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A family of non-invertible prime links

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The purpose of this paper is to show that there exists an infinite family of non-invertible prime 2-component links in S^3. It is then noted that the existence of such a family implies the existence of a family of non-invertible wild arcs in S^3, which are tame modulo an endpoint.

All our "knots" and "links" are oriented knots and links in S^3, which has its orientation fixed.

Let $l_1 \cup l_2$ be a non-splittable 2-component link. By the main theorem of [1], $l_1 \cup l_2$ has a unique factorisation into prime links, and by Theorem 1 of the same paper, one and only one of the prime links involved is a 2-component link $l_1^* \cup l_2^*$. (The other links are all 1-component links, that is, knots.) We call $l_1^* \cup l_2^*$ the *prime hub* of $l_1 \cup l_2$. Then we note the following

(a) $l_1 \cup l_2$ is F-isotopic (*cf.* [4]) to its prime hub, and

(b) $\lambda(l_1^*, l_2^*) = \lambda(l_1, l_2)$, where $\lambda(l_1, l_2)$ denotes the linking number of l_1 with l_2.

Let k_1 be one of the non-invertible pretzel knots of Trotter [5], and let k_2 be an unknotted simple closed curve in $S^3 - k_1$ which has linking number $+1$ with k_1. Then $k_1 \cup k_2$ is a non-invertible link whose prime hub consists of two simply linked circles, and is therefore invertible.

The situation is not lost, however; it is still possible to construct non-invertible prime links from our "basic link" $k_1 \cup k_2$. We proceed as follows.

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Let V be a closed regular neighbourhood of k_1 in $S^3 - k_2$, whose orientation is chosen so that $\lambda(V, k_2) = +1$. Let $k_1(r)$ be a knot lying in the interior of V with winding number $r > 1$; that is, a meridian of $\mathrm{Bd}V$ has linking number r with $k_1(r)$. Denote by L_r the non-splittable link $k_1(r) \cup k_2$. Because $\lambda(k_1(r), k_2) = r$, L_r and L_s represent different F-isotopy classes if $r \neq s$. The prime hubs H_r and H_s of L_r and L_s are therefore distinct if $r \neq s$.

We will show that L_r is non-invertible. Indeed, we will show that L_r and $L_r^{-1} = k_1(r)^{-1} \cup k_2^{-1} = k_1(-r) \cup k_2^{-1}$ are not even F-isotopic.

If there is an F-isotopy from L_r to L_r^{-1}, there exists a closed solid torus $V' \subset S^3 - k_2^{-1}$, and an orientation-preserving homeomorphism $h : (S^3, V) \to (S^3, V')$ such that

(i) $k_1(r)^{-1}$ has linking number r with a meridian of $\mathrm{Bd}V'$, and

(ii) k_2^{-1} has linking number $+1$ with a core of V'. (*cf.* [4], Theorem 3.)

Now V and V' have the same oriented "knot type" in S^3, that is their oriented longitudes are equivalently knotted. But statement (ii) implies that h takes a longitude of V to the inverse of a longitude of V', so that a longitude of V represents an invertible knot type in S^3. Since an oriented longitude of V has the same (oriented) type as the pretzel knot k_1, this is impossible. Thus L_r is non-invertible.

If the prime hub $H_r = k_1(r)^* \cup k_2^*$ is non-invertible, the family $\{H_r : r > 1\}$ will satisfy our requirements.

Since L_r and H_r are F-isotopic, and $k_1(r)$ has winding number $r > 1$ in V, we apply Theorem 3 of [4] to obtain an oriented closed solid torus $V^* \subset S^3 - k_2^*$, which has linking number $+1$ with $k_2^* = k_2$, which contains $k_1(r)^*$ in its interior with winding number r, and which is of the same oriented knot type as V . Replacing $k_1(r)$ by $k_1(r)^*$ and V by V^* in the argument above, we see that H_r and H_r^{-1} are not F-isotopic, that is H_r is non-invertible. This proves the result stated in the abstract.

APPLICATION. For $n > 1$, it is known that there exists an uncountable family A_n of non-invertible wild arcs which are tame modulo one endpoint, where they have penetration index $2n + 1$, (see [2], p. 91).

We can now complete this result by stating the following theorem for $n = 1$. The proof will appear in [3].

THEOREM. *There exists an uncountable family* A_1 *of non-invertible arcs which are tame modulo one endpoint, at which they have penetration index three.*

QUESTION. It has been shown by Whitten [6] that there exist non-invertible two-component links in S^3 which have invertible components. Does there exist an infinite family of prime links with this property?

References

[1] Yoko Hashizume, "On the uniqueness of the decomposition of a link", *Osaka Math. J.* 10 (1958), 283-300.

[2] James M. McPherson, "Wild knots and arcs in a 3-manifold", Ph.D. thesis at the University of New South Wales, Kensington, 1970.

[3] James M. McPherson, "Wild arcs in 3-manifolds, I: Families of Fox-Artin arcs", submitted.

[4] N. Smythe, "Topological invariants of isotopy of links, I", *Amer. J. Math.* 92 (1970), 86-98.

[5] H.F. Trotter, "Non-invertible knots exist", *Topology* 2 (1964), 275-280.

[6] W.C. Whitten, Jr, "A pair of non-invertible links", *Duke Math. J.* 36 (1969), 695-698.

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