $\binom{n+1}{s+1}$, and the number which do not contain this digit is $\binom{n+1}{s+2}$. Hence the highest power to which ϕ_1 occurs in the complete symmetric function is

$$\binom{n+1}{s+1}(n-s-1) + \binom{n+1}{s+2}(s+1) = \binom{n+1}{s+1}\frac{n(2s+3) - 2(s+1)^2}{s+2}.$$

This then is the value of d_{s+1}; when $s = 0$ and $s = 1$ it agrees with those values previously found for d_1 and d_2.

There is an exception to this result (as already noticed in the case $s = 1$); namely when $s + 1 = \frac{1}{2}n$. If $n = 2p$ and $s = p - 1$ the value of d_p is one-half of that given by this formula; hence, in this special case, the value of d_p is

$$\tfrac{1}{2}\binom{2p+1}{p}\frac{2p(2p+1) - 2p^2}{p+1} = p\binom{2p+1}{p}.$$

The scrolls R.

10. We will now obtain, by elementary correspondence theory, a formula for the order N_{p+1} of the scroll R_{p+1} which is generated by the secant $[p]$'s of ϑ.

It is to be expected that, of the singly-infinite family of trisecants of ϑ, a finite number will touch ϑ; we first obtain this number, since it is of great importance in the calculation of N_{p+1}. Suppose that a trisecant touches ϑ in T and meets it again in O; this trisecant, like any other trisecant of ϑ, is the line common to $n - 1$ primes which osculate C at points all belonging to the same set of J. Those primes which osculate C at the three remaining points of this set of J are such that one of them meets the trisecant in O while the other two both meet it in T. But it is impossible for $n + 1$ different osculating primes of C to pass through T, since C is only of class n; hence those two osculating primes of C which meet the trisecant in T must coincide with one and the same prime Σ, and the point in which Σ osculates C must be a double point of the involution J. The $(n + 2)$-hedron arising from this set of J consists of n primes, whose point of intersection is O, and of the prime Σ counted twice, and it follows that each of the n trisecants through O touches ϑ at its point of intersection with Σ. Thus the point of contact of ϑ with any one of its trisecants which is also a tangent lies in a prime which osculates C at a double point of J while, conversely, any prime which osculates C at a double point of J contains n points on ϑ at which the tangents meet ϑ again; the n tangents in fact are concurrent in a point of ϑ. Now the involution J, being a g^1_{n+2} on a rational curve, has $2n + 2$ double points; hence *there are* $2n(n + 1)$ *trisecants of* ϑ *which are also tangents of* ϑ, and they are distributed in $2n + 2$ concurrent sets of n.

11. We can now find the order N_2, of the ruled surface R_2 generated by the trisecants of ϑ, immediately. Suppose that two points of ϑ correspond to one another when the line joining them is

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a trisecant of ϑ; then to any given point of ϑ there correspond $2n$ other points of ϑ. The correspondence is clearly *symmetrical*, and we have seen that the number of its united points is $2n(n+1)$. But the order of the ruled surface generated by the joins of pairs of points which correspond to one another in a symmetrical $(2n, 2n)$ correspondence, with $2n(n+1)$ united points, between the points of a curve of order $\frac{1}{2}n(n+1)$, is equal to[1]

$$\tfrac{1}{2}\{2n \,.\, \tfrac{1}{2}n(n+1) + 2n \,.\, \tfrac{1}{2}n(n+1) - 2n(n+1)\} = n(n+1)(n-1),$$

the factor $\frac{1}{2}$ occurring before the bracket because of the symmetry of the correspondence. For the particular ruled surface R_2, however, each generator joins three distinct pairs of corresponding points, so that this result must be divided by 3. The order of R_2 is therefore $\frac{1}{3}n(n^2-1)$. We can obtain a difference equation for N_{p+1}, and this can be solved once the order of R_2 has been found.

We make use of the following general result. Suppose there is an (α, β) correspondence between the points of a curve ϑ of order N_1 and the generating spaces of a scroll R_p of order N_p; *i.e.* to every generating space of the scroll there correspond α points of ϑ, while to every point of ϑ there correspond β generating spaces of R_p. Suppose, further, that there are i points of ϑ which lie in generating spaces of R_p that correspond to them. Then those spaces which join points of ϑ to corresponding generating spaces of R_p generate a scroll R_{p+1} whose order N_{p+1} is given by

$$\alpha N_p + \beta N_1 - i = \mu N_{p+1},$$

μ being the number of corresponding pairs, of generating spaces of N_p and points of ϑ, that lie in a general generating space of N_{p+1}. In the applications which we shall make of this formula μ will in fact always be greater than 1.

12. For the sake of greater clearness we first obtain the order of R_3 before proceeding to obtain the general formula for N_{p+1}. In order to do this we set up a correspondence between the points and trisecants of ϑ, a point not lying in general on any trisecant that corresponds to it; we say that a point and trisecant of ϑ correspond when the plane joining them is a secant plane of ϑ. Since each trisecant lies in $n-1$ secant planes, each of which contains three

[1] See, for example, Edge: *The Theory of Ruled Surfaces* (Cambridge, 1931), pp. 17-18; or Baker, *Principles of Geometry*, 6 (Cambridge, 1933), p. 16.

points corresponding to the trisecant, the number of points of ϑ that correspond to a given trisecant is $3(n-1)$. Also through each point of ϑ there pass $\binom{n}{2}$ secant planes each of which contains two trisecants corresponding to the point; hence the number of trisecants corresponding to a given point of ϑ is $n(n-1)$, Moreover each secant plane is obtained in 12 different ways as the join of a corresponding point and trisecant. There remains only the question of how many points of ϑ there are which lie on trisecants that correspond to them. Now, when any trisecant t is given, a secant plane through t contains three further trisecants t_1, t_2, t_3; if then a vertex of the triangle formed by these three further trisecants, say the vertex $t_2 t_3$, lies on t, then either t_2 or t_3, say t_2, must coincide with t; this causes the vertex $t_1 t_2$ also to lie on t; the trisecants t_3 and t_1 are then tangents of ϑ, both of them meeting ϑ in their point of intersection. Thus when a trisecant t passes through a corresponding point of ϑ the tangent at this point is one of those trisecants which touch ϑ, and there is on t also a second point of ϑ that corresponds to it; the tangents of ϑ at these two points of t intersect, and their point of intersection is on ϑ. Conversely: take any pair of tangents of ϑ which both meet ϑ again in a point O; let T_1 and T_2 be their points of contact. Then the line $T_1 T_2$ is a trisecant of ϑ, and both T_1 and T_2 are points of ϑ which correspond to this trisecant. Now each concurrent set of n tangents gives rise to $\frac{1}{2}n(n-1)$ pairs of points T_1 and T_2: also there are $2n+2$ of these concurrent sets of n tangents; the number of points of ϑ lying on trisecants which correspond to them is therefore $2n(n+1)(n-1)$. We may obtain this number either by considering $2n+2$ sets of $\frac{1}{2}n(n-1)$ trisecants and counting each trisecant twice as containing two corresponding points, or else we may obtain it by considering $2n+2$ sets of n points and counting each point $n-1$ times as lying on $n-1$ corresponding trisecants. The trisecants being generators of R_2, whose order N_2 has been found to be $\frac{1}{3}n(n^2-1)$, we may now write down the equation

$$3(n-1)N_2 + n(n-1)\,.\,\tfrac{1}{2}n(n+1) - 2n(n+1)(n-1) = 12N_3,$$

giving $N_3 = \frac{1}{8}n(n^2-1)(n-2)$.

It is now noticeable that the order N_3 of R_3 is $3\binom{n+1}{4}$, the order N_2 of R_2 is $2\binom{n+1}{3}$, and the order N_1 of ϑ is $\binom{n+1}{2}$; it is

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natural then to surmise that the order N_{p+1} of R_{p+1} is $(p+1)\binom{n+1}{p+2}$, and this will indeed prove to be so.

13. To obtain the order of R_{p+1} we establish a correspondence between the points and the secant $[p-1]$'s of ϑ, a point of ϑ not lying, in general, in a corresponding secant $[p-1]$; a point and secant $[p-1]$ are to correspond when the $[p]$ which joins them is a secant $[p]$. When a secant $[p-1]$ is given there are, since the $[p-1]$ is the intersection of primes which osculate C at $n-p+1$ points belonging to a set of J, $n-p+1$ secant $[p]$'s passing through it. Each of these secant $[p]$'s has $\frac{1}{2}(p+1)(p+2)$ intersections with ϑ of which $\frac{1}{2}p(p+1)$ are on the secant $[p-1]$; there are $p+1$ others which correspond, in the sense above explained, to the secant $[p-1]$. Hence to each secant $[p-1]$ there correspond $(p+1)(n-p+1)$ points of ϑ. Also through any given point of ϑ there pass $\binom{n}{p}$ secant $[p]$'s each of which contains two secant $[p-1]$'s that do not pass through the point: for in any secant $[p]$ there are $p+2$ secant $[p-1]$'s, and any point in which the $[p]$ meets ϑ lies in p of these $[p-1]$'s. Thus to any given point of ϑ there correspond $2\binom{n}{p}$ secant $[p-1]$'s. Further: each secant $[p]$ may be obtained in $(p+1)(p+2)$ different ways as the join of a corresponding point and secant $[p-1]$ of ϑ. We have now to consider how many points of ϑ there are which lie in corresponding secant $[p-1]$'s; this can be found exactly in the same way as in the special case $p=2$. For if we take any secant $[p-1]$, any secant $[p]$ passing through it meets ϑ in $p+1$ further points, these being the vertices of a simplex in the $[p]$. A vertex of this simplex cannot lie in σ, the secant $[p-1]$ from which we started, unless a bounding prime of the simplex coincides with σ; this causes p of the vertices of the simplex to lie in σ, and the tangents of ϑ at these p points must all meet ϑ again in the remaining vertex of the simplex. Hence, when a point of ϑ lies in a corresponding secant $[p-1]$, the tangent at this point meets ϑ again; furthermore the secant $[p-1]$ contains altogether p corresponding points of ϑ, the tangents at these p points all meeting ϑ again in the same point. Conversely: take any set of p tangents all passing through the same point O of ϑ and touching ϑ in $T_1, T_2, \ldots, T_p$ respectively. Then the $[p-1]$ determined by these p points of contact must be a secant $[p-1]$, and each of the points of contact

corresponds, in the correspondence we are now considering, to this secant $[p-1]$. Since there are $2n+2$ sets of n concurrent tangents of ϑ, each set giving rise to $\binom{n}{p}$ sets of p tangents, it follows that the number of times a point of ϑ lies in a corresponding secant $[p-1]$ is $2(n+1)\binom{n}{p}p$. We may obtain this number either by considering $2n+2$ sets of $\binom{n}{p}$ secant $[p-1]$'s and counting each secant $[p-1]$ p times as containing p corresponding points, or we may obtain it by considering $2n+2$ sets of n points and counting each point $\binom{n-1}{p-1}$ times as lying on this number of corresponding secant $[p-1]$'s. The secant $[p-1]$'s being generating spaces of R_p we have the difference equation

$$(p+1)(n-p+1)N_p+2\binom{n}{p}\cdot\tfrac{1}{2}n(n+1)-2(n+1)\binom{n}{p}p=(p+1)(p+2)N_{p+1},$$

and it can be verified immediately that it is satisfied if

$$N_p=p\binom{n+1}{p+1}.$$

It therefore follows, by induction, that this is actually the order of the scroll R_p. In particular the order of the primal R_{n-1} is n^2-1.

14. The result that has just been obtained for the order of the scroll R_p affords another means of obtaining the class of the developable D_{s+1}, thus corroborating the value already found for d_{s+1}. We have seen that a generating prime of D_{s+1} is defined as the join of a generating $[s]$ of R_{s+1} to a generating $[n-s-2]$ of R_{n-s-1}; since the $[s]$ is common to $n-s$ of the faces of an $(n+2)$-hedron, and the $[n-s-2]$ is common to the remaining $s+2$ faces, the primes of the developable D_{s+1} join corresponding spaces in a (1, 1) correspondence between the generating spaces of R_{s+1} and those of R_{n-s-1}. Moreover no pair of corresponding spaces can intersect, for any point common to two corresponding spaces would be common to all the faces of an $(n+2)$-hedron, and there cannot be more than n osculating primes of C passing through the same point. Hence we have the formula[1]

$$d_{s+1}=N_{s+1}+N_{n-s-1},$$

[1] Edge: *Proc. London Math. Soc.* (2), 33 (1932), 53-54.

giving

$$d_{s+1} = (s+1)\binom{n+1}{s+2} + (n-s-1)\binom{n+1}{n-s} = \binom{n+1}{s+1}\frac{n(2s+3)-2(s+1)^2}{s+2},$$

as before. There is again the exception when $s+1 = n-s-1$; *i.e.* when n is even and $s = \frac{1}{2}n - 1$. In this case d_{s+1} is generated by joining generating spaces of N_{s+1} which correspond to one another in a *symmetrical* (1, 1) correspondence, so that the value of d_{s+1} given by the formula must be halved in this particular case.

The genera of the scrolls and developables.

15. We can obtain, by means of Zeuthen's correspondence formula[1], a relation connecting the genera π_p and π_{p+1} of the scrolls R_p and R_{p+1}, and hence, since we know the genus π_1 of ϑ, we can obtain π_p. Since there is a (1, 1) correspondence between the generating spaces of R_{p+1} and those of R_{n-p-1}, π_{p+1} and π_{n-p-1} must be equal to one another, and also to the genus of the developable D_{p+1}.

Let us set up a correspondence between the points of ϑ and its trisecants—the generators of R_2; a point and a trisecant of ϑ are to correspond to one another when the point lies on the trisecant. Then to any point of ϑ there correspond n trisecants. Suppose now that O is a point of ϑ such that two of the n trisecants which pass through it coincide with the trisecant OT_1T_2. The plane containing any pair of intersecting trisecants of ϑ contains also another pair, and meets ϑ in the six vertices of the quadrilateral formed by the four trisecants; in the limiting case of the two intersecting trisecants coinciding with OT_1T_2 the other two trisecants which lie in the plane must be the tangents of ϑ at T_1 and T_2, and these two tangents therefore intersect in a point of ϑ. Conversely: if the tangents of ϑ at two points T_1 and T_2 intersect in a point of ϑ, the plane containing them must be a secant plane of ϑ; the line T_1T_2 meets ϑ again in a point O, and two of the n trisecants through O coincide with OT_1T_2.

[1] Zeuthen: *Lehrbuch der abzählenden Methoden der Geometrie* (1914), 104-107; or *Math. Annalen* 3 (1871), 150. See also Baker: *Principles of Geometry* 6 (Cambridge 1933), 19. The formula for an (α, α') correspondence between two curves of genera p and p' is

$$\eta - \eta' = 2\alpha(p'-1) - 2\alpha'(p-1),$$

where η, η' are the numbers of branch points of the correspondence on the two curves. The formula applies not merely to curves, but to any loci generated by singly-infinite sets of elements.

We have seen however that trisecants which are also tangents of ϑ are distributed in $2n+2$ concurrent sets of n; hence the number of points O on ϑ which are such that two of their n corresponding trisecants coincide is $(2n+2)\binom{n}{2}$. Also, to any trisecant there correspond three points of ϑ, and there are $n(2n+2)$ trisecants for which two of these three points coincide. Hence, if π_2 is the genus of R_2, Zeuthen's formula gives, since the genus of ϑ is $\frac{1}{2}n(n-1)$,

$$(2n+2)\binom{n}{2} - n(2n+2) = 6(\pi_2 - 1) - 2n\{\tfrac{1}{2}n(n-1) - 1\},$$

so that
$$\pi_2 = 1 + \tfrac{1}{6}n(n+1)(2n-5).$$

16. Having obtained the genus of R_2 we can proceed to obtain the genus of R_3. We set up a correspondence between the trisecants and secant planes of ϑ, the trisecants being the generators of R_2 and the secant planes the generating planes of R_3; a trisecant and a secant plane correspond when the trisecant lies in the secant plane. To a given trisecant there correspond $n-1$ secant planes. If t is a trisecant of ϑ for which two of the $n-1$ corresponding secant planes coincide with a plane through t which meets ϑ again in T_1, T_2, T_3, then the tangents of ϑ at T_1, T_2, T_3 must be concurrent in a point of ϑ; it follows that the number of trisecants of ϑ for which two of the $n-1$ corresponding secant planes coincide is $(2n+2)\binom{n}{3}$. Also, to a given secant plane there correspond four trisecants. If the plane is such that two of these four trisecants coincide then it must be a bitangent plane of ϑ, the other two trisecants in the plane, apart from the two which coincide, being tangents which meet in a point of ϑ. The number of such bitangent planes is $(2n+2)\binom{n}{2}$. If we now apply Zeuthen's formula to this correspondence between trisecants and secant planes of ϑ we obtain

$$(2n+2)\binom{n}{3} - (2n+2)\binom{n}{2} = 8(\pi_3 - 1) - 2(n-1)(\pi_2 - 1).$$

This equation gives, after substituting its value for π_2,

$$\pi_3 = 1 + \frac{1}{24}(n-1)n(n+1)(3n-10).$$

It is now seen that for $p = 1, 2, 3$ we have

$$\pi_p = 1 + \binom{n+2}{p+1}\frac{pn - p^2 - 1}{n+2}.$$

It may therefore be suspected that this formula holds for all values of p, and this suspicion is strengthened when it is observed that this value for π_p fulfils the condition $\pi_{p+1} = \pi_{n-p-1}$. To make the suspicion into a certainty we obtain the relation between π_p and π_{p+1} by considering a correspondence between the scrolls R_p and R_{p+1}, the former scroll being generated by the secant $[p-1]$'s of ϑ and the latter by the secant $[p]$'s of ϑ.

Suppose then that a secant $[p-1]$ and a secant $[p]$ of ϑ correspond to one another when the secant $[p-1]$ lies in the secant $[p]$; then to a given secant $[p-1]$ there correspond $n-p+1$ secant $[p]$'s; while to a given secant $[p]$ there correspond $p+2$ secant $[p-1]$'s. Also, by using arguments similar to those above, it is found that there are $(2n+2)\binom{n}{p+1}$ secant $[p-1]$'s of ϑ for which two of the corresponding secant $[p]$'s coincide, while there are $(2n+2)\binom{n}{p}$ secant $[p]$'s of ϑ for which two of the corresponding secant $[p-1]$'s coincide. Zeuthen's formula therefore gives

$$(2n+2)\binom{n}{p+1}-(2n+2)\binom{n}{p} = 2(p+2)(\pi_{p+1}-1) - 2(n-p+1)(\pi_p-1),$$

and it is easily verified that this difference equation is satisfied by the above formula for π_p.

It has been stated that the genus of the developable D_p is the same as that of the scrolls R_p and R_{n-p}, this being so because there is a (1, 1) correspondence between the primes of D_p and the generating $[p-1]$'s of R_p. There is however an exception to this statement, for if n is even and equal to $2p$ there is not a (1, 1) correspondence but a (1, 2) correspondence between the primes of D_p and the generating spaces of R_p. The genus of R_p is still that given by the formula for π_p; that of D_p can be deduced at once by Zeuthen's formula. If ϖ is the genus of D_p the formula gives

$$4(\varpi - 1) = 2(\pi_p - 1),$$

for it is not possible for the two generating spaces of R_p that correspond to a prime of D_p ever to coincide, so that $\eta = \eta' = 0$. Hence, when $p = \frac{1}{2}n$,

$$\varpi - 1 = \tfrac{1}{2}(\pi_p - 1) = \tfrac{1}{2}\binom{2p+2}{p+1}\frac{p^2-1}{2p+2} = \tfrac{1}{2}(p-1)\binom{2p+1}{p}.$$

The cases $n = 3, 4, 5$.

17. The results that we have obtained include, in particular, the following.

If we take a g_5^1 on a twisted cubic in [3] the osculating planes of the curve at the five points of a set of the g_5^1 form a pentahedron, the locus of whose vertices is a curve ϑ of order 6 and genus 3. The curve ϑ has a scroll R_2^8 of trisecants on which it is a triple curve, while the planes which join the points of ϑ to their conjugate trisecants form a developable of class 14. This developable and the scroll R_2^8 are also of genus 3. There are 24 generators of R_2^8 which touch ϑ, and these are distributed in eight concurrent sets of three.

A g_6^1 on a rational normal quartic curve in [4] gives rise to an infinity of hexahedra the locus of whose vertices is a curve ϑ of order 10 and genus 6. The trisecants of ϑ generate a scroll R_2^{20} on which ϑ is a quadruple curve, while the secant planes of ϑ, each of which meets ϑ in six points, generate a scroll R_3^{15} on which R_2^{20} is a triple surface and ϑ a sextuple curve. The primes which join the points of ϑ to their conjugate secant planes give a developable D_1 of class 25, while the primes which join conjugate pairs of generators of R_2^{20} give a developable D_2 of class 20. The scroll R_3^{15} and the developable D_1 are, like ϑ, of genus 6; the developable D_2 is also of genus 6, while R_2^{20} is of genus 11. There are 40 generators of R_2^{20} which touch ϑ, and these are distributed in ten concurrent sets of four.

A g_7^1 on a rational normal quintic curve in [5] gives rise to a curve ϑ of order 15 and genus 10. The trisecants of ϑ generate a scroll R_2^{40} on which ϑ is a quintuple curve; there are 60 generators of R_2^{40} which touch ϑ, these being distributed in twelve concurrent sets of five. The secant planes of ϑ generate a scroll R_3^{45} on which ϑ is of multiplicity 10 and R_2^{40} of multiplicity 4. The secant solids of ϑ generate a scroll R_4^{24} on which ϑ, R_2^{40} and R_3^{45} are of respective multiplicities 10, 6 and 3. The primes which join conjugate points and secant solids of ϑ give a developable D_1 of class 39, while the primes which join conjugate trisecants and secant planes of ϑ give a developable D_2 of class 85. The scroll R_4^{24} and the developable D_1 are, like ϑ, of genus 10, while the scrolls R_2^{40} and R_3^{45}, and the developable D_2, are all of genus 26.