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TRACES ON JORDAN ALGEBRAS

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In the theory of Jordan algebras one encounters several definitions of the trace, and it is sometimes unclear whether the different notions are equivalent or not. If we restrict attention to the so-called JB-algebras studied in [**2**] and their weakly closed analogues JBW-algebras [**8**], we shall in the present note show that the different concepts are all equivalent for JBW-algebras, and that the conditions not involving projections are equivalent for JB-algebras. Among the seven equivalent conditions we shall consider, the second (ii) was used by Alfsen and Shultz [**1**] to show that if the JBW-algebra is the self-adjoint part of a von Neumann algebra, then the condition characterizes traces on the enveloping von Neumann algebra. Condition (iii) appears in Robertson's paper [**7**] together with the implication (ii) $\Rightarrow$ (iii). The inequality (iv) is a Jordan analogue of Gardner's inequality $|\varphi(x)| \leqq \varphi(|x|)$, [**3**], characterizing traces on C^*-algebras. We include a short proof of Gardner's inequality, from which (iv) follows naturally. Conditions (v) and (vi) were used by Topping [**9**] and Janssen [**4**] respectively in the different Jordan algebras they studied.

We refer the reader to [**2**] and [**8**] for the theory of JB- and JBW-algebras we shall need. Just recall that a JB-algebra is a real Jordan algebra A which is a Banach space with respect to a norm having the properties $\|x^2\| = \|x\|^2$ and $\|x^2\| \leqq \|x^2 + y^2\|$ for all x, y in A. If A is furthermore a Banach dual space, then A is called a JBW-algebra. Since the second dual of a JB-algebra is a JBW-algebra [**8**] it will be easy to obtain our results for JB-algebras from those on JBW-algebras.

PROPOSITION. (Gardner). *Let φ be a state on a C^*-algebra A. Then φ is a trace (i.e., $\varphi(xy) = \varphi(yx)$ for all x, y in A) if and only if $|\varphi(x)| \leqq \varphi(|x|)$ for all x in A.*

Proof. It suffices to show the condition is sufficient. Assuming, as we may, that A has a unit, the condition implies that $|\varphi(ux)| \leqq \varphi(x)$ for every unitary u in A and x positive in A. Since the unit ball of A is the closed convex hull of the set of unitaries [**6**, 1.1.12] it follows that $|\varphi(yx)| \leqq \|y\|\varphi(x)$ for every y in A. But then the functional $y \rightarrow \varphi(yx)$ attains its norm at 1 and is therefore positive [**6**, 3.1.4]. In particular, if $y = y^*$ then $\varphi(yx) = \overline{\varphi(yx)} = \varphi(xy)$, and the result follows since A is spanned by its positive elements.

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In order to formulate our main result we shall need the following notation from the theory of Jordan algebras [**5**]. The Jordan triple product is defined by

$$\{xyz\} = (x \circ y) \circ z - (z \circ x) \circ y + (y \circ z) \circ x,$$

which reduces for special Jordan algebras with $x \circ y = \frac{1}{2}(xy + yx)$ to $\{xyz\} = \frac{1}{2}(xyz + zyx)$. The linear mapping $y \rightarrow \{xyx\}$ is denoted by U_x.

THEOREM. *Let φ be a state on a JBW-algebra A. Then the following seven conditions are equivalent.*

(i) $\varphi(U_x y^2) = \varphi(U_y x^2)$ *for all x, y in A.*
(ii) $\varphi(x) = \varphi(U_p x) + \varphi(U_{1-p} x)$ *for all x in A and projections p in A.*
(iii) $\varphi(x \circ y) \geqq 0$ *for all positive x, y in A.*
(iv) $\varphi(x \circ y) \leqq \|x\|\varphi(|y|)$ *for all x, y in A.*
(v) $\varphi(U_s x) = \varphi(x)$ *for all x in A and s in A with $s^2 = 1$.*
(vi) $\varphi(x \circ (y \circ z)) = \varphi(x \circ y) \circ z)$ *for all x, y, z in A.*
(vii) $\varphi(U_y x) = \varphi(y^2 \circ x)$ *for all x, y in A.*

Proof. (i) $\Rightarrow$ (ii). If $x \geqq 0$ then by (i)

$$\varphi(U_p x) + \varphi(U_{1-p} x) = \varphi(U_{x^{1/2}} p) + \varphi(U_{x^{1/2}}(1 - p)) = \varphi(x).$$

Since the positive part A_+ of A spans A we are done.

(ii) $\Rightarrow$ (iii). By [**2**, eq. 2.36] if p is a projection in A and $x \in A_+$ we have

$$p \circ x = \frac{1}{2}(x + U_p x - U_{1-p} x).$$

Since $U_p U_p = U_p$ and $U_p U_{1-p} = 0$ by [**2**, eq. 2.35] we have $U_p(p \circ x) = U_p x$ and $U_{1-p}(p \circ x) = 0$. Thus by (ii) and [**2**, Proposition 2.7] we have

$$\varphi(p \circ x) = \varphi(U_p(p \circ x)) + \varphi(U_{1-p}(p \circ x)) = \varphi(U_p x) \geqq 0.$$

Since the cone generated by projections is norm dense in A_+ we get $\varphi(x \circ y) \geqq 0$ for every y in A_+, as desired.

(iii) $\Rightarrow$ (iv). A functional is positive if and only if it takes its norm at 1. Then from (iii) we have $\varphi(x \circ y) \leqq \|x\|\varphi(y)$ whenever $y \in A_+$. But then in the general case, if y_+ and y_- are the positive and negative parts of y,

$$\varphi(x \circ y) = \varphi(x \circ y_+) - \varphi(x \circ y_-) \leqq \|x\|\varphi(y_+ + y_-) = \|x\|\varphi(|y|).$$

(iv) $\Rightarrow$ (v). If $x \in A_+$ and $t \in [-1, 1]$ let

$$a = (1 + t)x + (1 - t)U_s x + 2(1 - t^2)^{1/2} s \circ x$$
$$b = (1 + t)^{1/2} 1 + (1 - t)^{1/2} s.$$

Since s is a symmetry, a straightforward application of the identity $U_y z = 2y \circ (y \circ z) - y^2 \circ z$ shows that $a = U_b x$, so $a \geqq 0$ by [**2**, Proposition 2.7]. In a special Jordan algebra we have the identity $y \circ U_y z = \{y^2 z y\}$, hence this identity holds in any Jordan algebra by Macdonald's theorem [**5**, p. 41]. Thus if s is a symmetry,

$$s \circ U_s z = \{1zs\} = s \circ z.$$

It follows that

$$\begin{aligned} s \circ a &= (1+t)s \circ x + (1-t)s \circ x + 2(1-t^2)^{1/2} s \circ (s \circ x) \\ &= 2s \circ x + (1-t^2)^{1/2}(x + U_s x). \end{aligned}$$

By assumption $\varphi(s \circ a) \leqq \varphi(a)$, which implies that

$$0 \leqq \varphi(a - s \circ a) = (1 - (1-t^2)^{1/2})\varphi(x + U_s x - 2s \circ x) + t\varphi(x - U_s x)$$

for all t in a neighborhood of 0. Since $1 - (1-t^2)^{1/2} \sim \frac{1}{2}t^2$ it follows that $\varphi(x - U_s x) = 0$.

(v) $\Rightarrow$ (vi). If $x, y, z \in A$ the following identity is easily verified if A is a special Jordan algebra, hence it holds for general A by Macdonald's theorem [**5**, p. 41].

$$(1) \quad U_y\{yxz\} = 2y^2 \circ \{xzy\} - \{xzy^3\}.$$

From the definition of the Jordan triple product we have

$$\begin{aligned} 2(x \circ y) \circ z &= \{xyz\} + \{yxz\} \\ 2x \circ (y \circ z) &= \{xzy\} + \{xyz\}. \end{aligned}$$

Therefore to show (vi) it suffices to show

$$(2) \quad \varphi(\{yxz\}) = \varphi(\{xzy\}),$$

and since linear combinations of symmetries are dense in A it suffices to show (2) when y is a symmetry. But then by (v) and (1) we have

$$\varphi(\{yxz\}) = \varphi(U_y\{yxz\}) = \varphi(2y^2 \circ \{xzy\} - \{xzy^3\}) = \varphi(\{xzy\}).$$

(vi) $\Rightarrow$ (vii). Take x, y in A. Then

$$U_y x = 2y \circ (y \circ x) - y^2 \circ x,$$

so by (vi) we have

$$\varphi(y^2 \circ x) = \varphi(y \circ (y \circ x)) = \tfrac{1}{2}\varphi(U_y x + y^2 \circ x),$$

and (vii) follows.

(vii) $\Rightarrow$ (i). By (vii) $\varphi(U_y x^2) = \varphi(y^2 \circ x^2) = \varphi(U_x y^2)$ for all x, y in A.

COROLLARY. *Let φ be a state on a JB-algebra. Then conditions* (i), (iii), (iv), (vi), *and* (vii) *in the theorem are all equivalent.*

Proof. Since Kaplansky's density theorem holds in JBW-algebras, and the second dual of a JB-algebra is a JBW-algebra [**8**], the corollary is immediate from the theorem and the fact that multiplication is strongly continuous on bounded sets [**2**, Proposition 3.7].

Remark. It is easy to see that condition (ii) in the JB-algebra case can be replaced by

$$\text{(ii}'\text{)}\quad \varphi(U_a x + U_b x) = \varphi(U_{(a^2+b^2)^{1/2}} x)$$

for all a, b, x in A. No such substitute seems to be available for condition (v), and contrary to the C^*-algebra case where we use unitaries, the condition is much too weak to characterize the trace in a general JB-algebra. Just remember that any symmetry has the form $2p - 1$ for some projection p in A, and A may have no nontrivial projections.

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