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A MAXIMUM PRINCIPLE FOR DIRICHLET-FINITE HARMONIC FUNCTIONS ON RIEMANNIAN SPACES

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Representations of harmonic functions by means of integrals taken over the harmonic boundary Δ_R of a Riemann surface R enable one to study the classification theory of Riemann surfaces in terms of topological properties of Δ_R (cf. [**6**; **4**; **1**; **7**]). In deducing such integral representations, essential use is made of the fact that the functions in question attain their maxima and minima on Δ_R.

The corresponding maximum principle in higher dimensions was discussed for bounded harmonic functions in [**3**]. In the present paper we consider Dirichlet-finite harmonic functions. We shall show that every such function on a subregion G of a Riemannian N-space R attains its maximum and minimum on the set $(\bar{G} \cap \Delta_R) \cup \overline{\partial G}$, where ∂G is the relative boundary of G in R and the closures are taken in Royden's compactification R^*. As an application we obtain the harmonic decomposition theorem relative to a compact subset K of R^* with a smooth $\partial(K \cap R)$.

We start by stating in § 1 some preliminary results, using the notation and terminology of [**3**]. In § 2 we prove a topological correspondence of Royden's compactification G^* of a subregion G and its closure $\bar{G}$ in R^*. The maximum principle for Dirichlet-finite harmonic functions and the harmonic decomposition theorem are established in § 3.

1. Given a Riemannian N-space R, Royden's algebra $\mathbf{M}(R)$ consists of bounded real-valued continuous functions on R with finite Dirichlet integrals over R. Royden's compactification R^* of R is defined by the following properties:

(i) R^* is a compact Hausdorff space,

(ii) R is an open dense subspace of R^*,

(iii) every function in $\mathbf{M}(R)$ has a continuous extension to R^*,

(iv) $\mathbf{M}(R)$ separates closed sets in R^*.

The vector lattice $\tilde{\mathbf{M}}(R)$ of Dirichlet-finite real-valued continuous functions on R is complete in the CD-topology: if $f = \text{CD-lim}_n f_n$ on R for $f_n \in \tilde{\mathbf{M}}(R)$, i.e., $D_R(f - f_n) \to 0$ as $n \to \infty$ and $\{f_n\}$ converges to f uniformly on compact subsets of R, then $f \in \tilde{\mathbf{M}}(R)$. If we further have uniform boundedness of

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$\{f_n\}$ on R we write $f = \text{BD-lim}_n f_n$ on R. Clearly $\mathbf{M}(R)$ is BD-complete. For a detailed discussion we refer the reader to [**1**; **7**].

Let $\mathbf{M}_0(R)$ be the algebra of functions in $\mathbf{M}(R)$ with compact supports in R and $\mathbf{M}_\Delta(R)$ the BD-closure of $\mathbf{M}_0(R)$ in $\mathbf{M}(R)$. The harmonic boundary $\Delta_R = \{p \in R^* | f(p) = 0 \text{ for all } f \in \mathbf{M}_\Delta(R)\}$ is a compact subset of the Royden boundary $\Gamma_R = R^* - R$. The CD-closure $\tilde{\mathbf{M}}_\Delta(R)$ of $\mathbf{M}_0(R)$ in $\tilde{\mathbf{M}}(R)$ also plays an important role in our discussion.

The following theorem was proved in [**3**]. *For any non-empty compact subset E of $\Gamma_R - \Delta_R$, there exists an Evans superharmonic function, i.e., a positive continuous function v on R^*, superharmonic on R, such that $v \equiv 0$ on Δ_R, $v \equiv \infty$ on E, and v has a finite Dirichlet integral on R.*

2. Let G be a subregion of a given Riemannian N-space R. We can construct two compactifications of G, viz., Royden's compactification G^* of G and the closure $\bar{G}$ of G in R^*. First we shall show that there is a topological relation between them (cf. [**7**]).

PROPOSITION 1. *There exists a unique continuous mapping η of G^* onto $\bar{G}$ such that*

(i) *$\eta(p) = p$ for $p \in G$,*

(ii) *$f(p^*) = f(\eta(p^*))$ for $p^* \in G^*$ and $f \in \mathbf{M}(R)$.*

Proof. Observe that $f|_G$ belongs to $\mathbf{M}(G)$ for every $f \in \mathbf{M}(R)$ and so $f|_G$ has a continuous extension to G^*.

For each $p^* \in G^*$ define a character x_{p^*} on $\mathbf{M}(G)$ by $x_{p^*}(g) = g(p^*)$ for all $g \in \mathbf{M}(G)$. We can consider x_{p^*} as a character on $\mathbf{M}(R)$ by the above observation. We shall first show that there exists a unique point $\eta(p^*) \in \bar{G}$ such that $x_{p^*}(f) = f(\eta(p^*))$ for all $f \in \mathbf{M}(R)$. Since $\mathbf{M}(R)$ separates points in R^*, the uniqueness of such an $\eta(p^*)$ is obvious.

Let $I = \{f \in \mathbf{M}(R) | x_{p^*}(f) = 0\}$. It is easy to see that I is a non-trivial maximal ideal of the algebra $\mathbf{M}(R)$. Suppose that there exists an $f_{\bar{p}} \in I$ for each $\bar{p} \in \bar{G}$ such that $f_{\bar{p}}(\bar{p}) \neq 0$. On squaring and then multiplying by a constant we may assume that $f_{\bar{p}} \geqq 0$ on R and $f_{\bar{p}}(\bar{p}) > 1$. Since $\bar{G}$ is compact, there exists a finite subset $\{\bar{p}_1, \ldots, \bar{p}_n\}$ of $\bar{G}$ such that

$$f = \sum_{i=1}^{n} f_{\bar{p}_i} > 1$$

on $\bar{G}$. Define $\bar{f}$ on R^* by $\bar{f}(p) = f(p)$ for $f(p) > 1$ and $\bar{f}(p) = 1$ for $f(p) \leqq 1$. Clearly $\bar{f} \equiv f$ on G and $\bar{f} \in \mathbf{M}(R)$. Hence $x_{p^*}(\bar{f}) = x_{p^*}(f) = 0$ since $f \in I$ and $1 = \bar{f} \cdot (1/\bar{f}) \in I$, which violates the maximality of I. We have shown that there exists a unique point $\eta(p^*) \in \bar{G}$ such that $f(\eta(p^*)) = 0$ for all $f \in I$. For any $f \in \mathbf{M}(R)$, $f - f(p^*) \in I$ and $f(\eta(p^*)) = f(p^*)$.

We can define a mapping η: $G^* \to \bar{G}$ such that

$$f(p^*) = f(\eta(p^*))$$

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for all $p^* \in G^*$ and $f \in \mathbf{M}(R)$. Since $\mathbf{M}(R)$ separates points in R^*, $\eta(p) = p$ for $p \in G$. To prove the continuity of η choose an arbitrary net

$$\{p_\lambda^* \in G^* |\ \lambda \in \Lambda \text{ and } \Lambda \text{ is a directed set}\}$$

which converges to p^* in G^*. Since every $f \in \mathbf{M}(R)$ can be considered as a continuous function on G^*, the net $\{f(p_\lambda^*) |\ \lambda \in \Lambda\}$ converges to $f(p^*)$. Since $f(p_\lambda^*) = f(\eta(p_\lambda^*))$ and $f(p^*) = f(\eta(p^*))$, the net $\{f(\eta(p_\lambda^*)) |\ \lambda \in \Lambda\}$ converges to $f(\eta(p^*))$ for all $f \in \mathbf{M}(R)$, and $\eta(p^*)$ is a cluster point of the net $\{\eta(p_\lambda^*) |\ \lambda \in \Lambda\}$ in $\bar{G}$ in view of the Urysohn property of $\mathbf{M}(R)$. Since $\bar{G}$ is compact, it suffices to show that $\eta(p^*)$ is the unique cluster point in $\bar{G}$. On the contrary, suppose that there exists a subnet $\{\eta(p_{\lambda_i}^*)\}$ which converges to $\bar{p}$ in $\bar{G}$ with $\bar{p} \neq \eta(p^*)$. Choose an $f \in \mathbf{M}(R)$ such that $f(\bar{p}) \neq f(\eta(p^*))$. On the other hand,

$$f(\bar{p}) = \lim_{\lambda_i} f(\eta(p_{\lambda_i}^*)) = \lim_{\lambda} f(\eta(p_\lambda^*)) = f(\eta(p^*)),$$

a contradiction.

It remains to show that η is surjective. Let $\bar{p}$ be an arbitrary point in $\bar{G}$. Since G is dense in $\bar{G}$, there exists a net $\{\bar{p}_\lambda |\ \lambda \in \Lambda\}$ in G which converges to $\bar{p}$ in $\bar{G}$. Since $\bar{p}_\lambda \in G \subset G^*$ and G^* is compact, we may assume that the net $\{\bar{p}_\lambda |\ \lambda \in \Lambda\}$ converges to a point p^* in G^*. For every $f \in \mathbf{M}(R)$,

$$f(\bar{p}) = \lim_\lambda f(\bar{p}_\lambda) = f(p^*)$$

and so $\bar{p} = \eta(p^*)$.

In general, the projection $\eta: G^* \to \bar{G}$ is not a homeomorphism but its restriction to a certain subset of G^* yields a homeomorphism. This result is essential for the proof of the maximum principle for Dirichlet-finite harmonic functions.

We are ready to show the following result (cf. [**7**]).

PROPOSITION 2. *Let* $\beta(G) = (\bar{G} - \overline{\partial G}) \cap \Gamma_R$. *Then the projection*

$$\eta: \{p^* \in G^* |\ \eta(p^*) \in G \cup \beta(G)\} \to G \cup \beta(G)$$

is a surjective homeomorphism.

Proof. In view of the previous proposition, all we have to verify is that η is injective and η^{-1} is continuous.

First we shall show that η is injective. Suppose that there existed two points p_1^*, p_2^* in G^* such that $\eta(p_1^*) = \eta(p_2^*) = \bar{p} \in G \cup \beta(G)$. Choose a $g \in \mathbf{M}(G)$ such that $g(p_1^*) \neq g(p_2^*)$. Since $\{\bar{p}\}$ and $\overline{(R - G)}$ are disjoint closed subsets of R^*, there exists a function $f \in \mathbf{M}(R)$ such that $f(\bar{p}) = 1$ and $f \equiv 0$ on $\overline{(R - G)}$. Clearly $fg \in \mathbf{M}(G)$. Since $f \equiv 0$ on $R - G$, we can consider fg as an element of $\mathbf{M}(R)$. By Proposition 1,

$$(fg)(\bar{p}) = (fg)(\eta(p_i^*)) = (fg)(p_i^*) = f(p_i^*)g(p_i^*) = f(\bar{p})g(p_i^*) = g(p_i^*), \quad i = 1, 2.$$

This contradicts the choice of g. Thus η is injective.

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To prove the continuity of η^{-1}, take a net $\{p_\lambda | \lambda \in \Lambda\}$ in $G \cup \beta(G)$ which converges to a point p in $G \cup \beta(G)$. Since G^* is compact, the net $\{\eta^{-1}(p_\lambda) | \lambda \in \Lambda\}$ has a cluster point in G^*, i.e., there exist a point $p^* \in G^*$ and a subnet $\{\eta^{-1}(p_{\lambda_i})\}$ of the net $\{\eta^{-1}(p_\lambda) | \lambda \in \Lambda\}$ which converges to p^* in G^*. By the continuity of η: $G^* \to \bar{G}$, the net $\{p_{\lambda_i}\}$ converges to $\eta(p^*)$. Hence $p = \eta(p^*)$ since $\bar{G}$ is a Hausdorff space. Thus $\eta^{-1}(p)$ is a cluster point of the net $\{\eta^{-1}(p_\lambda) | \lambda \in \Lambda\}$. As in the proof of the previous proposition, it suffices to show that $\eta^{-1}(p)$ is the only cluster point in G^*. Suppose that there existed another cluster point $q^* \in G^*$ and a subnet $\{\eta^{-1}(p_{\lambda_j})\}$ of the net

$$\{\eta^{-1}(p_\lambda) | \lambda \in \Lambda\}$$

such that $\{\eta^{-1}(p_{\lambda_j})\}$ converges to q^* in G^*. For every $f \in \mathbf{M}(R)$,

$$f(q^*) = \lim_j f(\eta^{-1}(p_{\lambda_j})) = \lim_j f(p_{\lambda_j}) = f(p)$$

and similarly $f(p^*) = f(p)$. Thus we have

$$f(p^*) = f(q^*) \qquad \text{or equivalently} \qquad f(\eta(p^*)) = f(\eta(q^*))$$

for all $f \in \mathbf{M}(R)$. Hence $\eta(q^*) = \eta(p^*) = p \in G \cup \beta(G)$ and $q^* = p^*$ since η^{-1} is well-defined on $G \cup \beta(G)$.

The proof of the proposition is herewith complete.

COROLLARY. *Every $f \in \mathbf{M}(G)$ has a continuous extension to $G \cup \beta(G)$.*

3. Let $\mathcal{O}_G$ be the class of Riemannian spaces on which there exist no Green's functions. It is known that the class $\mathrm{HD}(R)$ of Dirichlet-finite harmonic functions on R consists of constants for $R \in \mathcal{O}_G$ (cf. [**8**]). Throughout our discussion we understand that $\mathrm{HD}(R) = \{0\}$ for $R \in \mathcal{O}_G$. Thus the class $\mathrm{HBD}(R) = \{u \in \mathrm{HD}(R) | \sup_R |u| < \infty\}$ is identical with $\mathrm{HD}(R)$ for $R \in \mathcal{O}_G$. Our next question is: How many HBD-functions are there in the space $\mathrm{HD}(R)$ for an arbitrary Riemannian space R?

First we prove the following result.

LEMMA. *Every $f \in \tilde{\mathbf{M}}(R)$ has a unique decomposition in the form*

$$f = u + g,$$

where $u \in \mathrm{HD}(R)$ and $g \in \tilde{\mathbf{M}}_\Delta(R)$. In particular, u can be chosen as the CD-*limit of a sequence in the space* $\mathrm{HBD}(R)$.

Proof. By our convention $\mathrm{HD}(R) = \{0\}$ for $R \in \mathcal{O}_G$ it suffices to prove the assertion for $R \notin \mathcal{O}_G$.

First we assume that $f \geqq 0$ on R. For each $n \geqq 1$ set $f_n = f \cap n \in \mathbf{M}(R)$. Let $\{R_m\}_0^\infty$ be a regular exhaustion of R such that R_0 and R_1 are parametric balls at a fixed point $p_0 \in R$.

Since $f_n \in \mathbf{M}(R)$, it has the unique decomposition

$$f_n = u_n + g_n,$$

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where $u_n \in \mathrm{HBD}(R)$ and $g_n \in \mathbf{M}_\Delta(R)$. Here $u_n = \text{BD-lim}_m u_{nm}$ on R, with $u_{nm} \in \mathbf{M}(R)$, $u_{nm} \in \mathrm{H}(R_m)$, and $u_{nm} = f_n$ on $R - R_m$ (cf. [**3**]).

Let w_m be the harmonic measure of ∂R_m with respect to $R_m - \bar{R}_1$, i.e., $w_m \equiv 1$ on $\bar{R}_1$, $w_m \in \mathrm{H}(R_m - \bar{R}_1)$, and $w_m \equiv 0$ on $R - R_m$. By Green's formula,

$$\begin{aligned} D_R(f_n - u_{nm}, w_m) &= D_{R_m - R_1}(f_n - u_{nm}, w_m) \\ &= \int_{\partial(R_m - R_1)} (f_n - u_{nm}) *dw_m \\ &= \int_{\partial R_1} u_{nm} *dw_m - \int_{\partial R_1} f_n *dw_m. \end{aligned}$$

Hence in view of $*dw_m \leqq 0$ on ∂R_1 and $u_{nm} \geqq 0$ on R, we have

$$\begin{aligned} \left(\inf_{\partial R_1} u_{nm}\right) \cdot D_R(w_m) &= \left(\inf_{\partial R_1} u_{nm}\right) \cdot \left(- \int_{\partial R_1} *dw_m\right) \\ &\leqq - \int_{\partial R_1} u_{nm} *dw_m \\ &\leqq - \int_{\partial R_1} f_n *dw_m + |D_R(f_n - u_{nm}, w_m)| \\ &\leqq \left(\sup_{\partial R_1} f_n\right) \cdot D_R(w_m) + 2D_R(f_n)^{\frac{1}{2}} D_R(w_m)^{\frac{1}{2}}. \end{aligned}$$

On the other hand, $w = \text{BD-lim}_m w_m$ exists on R and $D_R(w_m) \geqq D_R(w) > 0$ since $R \notin \mathcal{O}_G$. Thus we obtain

$$\begin{aligned} u_{nm}(p_0) &\leqq k \inf_{\partial R_0} u_{nm} \leqq k \inf_{\partial R_1} u_{nm} \\ &\leqq k \left\{\sup_{\partial R_1} f_n + 2D_R(f_n)^{\frac{1}{2}} \cdot D_R(w_m)^{-\frac{1}{2}}\right\} \leqq k \left\{\sup_{\partial R_1} f + 2D_R(f)^{\frac{1}{2}} \cdot D_R(w)^{-\frac{1}{2}}\right\} \end{aligned}$$

for all m and $n \geqq 1$, where $k = k(\bar{R}_0, R_1)$ is Harnack's constant for R_1. Since $u_n = \text{BD-lim}_m u_{nm}$ on R, the sequence $\{u_n(p_0)\}$ is bounded. On taking a subsequence if necessary we may assume that $\{u_n(p_0)\}$ is convergent. Since $(f_{n+p} - f_n) = (u_{n+p} - u_n) + (g_{n+p} - g_n)$ is the decomposition of $f_{n+p} - f_n$ in a lemma in [**3**, § 2, Lemma] we have

$$D_R(f_{n+p} - f_n) = D_R(u_{n+p} - u_n) + D_R(g_{n+p} - g_n).$$

Because of $\lim_n D_R(f_{n+p} - f_n) = 0$, the sequences $\{u_n\}$ and $\{g_n\}$ are D-Cauchy on R. Thus by the convergence theorem in [**8**, p. 128],

$$u = \text{CD-lim}_n u_n$$

exists on R and $u \in \mathrm{HD}(R)$. Since $f = \text{CD-lim}_n f_n$ on R, $g = \text{CD-lim}_n g_n$ exists on R and $g \in \tilde{\mathbf{M}}_\Delta(R)$ in view of the CD-completeness of $\tilde{\mathbf{M}}_\Delta(R)$.

For an arbitrary $f \in \tilde{\mathbf{M}}(R)$ we can construct decompositions of $f \cup 0$ and $-f \cap 0$ separately and combine them to obtain

$$f = u + g,$$

where $g \in \tilde{\mathbf{M}}_\Delta(R)$ and $u \in \mathrm{HD}(R)$ is the CD-limit of a sequence in the space $\mathrm{HBD}(R)$.

To prove uniqueness let $f = u' + g'$ be another decomposition. Then $v \equiv u - u' = g' - g \in \mathrm{HD}(R) \cap \tilde{\mathbf{M}}_\Delta(R)$. Choose a sequence $\{v_m\}$ in $\mathbf{M}_0(R)$ such that $v = \text{CD-lim}_m v_n$ on R. Then $D_R(v, v_m) = 0$ by Green's formula and v is a constant on R. Since $v \equiv 0$ on Δ_R, $v \equiv 0$ on R, as desired.

Using the above lemma we shall prove the following result (cf. [**5**; **4**]).

PROPOSITION 3. *For an arbitrary Riemannian space R, the space* $\mathrm{HBD}(R)$ *is* CD-*dense in* $\mathrm{HD}(R)$.

Proof. As we remarked earlier, we may assume that $R \notin \mathcal{O}_G$. By virtue of $\mathrm{HD}(R) \subset \tilde{\mathbf{M}}(R)$, every $u \in \mathrm{HD}(R)$ has a unique decomposition by the above lemma. Since $u = u + 0$ is such a decomposition, u is the CD-limit of a sequence in $\mathrm{HBD}(R)$. This completes the proof.

As a direct consequence we have the following result (cf. [**5**]).

COROLLARY. *The Virtanen identity*

$$\mathcal{O}_{\mathrm{HD}} = \mathcal{O}_{\mathrm{HBD}}$$

is valid for Riemannian spaces.

We are now ready to establish the maximum principle for HD-functions. It is one of the most important theorems in the study of HD-functions. In the case of a Riemann surface, the proof offers no difficulties since the double of a subregion can be used (cf. [**7**]).

THEOREM 1. *Let G be a subregion of an arbitrary Riemannian space R. If $u \in \mathrm{HD}(G)$ has the property*

$$m \leqq \liminf_{p \in G, p \to q} u(p) \leqq \limsup_{p \in G, p \to q} u(p) \leqq M$$

for all $q \in (\bar{G} \cap \Delta_R) \cup \overline{\partial G}$, then

$$m \leqq u \leqq M$$

throughout the subregion G.

Proof. It suffices to show that $u \geqq m$ on G whenever

$$\liminf_{p \in G, p \to q} u(p) \geqq m$$

for all $q \in (\bar{G} \cap \Delta_R) \cup \overline{\partial G}$. We may assume that $m > -\infty$. Observe that every $g \in \tilde{\mathbf{M}}(G)$ has a continuous extension to G^* and therefore to $G \cup \beta(G)$ by the corollary in § 2.

Set

$$E_n = \left\{ q \in \bar{G} - G \;\middle|\; \liminf_{p \in G, p \to q} u(p) \leqq m - \frac{1}{n} \right\}$$

for all $n \geqq 1$. It is easily seen that E_n is a closed set in $\Gamma_R - \Delta_R$. Let v_n be the Evans superharmonic function on R such that $v_n \equiv \infty$ on E_n and $v_n \equiv 0$ on Δ_R. For each $\epsilon > 0$ we have

$$\liminf_{p \in G, p \to q} (u + \epsilon v_n)(p) > m - \frac{1}{n}$$

for all $q \in \bar{G} - G - E_n$ since $\epsilon v_n > 0$ on R.

By the above theorem there exists a sequence $\{u_n\}$ in HBD(G) such that $u = \text{CD-lim}_n\, u_n$ on G. Since $u + \epsilon v_n = \text{CD-lim}_k(u_k + \epsilon v_n)$ on G and these functions are continuously extendable to $G \cup \beta(G)$,

$$(u + \epsilon v_n)(q) = \lim_k (u_k + \epsilon v_n)(q)$$

for all $q \in E_n \subset \beta(G)$. Since u_k is bounded on G and $v_n \equiv \infty$ on E_n, we have

$$\liminf_{p \in G, p \to q} (u_k + \epsilon v_n)(p) = (u_k + \epsilon v_n)(q) = \infty$$

for all $q \in E_n$. Thus we obtain

$$\liminf_{p \in G, p \to q} (u + \epsilon v_n)(p) > m - \frac{1}{n}$$

for all $q \in \bar{G} - G$ and $n \geqq 1$. Here $u + \epsilon v_n$ is superharmonic on G and therefore $u + \epsilon v_n > m - 1/n$ on G. On letting $\epsilon \to 0$ and then $n \to \infty$ we obtain the assertion.

Among various consequences of the above theorem we state here the harmonic decomposition theorem (cf. [**6**; **4**]). Recall that a compact subset K on R^* is called a distinguished compact set if $K = \overline{(K \cap R)}$ and $\partial(K \cap R)$ is smooth.

THEOREM 2. *Let K be a distinguished compact subset of R^* and f a Dirichlet finite Tonelli function on R. Then*

(i) *f has a unique decomposition $f = u + g$, where $u \in \tilde{\mathbf{M}}(R) \cap \text{HD}(R - K)$ and $g \in \tilde{\mathbf{M}}_\Delta(R)$ with $g \equiv 0$ on K,*

(ii) *every $h \in \tilde{\mathbf{M}}_\Delta(R)$ with $h \equiv 0$ on K is orthogonal to u, i.e. $D_R(u, h) = 0$,*

(iii) *the Dirichlet principle is valid: $D_R(f) = D_R(u) + D_R(g)$,*

(iv) *$|u| \leqq \sup_{\partial(K \cap R) \cup \Delta_R} |f|$ on $R - K$,*

(v) *if v is a superharmonic (subharmonic) function on $R - K$ such that $v \geqq f$ $(v \leqq f)$ on $R - K$, then $v \geqq u$ $(v \leqq u)$ on $R - K$. Here we assume that $K \cup \Delta_R \neq \emptyset$.*

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