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Abstract

Let K be a finitely generated extension of $\mathbb{Q}$. We consider the family of ℓ-adic representations (ℓ varies through the set of all prime numbers) of the absolute Galois group of K, attached to ℓ-adic cohomology of a separated scheme of finite type over K. We prove that the fields cut out from the algebraic closure of K by the kernels of the representations of the family are linearly disjoint over a finite extension of K. This gives a positive answer to a question of Serre.

1. Introduction

Let Γ be a profinite group and $(\Gamma_i)_{i\in I}$ a family of groups. For every i let $\rho_i\colon \Gamma\to\Gamma_i$ be a homomorphism. Following Serre (cf. [Ser10, p. 1]), we shall say that the family $(\rho_i)_{i\in I}$ is *independent*, provided the homomorphism

$$\Gamma \xrightarrow{\rho} \prod_{i\in I}\rho_i(\Gamma)$$

induced by the ρ_i is surjective. Let $\Gamma'\subset\Gamma$ be a closed subgroup. We call the family $(\rho_i)_{i\in I}$ *independent over* Γ', if $\rho(\Gamma')=\prod_{i\in I}\rho_i(\Gamma')$. Finally we call the family $(\rho_i)_{i\in I}$ *almost independent*, if there exists an open subgroup $\Gamma'\subset\Gamma$, such that $(\rho_i)_{i\in I}$ is independent over Γ'. Of particular interest is the special case where $\Gamma=\mathrm{Gal}_K$ is the absolute Galois group of a field K, and $(\rho_\ell)_{\ell\in\mathbb{L}}$ is a family of ℓ-adic representations of Gal_K, indexed by the set $\mathbb{L}$ of all prime numbers.

Important examples of such families of representations arise as follows: let K be a field of characteristic zero and let X/K be a separated K-scheme of finite type. Denote by $\widetilde{K}$ an algebraic closure of K. For every $\ell\in\mathbb{L}$ and every $q\geqslant 0$ we consider the representation of the absolute Galois group $\mathrm{Gal}(\widetilde{K}/K)$

$$\rho_{\ell,X}^{(q)}\colon \mathrm{Gal}(\widetilde{K}/K)\longrightarrow \mathrm{Aut}_{\mathbb{Q}_\ell}(\mathrm{H}^q(X_{\widetilde{K}},\mathbb{Q}_\ell))$$

afforded by the étale cohomology group $\mathrm{H}^q(X_{\widetilde{K}},\mathbb{Q}_\ell)$, and also the representation

$$\rho_{\ell,X,c}^{(q)}\colon \mathrm{Gal}(\widetilde{K}/K)\longrightarrow \mathrm{Aut}_{\mathbb{Q}_\ell}(\mathrm{H}^q_c(X_{\widetilde{K}},\mathbb{Q}_\ell))$$

afforded by the étale cohomology group with compact support $\mathrm{H}^q_c(X_{\widetilde{K}},\mathbb{Q}_\ell)$. One can wonder in which circumstances the families $(\rho_{\ell,X}^{(q)})_{\ell\in\mathbb{L}}$ and $(\rho_{\ell,X,c}^{(q)})_{\ell\in\mathbb{L}}$ are almost independent.

In the recent paper [Ser10] Serre considered the special case where K is a number field. He proved a general independence criterion for certain families of ℓ-adic representations over

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a number field (cf. [Ser10, § 2, Théorèm 1]), and used this criterion together with results of Katz–Laumon and of Berthelot (cf. [Ill]) in order to prove the following (cf. [Ser10, § 3]).

Let K be a number field and X/K a separated scheme of finite type. Then the families of representations $(\rho_{\ell,X}^{(q)})_{\ell\in\mathbb{L}}$ and $(\rho_{\ell,X,c}^{(q)})_{\ell\in\mathbb{L}}$ are almost independent.

The special case of an abelian variety X over a number field K had been dealt with earlier in a letter from Serre to Ribet (cf. [Ser00]). In [Ser10, p. 4] Serre asks the following question.

Does this theorem remain true, if one replaces the number field K by a finitely generated transcendental extension K of $\mathbb{Q}$?

This kind of problem also shows up in Serre's article [Ser94, 10.1] and in Illusie's manuscript [Ill]. The aim of our paper is to answer this question affirmatively. In order to do this we prove an independence criterion for families of ℓ-adic representations of the étale fundamental group $\pi_1(S)$ of a normal $\mathbb{Q}$-variety S (cf. Theorem 3.4 below). This criterion allows us to reduce the proof of the following Theorem 1.1 to the number field case, where it is known to hold true thanks to the theorem of Serre (cf. [Ser10]) mentioned above. We do take Tate twists into account. For every $\ell\in\mathbb{L}$ we denote by $\varepsilon_\ell\colon \mathrm{Gal}_K\to\mathrm{Aut}_{\mathbb{Q}_\ell}((\varprojlim_{i\in\mathbb{N}}\mu_{\ell^i})\otimes\mathbb{Q}_\ell)\subset\mathbb{Q}_\ell^\times$ the cyclotomic character, by $\varepsilon_\ell^{\otimes -1}$ its contragredient and define for every $d\in\mathbb{Z}$

$$\rho_{\ell,X}^{(q)}(d):=\rho_{\ell,X}^{(q)}\otimes\varepsilon_\ell^{\otimes d}\quad\text{and}\quad\rho_{\ell,X,\mathrm{c}}^{(q)}(d):=\rho_{\ell,X,\mathrm{c}}^{(q)}\otimes\varepsilon_\ell^{\otimes d}.$$

Theorem 1.1. *Let K be a finitely generated extension of $\mathbb{Q}$. Let X/K be a separated scheme of finite type. Then for every $q\in\mathbb{N}$ and every $d\in\mathbb{Z}$ the families $(\rho_{\ell,X}^{(q)}(d))_{\ell\in\mathbb{L}}$ and $(\rho_{\ell,X,c}^{(q)}(d))_{\ell\in\mathbb{L}}$ of representations of Gal_K are almost independent.*

Note that outside certain special cases it is not known whether the representations in Theorem 1.1 are semisimple. Hence we cannot use techniques like the semisimple approximation of monodromy groups in the proof of Theorem 1.1.

Theorem 1.1 has an important consequence for the arithmetic of abelian varieties. Let A/K be an abelian variety. For every $\ell\in\mathbb{L}$ consider the Tate module $T_\ell(A):=\varprojlim_i A(\widetilde{K})[\ell^i]$, define $V_\ell(A):=T_\ell(A)\otimes_{\mathbb{Z}_\ell}\mathbb{Q}_\ell$ and let

$$\eta_{\ell,A}\colon \mathrm{Gal}(\widetilde{K}/K)\longrightarrow\mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell(A))$$

be the ℓ-adic representation attached to A. Then the $\mathbb{Q}_\ell[\mathrm{Gal}_K]$-modules $V_\ell(A)$ and $\mathrm{H}^1(A^\vee_{\widetilde{K}},\mathbb{Q}_\ell(1))$ are isomorphic, i.e. the representation $\eta_{\ell,A}$ is isomorphic to $\rho_{\ell,A^\vee}(1)$. Hence Theorem 1.1 implies that the family $(\eta_{\ell,A})_{\ell\in\mathbb{L}}$ is almost independent. Denote by $K(A[\ell^\infty])$ the fixed field in $\widetilde{K}$ of the kernel of $\eta_{\ell,A}$. Then $K(A[\ell^\infty])$ is the field obtained from K by adjoining the coordinates of the ℓ-power division points in $A(\widetilde{K})$. Using Remark 3.1 below we see that Theorem 1.1 has the following corollary.

Corollary 1.2. *Let K be a finitely generated extension of $\mathbb{Q}$ and A/K an abelian variety. Then there is a finite extension E/K such that the family $(EK(A[\ell^\infty]))_{\ell\in\mathbb{L}}$ is linearly disjoint over E.*

This paper has an appendix with a more elementary proof of this corollary, which is based on our Theorem 3.4 below, but avoiding use of étale cohomology.

Notation and preliminaries

For a field K fix an algebraic closure $\widetilde{K}$ and denote by Gal_K the absolute Galois group of K. We denote by $\mathbb{L}$ the set of all prime numbers.

Let S be a scheme and $s \in S$ a point (in the underlying topological space). Then $k(s)$ denotes the residue field at s. A *geometric point* of S is a morphism $\overline{s}\colon \mathrm{Spec}(\Omega) \to S$ where Ω is an algebraically closed field. To give such a geometric point $\overline{s}$ is equivalent to giving a pair (s, i) consisting of a usual point $s \in S$ and an embedding $i\colon k(s) \to \Omega$. We then let $k(\overline{s})$ be the algebraic closure of $i(k(s))$ in Ω. Now assume S is an integral scheme and let K be its function field. Then we view S as equipped with the geometric generic point $\mathrm{Spec}(\widetilde{K}) \to S$ and denote by $\pi_1(S)$ the étale fundamental group of S with respect to this geometric point. For a scheme S over a field F and an extension F'/F we define $S_{F'} := S \times_F \mathrm{Spec}(F')$. A *variety* S/F is an integral separated F-scheme of finite type.

Now let S be a connected normal scheme with function field K. Assume for simplicity that $\mathrm{char}(K) = 0$. If E/K is an algebraic field extension, then $S^{(E)}$ denotes the normalization of S in E (cf. [EGAII, 6.3]). This notation is used throughout this manuscript. The canonical morphism $S^{(E)} \to S$ is universally closed and surjective. (This follows from the going-up theorem, cf. [EGAII, 6.1.10].) If E/K is a finite extension, then $S^{(E)} \to S$ is a finite morphism (cf. [Mil80, Proposition I.1.1]). We shall say that an algebraic extension E/K is *unramified along* S, provided the morphism $S^{(E')} \to S$ is étale for every *finite* extension E'/K contained in E. We denote by $K_{S,\mathrm{nr}}$ the maximal extension of K inside $\widetilde{K}$ which is unramified along S, and by S_{nr} the normalization of S in $K_{S,\mathrm{nr}}$. One can then identify $\pi_1(S)$ with $\mathrm{Gal}(K_{S,\mathrm{nr}}/K)$. Let E/K be a Galois extension. If $P \in S$ is a closed point and $\hat{P}$ is a point in $S^{(E)}$ above P, then we define $D_{E/K}(\hat{P}) \subset \mathrm{Gal}(E/K)$ to be the decomposition group of $\hat{P}$, i.e. the stabilizer of $\hat{P}$ under the action of $\mathrm{Gal}(E/K)$.

2. Finiteness properties of Jordan extensions

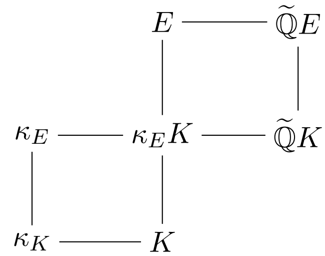
Let E/K be an algebraic field extension and $d \in \mathbb{N}$. We call the extension E/K *d-Jordanian*, if there exists a family $(K_i)_{i\in I}$ of intermediate fields of E/K such that K_i/K is Galois and $[K_i : K] \leqslant d$ for all $i \in I$ and such that E is a (possibly infinite) abelian Galois extension of the compositum $\prod_{i\in I} K_i$. The 1-Jordanian extensions of K are hence just the abelian extensions of K. If K is a number field and E/K is a d-Jordanian extension of K which is everywhere unramified, then E/K is finite. This has been shown by Serre in [Ser10, Théorème 2], making use of the Hermite–Minkowski theorem and the finiteness of the Hilbert class field. The aim of this section is to derive a similar finiteness property for d-Jordanian extensions of function fields over $\mathbb{Q}$. In Lemmata 2.6, 2.7 and 2.8 we follow closely the paper [KL81] of Katz and Lang on geometric class field theory, giving complete details for the convenience of the reader.

If E is any extension field of $\mathbb{Q}$, then we denote by κ_E the algebraic closure of $\mathbb{Q}$ in E,

$$\kappa_E := \{x \in E : x \text{ is algebraic over } \mathbb{Q}\},$$

and we call κ_E the *constant field* of E. We say that E/K is a *constant field extension*, if $\kappa_E K = E$.

Remark 2.1. Let K be a finitely generated extension of $\mathbb{Q}$. Let E/K be an algebraic extension. Then there is a diagram of fields as follows.



The field κ_K is a number field and κ_E/κ_K is an algebraic extension. If E/K is Galois, then κ_E/κ_K, $\kappa_E K/K$ and $\widetilde{\mathbb{Q}}E/\widetilde{\mathbb{Q}}K$ are Galois as well, and the restriction maps $\mathrm{Gal}(\widetilde{\mathbb{Q}}E/\widetilde{\mathbb{Q}}K) \to \mathrm{Gal}(E/\kappa_E K)$ and $\mathrm{Gal}(\kappa_E K/K) \to \mathrm{Gal}(\kappa_E/\kappa_K)$ are both bijective.

The aim of this section is to prove the following proposition.

Proposition 2.2. *Let $S/\mathbb{Q}$ be a normal variety with function field K. Let $d \in \mathbb{N}$. Let E/K be a d-Jordanian extension which is unramified along S. Then $E/\kappa_E K$ is a finite extension.*

Note that in the situation of Proposition 2.2 the extension κ_E/κ_K may well be infinite algebraic. The proof occupies the rest of this section.

Lemma 2.3. *Let $S/\mathbb{Q}$ be a normal variety with function field K. Let $d \in \mathbb{N}$. Let E/K be an algebraic extension which is unramified outside S. Assume that there is a family $(K_i)_{i\in I}$ of intermediate fields of E/K such that each K_i/K is Galois with $[K_i : K] \leqslant d$ and such that $E = \prod_{i\in I} K_i$. Then $E/\kappa_E K$ is finite and $\mathrm{Gal}(\kappa_E/\kappa_K)$ is a (possibly infinite) group of exponent $\leqslant d!$.*

Proof. The Galois group $\mathrm{Gal}(E/K)$ is a closed subgroup of $\prod_{i\in I} \mathrm{Gal}(K_i/K)$. By Remark 2.1 $\mathrm{Gal}(\kappa_E/\kappa_K)$ is a quotient of $\mathrm{Gal}(E/K)$, hence $\mathrm{Gal}(\kappa_E/\kappa_K)$ has exponent $\leqslant d!$. Again by Remark 2.1 it is now enough to show that $\widetilde{\mathbb{Q}}E/\widetilde{\mathbb{Q}}K$ is finite. The Galois group $\mathrm{Gal}(\widetilde{\mathbb{Q}}E/\widetilde{\mathbb{Q}}K)$ is a quotient of $\pi_1(S_{\widetilde{\mathbb{Q}}})$, and $\pi_1(S_{\widetilde{\mathbb{Q}}})$ is topologically finitely generated (cf. [SGA7, II.2.3.1]). Hence there are only finitely many intermediate fields L of $\widetilde{\mathbb{Q}}E/\widetilde{\mathbb{Q}}K$ with $[L : \widetilde{\mathbb{Q}}K] \leqslant d$ (cf. [FJ05, 16.10.2]). This implies that $\widetilde{\mathbb{Q}}E/\widetilde{\mathbb{Q}}K$ is finite. □

Lemma 2.4. *Let K be a finitely generated extension of $\mathbb{Q}$. Let E/K be a (possibly infinite) Galois extension. Assume that $\mathrm{Gal}(E/K)$ has finite exponent. Let $X = (X_1, \ldots, X_n)$ be a transcendence base of $K/\mathbb{Q}$ and R the integral closure of $\mathbb{Z}[X]$ in E. Then the residue field $k(\mathfrak{m}) = R/\mathfrak{m}$ is* finite *for every maximal ideal $\mathfrak{m}$ of R.*

Proof. Let R' be the integral closure of $\mathbb{Z}[X]$ in K. Let $\mathfrak{m}$ be a maximal ideal of R. Define $\mathfrak{m}' := \mathfrak{m} \cap R'$ and $\mathfrak{p} = \mathfrak{m} \cap \mathbb{Z}[X]$. There are diagrams of fields and residue fields

$$\mathbb{Q}(X) \text{ ------ } K \text{ ------ } E \quad \text{and} \quad k(\mathfrak{p}) \text{ ------ } k(\mathfrak{m}') \text{ ------ } k(\mathfrak{m}).$$

By the going-up theorem $\mathfrak{p}$ is a maximal ideal of $\mathbb{Z}[X]$, and $k(\mathfrak{p}) = \mathbb{Z}[X]/\mathfrak{p}$ is a *finite* field. Furthermore R' is a finitely generated $\mathbb{Z}[X]$-module (cf. [Mil80, Proposition I.1.1]). This implies that $k(\mathfrak{m}')$ is a finite field. The extension $k(\mathfrak{m})/k(\mathfrak{m}')$ is Galois and the Galois group $G := \mathrm{Gal}(k(\mathfrak{m})/k(\mathfrak{m}'))$ is a subquotient of $\mathrm{Gal}(E/K)$. Hence G is of finite exponent. On the other hand G must be procyclic, because it is a quotient of the Galois group $\hat{\mathbb{Z}}$ of the finite field $k(\mathfrak{m}')$. It follows that G is finite and that $k(\mathfrak{m})$ is a finite field. □

Lemma 2.5. *Let K be a finitely generated extension of $\mathbb{Q}$. Let E/K be a (possibly infinite) Galois extension. Let $X = (X_1, \ldots, X_n)$ be a transcendence base of $K/\mathbb{Q}$ and R the integral closure of $\mathbb{Z}[X]$ in E. Let $f \in R$ be a non-zero element. Then there exists a natural number N (depending on f) such that for every prime number p not dividing N there exists a maximal ideal $\mathfrak{m} \subset R$ which satisfies $f \notin \mathfrak{m}$ and $\mathrm{char}(k(\mathfrak{m})) = p$.*

Proof. Let $f \in R$ be a non-zero element and consider the closed set $V(f) = \{\mathfrak{p} \in \mathrm{Spec}(R) : f \in \mathfrak{p}\}$. The canonical morphism $\pi : \mathrm{Spec}(R) \to \mathrm{Spec}(\mathbb{Z}[X])$ is closed (cf. [EGAII, 6.1.10]), hence $\pi(V(f))$ is a closed subset of $\mathrm{Spec}(\mathbb{Z}[X])$. It is also a proper subset of $\mathrm{Spec}(\mathbb{Z}[X])$. It follows that there is a non-zero polynomial $g \in \mathbb{Z}[X]$ such that $D(g) \cap \pi(V(f)) = \emptyset$, where by definition $D(g) = \{\mathfrak{p} \in \mathrm{Spec}(\mathbb{Z}[X]) : g \notin \mathfrak{p}\}$. Choose $a \in \mathbb{Z}^n$ with $g(a) \neq 0$ and define $N := g(a)$. Now let p be a prime number not dividing $g(a)$. Consider the maximal ideal $\mathfrak{p} = (p, X_1 - a_1, \ldots, X_n - a_n)$ of $\mathbb{Z}[X]$. Then $\mathfrak{p} \in D(g)$. Finally let $\mathfrak{m}$ be a prime ideal of R such that $\pi(\mathfrak{m}) = \mathfrak{p}$. Then $f \notin \mathfrak{m}$ and $\mathrm{char}(k(\mathfrak{m})) = p$ as desired. □

We now show that a weak form of the Mordell–Weil theorem holds true over finitely generated extensions of fields like the field κ_E in Lemma 2.3. If B is a semiabelian variety over a field K, then we define $T(B) = \prod_{\ell \in \mathbb{L}} T_\ell(B)$ and $T(B)_{\neq p} := \prod_{\ell \in \mathbb{L} \setminus \{p\}} T_\ell(B)$ (for $p \in \mathbb{L}$), where $T_\ell(B) = \varprojlim_{i \in \mathbb{N}} B(\widetilde{K})[\ell^i]$ is the Tate module of B for every $\ell \in \mathbb{L}$. If M is a compact topological Gal_K-module, then we define the *module of coinvariants* M_{Gal_K} of M to be the largest Hausdorff quotient of M on which Gal_K acts trivially.

Lemma 2.6. *Let K be a finitely generated extension of $\mathbb{Q}$. Let E/K be a Galois extension. Assume that $\mathrm{Gal}(E/K)$ has finite exponent. Let B/E be a semiabelian variety. Then $T(B)_{\mathrm{Gal}_E}$ is finite.*

Proof. Let E'/E be a finite extension over which the torus part of B splits. Then there exists a finite Galois extension L/K such that $LE \supset E'$, and $\mathrm{Gal}(LE/K)$ has finite exponent again. The group $T(B)_{\mathrm{Gal}_E}$ is a quotient of $T(B)_{\mathrm{Gal}_{LE}}$. Hence we may assume right from the beginning that B is an extension of an abelian variety A by a split torus $\mathbb{G}_{m,E}^d$. Then there is an exact sequence of Gal_E-modules

$$0 \longrightarrow T(\mathbb{G}_m)^d \longrightarrow T(B) \longrightarrow T(A) \longrightarrow 0.$$

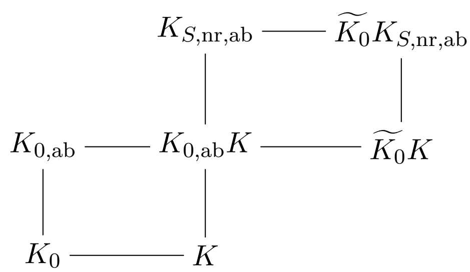
As the functor $-_{\mathrm{Gal}_E}$ is right exact, it is enough to prove that $T(A)_{\mathrm{Gal}_E}$ and $T(\mathbb{G}_m)_{\mathrm{Gal}_E}$ are both finite. We may thus assume that either B is an abelian variety over E (case 1) or $B = \mathbb{G}_{m,E}$ (case 2). We shall prove the finiteness of $T(B)_{\mathrm{Gal}_E}$ in both cases.

Choose a transcendence base $X = (X_1, \ldots, X_n)$ of $K/\mathbb{Q}$ and let R be the integral closure of $\mathbb{Z}[X]$ in E. In case 1 there is a nonempty open subscheme $U \subset \mathrm{Spec}(R)$ such that B extends to an abelian scheme $\mathcal{B}$ over U. In case 2 we define $U = \mathrm{Spec}(R)$ and put $\mathcal{B} := \mathbb{G}_{m,U}$. Let $\mathfrak{m}$ be a maximal ideal of R contained in U, define $p = \mathrm{char}(R/\mathfrak{m})$, and denote by $\overline{B} = \mathcal{B} \times_U \mathrm{Spec}(k(\mathfrak{m}))$ the special fibre at $\mathfrak{m}$. Let n be a positive integer which is coprime to p. Then the restriction of $\mathcal{B}[n]$ to $S := U[1/n]$ is a finite étale group scheme over S and $\mathfrak{m} \in S$. Let $\mathfrak{m}_{\mathrm{nr}}$ be a closed point of S_{nr} over $\mathfrak{m}$. Taking a projective limit over the cospecialization maps $B[n](\widetilde{E}) \cong \overline{B}[n](k(\mathfrak{m}_{\mathrm{nr}}))$, we obtain an isomorphism

$$T(B)_{\neq p} \cong T(\overline{B})_{\neq p},$$

which induces a surjection $T(\overline{B})_{\neq p, \mathrm{Gal}_{\mathbb{F}}} \to T(B)_{\neq p, \mathrm{Gal}_E}$, where we have put $\mathbb{F} = k(\mathfrak{m})$. The field $\mathbb{F}$ is *finite* by Lemma 2.4 and $\overline{B}$ is either an abelian variety over $\mathbb{F}$ (case 1) or the multiplicative group scheme over $\mathbb{F}$ (case 2). In both cases it is known that $T(\overline{B})_{\neq p, \mathrm{Gal}_{\mathbb{F}}}$ is finite (cf. [KL81, Theorem 1 (ter), p. 299]). This shows that $T(B)_{\neq p,\mathrm{Gal}_E}$ is finite, whenever there exists a maximal ideal $\mathfrak{m}$ of R contained in U with $\mathrm{char}(k(\mathfrak{m}))=p$. Now it follows by part (b) of Lemma 2.5 that there are two different prime numbers $p_1\neq p_2$ such that $T(B)_{\neq p_1,\mathrm{Gal}_E}$ and $T(B)_{\neq p_2,\mathrm{Gal}_E}$ are finite, and the assertion follows from that. □

Let K_0 be a field of characteristic zero and S/K_0 a normal geometrically irreducible variety with function field K. There is a canonical epimorphism $p\colon \pi_1(S)\to \mathrm{Gal}_{K_0}$ (with kernel $\pi_1(S_{\widetilde{K_0}})$) and, following Katz–Lang [KL81, p. 285], we define $\mathcal{K}(S/K_0)$ to be the kernel of the map $\pi_1(S)_{\mathrm{ab}}\to \mathrm{Gal}_{K_0,\mathrm{ab}}$ induced by p on the abelianizations. If we denote by $K_{S,\mathrm{nr},\mathrm{ab}}$ the maximal abelian extension of K which is unramified along S, then there is a diagram of fields



(cf. [KL81, p. 286]) and the groups $\mathrm{Gal}(K_{S,\mathrm{nr},\mathrm{ab}}/K_{0,\mathrm{ab}}K)$ and $\mathrm{Gal}(\widetilde{K_0}K_{S,\mathrm{nr},\mathrm{ab}}/\widetilde{K_0}K)$ are both isomorphic to $\mathcal{K}(S/K_0)$. The main result in the paper [KL81] of Katz and Lang is: *if K_0 is finitely generated and S/K_0 a smooth geometrically irreducible variety, then $\mathcal{K}(S/K_0)$ is finite.* On the other hand, if K_0 is algebraically closed and S/K_0 is a smooth proper geometrically irreducible curve of genus g, then $\mathcal{K}(S/K_0)\cong\widehat{\mathbb{Z}}^{2g}$ is infinite, unless $g=0$. In order to finish up the proof of Proposition 2.2 we have to prove the finiteness of $\mathcal{K}(S/K_0)$ in the case of certain algebraic extensions $K_0/\mathbb{Q}$ (like the field κ_E in Lemma 2.3) which are *not* finitely generated but much smaller than $\widetilde{\mathbb{Q}}$.

Lemma 2.7. *Let K be a finitely generated extension of $\mathbb{Q}$. Let E/K be a (possibly infinite) Galois extension. Assume that $\mathrm{Gal}(E/K)$ has finite exponent. Let C/E be a smooth proper geometrically irreducible curve and S the complement of a divisor D in C. Then $\mathcal{K}(S/E)$ is finite.*

Proof. There is a finite extension E'/E such that S has an E'-rational point and D is E'-rational. There is a finite extension E''/E' which is Galois over K. Then $\mathrm{Gal}(E''/K)$ must have finite exponent (because $\mathrm{Gal}(E/K)$ and $\mathrm{Gal}(E''/E)$ do). Furthermore $\mathcal{K}(S_{E''}/E'')$ surjects onto $\mathcal{K}(S/E)$ (cf. [KL81, Lemma 1, p. 291]). Hence we may assume from the beginning that S has an E-rational point and D is E-rational. The generalized Jacobian J of C with respect to the modulus D is a semiabelian variety. (If $S=C$, then J is just the usual Jacobian variety of C.) Furthermore there is an isomorphism

$$\pi_1(S_{\widetilde{E}})_{\mathrm{ab}}\cong T(J).$$

On the other hand $\pi_1(S_{\widetilde{E}})_{\mathrm{ab},\mathrm{Gal}_E}$ is isomorphic to $\mathcal{K}(S/E)$ (cf. [KL81, Lemma 1, p. 291]). Hence it is enough to prove that $T(J)_{\mathrm{Gal}_E}$ is finite. But this has already been done in Lemma 2.6. □

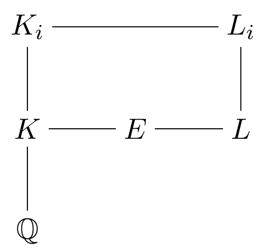
Lemma 2.8. *Let K be a finitely generated extension of $\mathbb{Q}$. Let E/K be a (possibly infinite) Galois extension. Assume that $\mathrm{Gal}(E/K)$ has finite exponent. Let S/E be a normal geometrically irreducible variety. Then $\mathcal{K}(S/E)$ is finite.*

Proof. There is a finite extension L/E and a sequence of elementary fibrations in the sense of Artin (cf. [SGA4, Exposé XI, 3.1–3.3])

$$\mathrm{Spec}(L) = U_0 \xleftarrow{f_1} U_1 \xleftarrow{f_2} U_2 \xleftarrow{f_3} \cdots \xleftarrow{f_n} U_n \subset S_L$$

where U_n is a non-empty open subscheme of S_L and $\dim(U_i) = i$ for all i. There exists $\omega \in L$ such that $L = E(\omega)$. The extension $K(\omega)/K$ is finite, hence contained in a finite Galois extension K'/K. Then EK'/K is Galois, EK'/E is finite and $L \subset EK'$. Replacing L by EK' we may assume that L/K is Galois, and then $\mathrm{Gal}(L/K)$ must be of finite exponent.

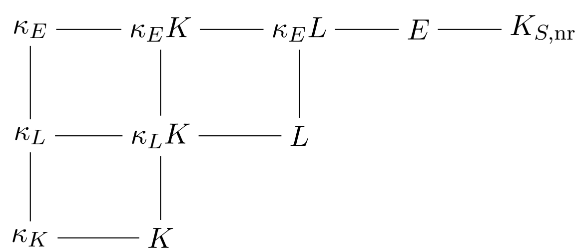
Let L_i be the function field of U_i. Then the generic fibre $S_{i+1} := U_{i+1} \times_{U_i} \mathrm{Spec}(L_i)$ of f_{i+1} is a curve over L_i which is the complement of a divisor in a smooth proper geometrically irreducible curve C_{i+1}/L_i. The extension L_i/L is finitely generated (of transcendence degree i). Hence $L_i = L(u_1, \ldots, u_s)$ for certain elements $u_1, \ldots, u_s \in L_i$. Let us define $K_i := K(u_1, \ldots, u_s)$. Then there is a diagram of fields



such that the vertical extensions are all finitely generated and $L_i = K_i L$. The extension L_i/K_i is Galois because L/K is Galois, and the restriction map $\mathrm{Gal}(L_i/K_i) \to \mathrm{Gal}(L/K)$ is injective. Hence $\mathrm{Gal}(L_i/K_i)$ is a group of finite exponent and K_i is finitely generated.

Lemma 2.7 implies that $\mathcal{K}(S_{i+1}/L_i)$ is finite for every $i \in \{0, \ldots, n-1\}$. By [KL81, Lemma 2] and [KL81, (1.4)] it follows that $\mathcal{K}(U_n/L)$ is finite. Then [KL81, Lemma 3] implies that $\mathcal{K}(S_L/L)$ is finite, and [KL81, Lemma 1] shows that $\mathcal{K}(S/E)$ is finite, as desired. □

Proof of Proposition 2.2. Let $S/\mathbb{Q}$ be a normal variety with the function field K. Let E/K be a d-Jordan extension contained in the extension $K_{S,\mathrm{nr}}/K$. There is an intermediate field L of E/K such that E/L is abelian and L is a compositum of Galois extensions of K, each of degree $\leqslant d$. By Lemma 2.3 $L/\kappa_L K$ is a finite extension. We have the following diagram of fields.



Now $S^{(L)}$ is the normalization of the geometrically irreducible κ_L-variety

$$S^{(\kappa_L K)} = S \times_{\kappa_K} \mathrm{Spec}(\kappa_L)$$

in the *finite* extension $L/\kappa_L K$. Hence $S^{(L)}$ is a geometrically irreducible variety over κ_L. (The crucial point is that $S^{(L)}$ is of finite type over κ_L.) The extension E/L is abelian and unramified along $S^{(L)}$. Hence $\mathrm{Gal}(E/\kappa_E L)$ is a quotient of $\mathcal{K}(S^{(L)}/\kappa_L)$. The field κ_K is a *number field* and $\mathrm{Gal}(\kappa_L/\kappa_K)$ is a group of exponent $\leqslant d!$, because it is a quotient of $\mathrm{Gal}(L/K)$ (cf. Remark 2.1).

Hence Lemma 2.8 implies that $\mathcal{K}(S^{(L)}/\kappa_L)$ is finite. It follows that $E/\kappa_E L$ is a finite extension. Now $\kappa_E L/\kappa_E K$ is finite, because $L/\kappa_L K$ is finite. It follows that $E/\kappa_E K$ is finite, as desired. □

3. Representations of the fundamental group

We start this section with two remarks and a lemma about families of representations of certain profinite groups. Then we prove an independence criterion for families of representations of the étale fundamental group $\pi_1(S)$ of a normal $\mathbb{Q}$-variety S (cf. Theorem 3.4). This criterion is the technical heart of the paper.

Remark 3.1. Let K be a field, Ω/K a Galois extension and $I \subset \mathbb{N}$. Let $(\Gamma_i)_{i\in I}$ be a family of profinite groups. For every $i \in I$ let $\rho_i : \mathrm{Gal}(\Omega/K) \to \Gamma_i$ be a continuous homomorphism. Let K_i be the fixed field of $\ker(\rho_i)$ in Ω. Then the following conditions are equivalent.

(i) The family $(\rho_i)_{i\in I}$ is independent.

(ii) The family $(K_i)_{i\in I}$ of fields is linearly disjoint over K.

(iii) If $s \geqslant 1$ and $i_1 < i_2 < \cdots < i_{s+1}$ are elements of I, then

$$K_{i_1} \cdots K_{i_s} \cap K_{i_{s+1}} = K.$$

Proof. As the homomorphisms ρ_i induce isomorphisms $\mathrm{Gal}(K_i/K) \cong \mathrm{im}(\rho_i)$, (i) is satisfied if and only if the natural map $\mathrm{Gal}(\Omega/K) \to \prod_{i\in I} \mathrm{Gal}(K_i/K)$ is surjective, and this is in turn equivalent to (ii) (cf. [FJ05, 2.5.6]). It is well-known that (ii) is equivalent to (iii) (cf. [FJ05, p. 36]). □

Remark 3.2. Let Γ be a profinite group and $n \in \mathbb{N}$. For every $\ell \in \mathbb{L}$ let Γ_ℓ be a profinite group and $\rho_\ell \colon \Gamma \to \Gamma_\ell$ a continuous homomorphism. Assume that for every $\ell \in \mathbb{L}$ there is an integer $n \in \mathbb{N}$ such that Γ_ℓ is isomorphic to a subquotient of $\mathrm{GL}_n(\mathbb{Z}_\ell)$.

(a) Let $\Gamma' \subset \Gamma$ be an open subgroup. If the family $(\rho_\ell)_{\ell\in\mathbb{L}}$ is independent, then there is a finite subset $I \subset \mathbb{L}$ such that the family $(\rho_\ell)_{\ell\in\mathbb{L}\smallsetminus I}$ is independent over Γ'.

(b) The following conditions (i) and (ii) are equivalent.

  (i) The family $(\rho_\ell)_{\ell\in\mathbb{L}}$ is almost independent.

  (ii) There exists a finite subset $I \subset \mathbb{L}$ such that $(\rho_\ell)_{\ell\in\mathbb{L}\smallsetminus I}$ is almost independent.

Proof. Let $\rho \colon \Gamma \to \prod_{\ell\in\mathbb{L}} \Gamma_\ell$ be the homomorphism induced by the ρ_ℓ. To prove (a) assume that $\rho(\Gamma) = \prod_{\ell\in\mathbb{L}} \rho_\ell(\Gamma)$. The subgroup $\rho(\Gamma')$ is open in $\prod_{\ell\in\mathbb{L}} \rho_\ell(\Gamma)$, because a surjective homomorphism of profinite groups is open (cf. [FJ05, p. 5]). It follows from the definition of the product topology that there is a finite subset $I \subset \mathbb{L}$ such that $\rho(\Gamma') \supset \prod_{\ell\in I}\{1\} \times \prod_{\ell\in\mathbb{L}\smallsetminus I} \rho_\ell(\Gamma)$. This implies that $(\rho_\ell)_{\ell\in\mathbb{L}\smallsetminus I}$ is independent over Γ' and finishes the proof of part (a). For part (b) see [Ser10, Lemme 3]. □

Let K be a field, $n \in \mathbb{N}$ and Ω/K a fixed Galois extension. For every $\ell \in \mathbb{L}$ let Γ_ℓ be a profinite group and $\rho_\ell \colon \mathrm{Gal}(\Omega/K) \to \Gamma_\ell$ a continuous homomorphism. Assume that Γ_ℓ is isomorphic to a subquotient of $\mathrm{GL}_n(\mathbb{Z}_\ell)$ for every $\ell \in \mathbb{L}$. Denote by K_ℓ the fixed field in Ω of the kernel of ρ_ℓ. Then K_ℓ is a Galois extension of K and ρ_ℓ induces an isomorphism $\mathrm{Gal}(K_\ell/K) \cong \rho_\ell(\mathrm{Gal}(\Omega/K))$. For every extension E/K contained in Ω and every $\ell \in \mathbb{L}$ we define $G_{\ell,E} := \rho_\ell(\mathrm{Gal}(\Omega/E))$ and $E_\ell := EK_\ell$. Then $G_{\ell,E}$ is isomorphic to a subquotient of $\mathrm{GL}_n(\mathbb{Z}_\ell)$ and ρ_ℓ induces an isomorphism

$$\mathrm{Gal}(E_\ell/E) \cong G_{\ell,E}.$$

Furthermore we define $G^+_{\ell,E}$ to be the subgroup of $G_{\ell,E}$ generated by its ℓ-Sylow subgroups. Then $G^+_{\ell,E}$ is normal in $G_{\ell,E}$. Finally we let E^+_ℓ be the fixed field of $\rho_\ell^{-1}(G^+_{\ell,E})\cap \mathrm{Gal}(\Omega/E)$. Then E^+_ℓ is an intermediate field of E_ℓ/E which is Galois over E, the group $\mathrm{Gal}(E_\ell/E^+_\ell)$ is isomorphic to $G^+_{\ell,E}$ and $\mathrm{Gal}(E^+_\ell/E)$ is isomorphic to $G_{\ell,E}/G^+_{\ell,E}$.

Lemma 3.3. *Let E/K be a Galois extension contained in $\widetilde{K}$ and let $\ell\in\mathbb{L}$.*

(a) *The extension E^+_ℓ/E is a* finite *Galois extension, and $\mathrm{Gal}(E^+_\ell/E)$ is isomorphic to a subquotient of $\mathrm{GL}_n(\mathbb{F}_\ell)$.*

(b) *If E/K is finite and $[E:K]$ is not divisible by ℓ, then $G^+_{\ell,E}=G^+_{\ell,K}$ and $EK^+_\ell=E^+_\ell$.*

Proof. The profinite group $G_{\ell,E}$ is a closed normal subgroup of $G_{\ell,K}$, and $G_{\ell,K}$ is isomorphic to a subquotient of $\mathrm{GL}_n(\mathbb{Z}_\ell)$. Hence there is a closed subgroup U_ℓ of $\mathrm{GL}_n(\mathbb{Z}_\ell)$ and a closed normal subgroup V_ℓ of U_ℓ such that there is an isomorphism $i: G_{\ell,E}\to U_\ell/V_\ell$. Furthermore there is a closed normal subgroup U^+_ℓ of U_ℓ containing V_ℓ such that $i(G^+_{\ell,K})=U^+_\ell/V_\ell$. The group U_ℓ/U^+_ℓ is isomorphic to $G_{\ell,E}/G^+_{\ell,E}$. Its order is coprime to ℓ. The kernel of the restriction map $r\colon \mathrm{GL}_n(\mathbb{Z}_\ell)\to \mathrm{GL}_n(\mathbb{F}_\ell)$ is a pro-ℓ group; hence the intersection of this kernel with U_ℓ is contained in U^+_ℓ. This shows that r induces an isomorphism $U_\ell/U^+_\ell\to r(U_\ell)/r(U^+_\ell)$. Altogether we see that

$$\mathrm{Gal}(E^+_\ell/E)\cong G_{\ell,E}/G^+_{\ell,E}\cong U_\ell/U^+_\ell\cong r(U_\ell)/r(U^+_\ell)$$

and part (a) follows, because $r(U_\ell)/r(U^+_\ell)$ is obviously a subquotient of $\mathrm{GL}_n(\mathbb{F}_\ell)$.

Every ℓ-Sylow subgroup of $G_{\ell,E}$ lies in an ℓ-Sylow subgroup of $G_{\ell,K}$, hence $G^+_{\ell,E}\subset G^+_{\ell,K}$. Assume from now on that $[E:K]$ is finite and not divisible by ℓ. Then every ℓ-Sylow subgroup of $G_{\ell,K}$ must map to the trivial group under the projection $G_{\ell,K}\to G_{\ell,K}/G_{\ell,E}$, because the order of the quotient group is coprime to ℓ. Hence every ℓ-Sylow subgroup of $G_{\ell,K}$ lies in $G_{\ell,E}$. This shows that $G^+_{\ell,K}=G^+_{\ell,E}$. The Galois group $\mathrm{Gal}(E_\ell/EK^+_\ell)$ is $G^+_{\ell,K}\cap G_{\ell,E}$ and the Galois group $\mathrm{Gal}(E_\ell/EK^+_\ell)$ is $G^+_{\ell,E}$. As $G^+_{\ell,K}=G^+_{\ell,E}$ it follows that $\mathrm{Gal}(E_\ell/EK^+_\ell)=\mathrm{Gal}(E_\ell/E^+_\ell)$, hence $EK^+_\ell=E^+_\ell$. ☐

Let S be a normal $\mathbb{Q}$-variety with function field K. We shall now study families of representations of the fundamental group $\pi_1(S)$ (viewing S as a scheme equipped with the generic geometric point $\mathrm{Spec}(\widetilde{K})\to K$). Recall that we may identify $\pi_1(S)$ with $\mathrm{Gal}(K_{S,\mathrm{nr}}/K)$.

Theorem 3.4. *Let $S/\mathbb{Q}$ be a normal variety with function field K. Let $P_{\mathrm{nr}}\in S_{\mathrm{nr}}$ be a closed point. For every $\ell\in\mathbb{L}$ let Γ_ℓ be a profinite group and $\rho_\ell\colon \pi_1(S)\to\Gamma_\ell$ a continuous homomorphism. We make two assumptions.*

(a) *Assume there is an integer $n\in\mathbb{N}$ such that for every $\ell\in\mathbb{L}$ the profinite group Γ_ℓ is isomorphic to a subquotient of $\mathrm{GL}_n(\mathbb{Z}_\ell)$.*

(b) *Assume that there exists an open subgroup D' of the decomposition group $D_{K_{S,\mathrm{nr}}/K}(P_{\mathrm{nr}})$ such that the family $(\rho_\ell)_{\ell\in\mathbb{L}}$ is independent over D'.*

Then the family $(\rho_\ell)_{\ell\in\mathbb{L}}$ is almost independent.

The proof of Theorem 3.4 occupies the rest of this section. From now on all the assumptions of Theorem 3.4 are in force, until the proof is finished. For every algebraic extension E/K contained in $K_{S,\mathrm{nr}}$ we define $G_{\ell,E}=\rho_\ell(\mathrm{Gal}(K_{S,\mathrm{nr}}/E))$, $G^+_{\ell,E}$, E_ℓ and E^+_ℓ exactly as before. Furthermore we shall write P_E for the point in $S^{(E)}$ below P_{nr}.

We tacitly assume in the following that $\widetilde{\mathbb{Q}}$ denotes the algebraic closure of K *inside* $\widetilde{K}$. Then already $K_{S,\mathrm{nr}}$ contains $\widetilde{\mathbb{Q}}$, because the constant field extensions of K are unramified along S. The structure morphism $S_{\mathrm{nr}} \to \mathrm{Spec}(\mathbb{Q})$ factors through $\mathrm{Spec}(\widetilde{\mathbb{Q}})$, because S_{nr} is normal. It follows in particular that $k(P_{\mathrm{nr}}) = \widetilde{\mathbb{Q}}$.

Lemma 3.5. *There is a finite Galois extension E/K contained in $K_{S,\mathrm{nr}}$ and a finite subset $I \subset \mathbb{L}$ such that the following statements about E and I hold true.*

(a) *For all $\ell \in \mathbb{L} \smallsetminus I$ the extension E_ℓ^+/E is a constant field extension, that is: $\kappa_{E_\ell^+} E = E_\ell^+$.*

(b) *The point P_E is a κ_E-rational point of $S^{(E)}$.*

(c) *The family $(\rho_\ell)_{\ell \in \mathbb{L} \smallsetminus I}$ is independent over $D_{K_{S,\mathrm{nr}}/E}(P_{\mathrm{nr}})$.*

Proof. Let $L := \prod_{\ell \in \mathbb{L}} K_\ell^+$ be the composite field of all the K_ℓ^+. By Lemma 3.3, for each $\ell \in \mathbb{L}$, the group $\mathrm{Gal}(K_\ell^+/K)$ is isomorphic to a subquotient of $\mathrm{GL}_n(\mathbb{F}_\ell)$, and $|\mathrm{Gal}(K_\ell^+/K)|$ is not divisible by ℓ. By [Ser10, Théorème 3'] (which is a generalization due to Serre of the classical theorem of Jordan) it follows that there is an integer d (independent of ℓ) such that for every $\ell \in \mathbb{L}$ the group $\mathrm{Gal}(K_\ell^+/K)$ has an abelian normal subgroup A_ℓ of index $[\mathrm{Gal}(K_\ell^+/K) : A_\ell] \leqslant d$. Let K'_ℓ be the fixed field of A_ℓ in K_ℓ^+. Then $K' := \prod_{\ell \in \mathbb{L}} K'_\ell$ is a compositum of Galois extensions of K and $[K'_\ell : K] \leqslant d$ for all $\ell \in \mathbb{L}$. The extension $K' K_\ell^+/K'$ is abelian for every $\ell \in \mathbb{L}$. It follows that L/K is a d-Jordanian extension. Furthermore L/K is contained in $K_{S,\mathrm{nr}}$. By Proposition 2.2, L is a *finite* extension of $\kappa_L K$. Note that $\kappa_L/\mathbb{Q}$ may well be an infinite extension. Hence there is an element $\omega \in L$ such that $L = \kappa_L K(\omega)$. Let E_1 be the Galois closure of $K(\omega)/K$ in L. Then E_1/K is a finite Galois extension and $\kappa_L E_1 = L$. Hence we have a diagram of fields

$$\begin{array}{ccc} \kappa_L K & \text{---} & L \\ | & & | \\ K & \text{---} & E_1 \end{array}$$

in which the vertical extensions are constant field extensions and in which the horizontal extensions are finite. Furthermore L contains K_ℓ^+ for every $\ell \in \mathbb{L}$.

Now consider the canonical isomorphism

$$r\colon D_{K_{S,\mathrm{nr}}/K}(P_{\mathrm{nr}}) \cong \mathrm{Gal}(k(P_{\mathrm{nr}})/k(P_K)).$$

Let λ_1 be the fixed field of $r(D')$ in $k(P_{\mathrm{nr}}) = \widetilde{\mathbb{Q}}$. Since D' is open in $D_{K_{S,\mathrm{nr}}/K}(P_{\mathrm{nr}})$, the field λ_1 is a finite extension of $k(P_K)$, so λ_1 is a finite extension of $\mathbb{Q}$. Choose a finite Galois extension λ/κ_K containing λ_1 and $k(P_{E_1})$, and define $E := \lambda E_1$. Then $S^{(E)} = S^{(E_1)} \times_{\kappa_{E_1}} \mathrm{Spec}(\lambda)$ and $\kappa_E = \lambda$. There is the following diagram of number fields.

$$\begin{array}{ccccccc} \lambda_1 & \text{---} & \lambda & = & \kappa_E & \text{---} & k(P_E) \\ | & & | & & & & \\ k(P_K) & \text{---} & k(P_{E_1}) & & & & \\ | & & | & & & & \\ \kappa_K & \text{---} & \kappa_{E_1} & & & & \end{array}$$

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The fibre of P_{E_1} under the projection $S^{(E)} \to S^{(E_1)}$ is $\mathrm{Spec}(\kappa_E \otimes_{\kappa_{E_1}} k(P_{E_1}))$, and this fibre splits up into the coproduct of $[k(P_{E_1}) : \kappa_{E_1}]$ many copies of $\mathrm{Spec}(\kappa_E) = \mathrm{Spec}(\lambda)$, because λ/κ_E is Galois and $\lambda \supset k(P_{E_1})$. Thus all points in $S^{(E)}$ over P_{E_1} are κ_E-rational. In particular P_E is κ_E-rational.

It follows that

$$r(D_{K_{S,\mathrm{nr}}/E}(P_{\mathrm{nr}})) = \mathrm{Gal}(k(P_{\mathrm{nr}})/k(P_E)) = \mathrm{Gal}(k(P_{\mathrm{nr}})/\kappa_E),$$

and this group is an open subgroup of $r(D') = \mathrm{Gal}(k(P_{\mathrm{nr}})/\lambda_1)$ because κ_E is a finite extension of λ_1. Hence $D_{K_{S,\mathrm{nr}}/E}(P_{\mathrm{nr}})$ is an open subgroup of D'.

As $(\rho_\ell)_{\ell\in\mathbb{L}}$ is independent over D' by one of our assumptions, it follows from part (a) of Remark 3.2 that there is a finite subset $I' \subset \mathbb{L}$ such that the family $(\rho_\ell)_{\ell\in\mathbb{L}\smallsetminus I'}$ is independent over $D_{K_{S,\mathrm{nr}}/E}(P_{\mathrm{nr}})$. Finally K_ℓ^+E/E is a constant field extension, because K_ℓ^+E is an intermediate field of LE/E and $LE = \kappa_L E$ is a constant field extension of E due to our construction. By Lemma 3.3 we see that $E_\ell^+ = K_\ell^+E$ for all $\ell \in \mathbb{L}$ which do not divide the index $[E:K]$. Hence assertions (a), (b) and (c) follow, if we put $I := I' \cup \{\ell \in \mathbb{L} : \ell \text{ divides } [E:K]\}$. □

LEMMA 3.6. *Let E and I be as in Lemma 3.5. Let $s \geqslant 1$. Let $\ell_1 < \cdots < \ell_{s+1}$ be some elements of $\mathbb{L} \smallsetminus I$. Then $E_{\ell_1} \cdots E_{\ell_s} \cap E_{\ell_{s+1}}$ is a regular extension of κ_E (i.e. the algebraic closure of $\mathbb{Q}$ in $E_{\ell_1} \cdots E_{\ell_s} \cap E_{\ell_{s+1}}$ is κ_E).*

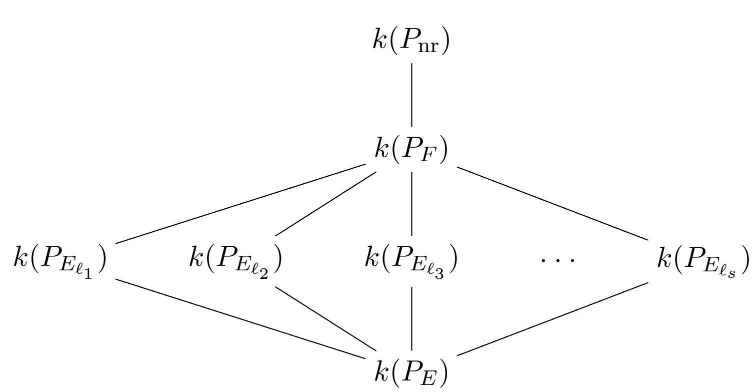
Proof. The canonical isomorphism

$$r\colon D_{K_{S,\mathrm{nr}}/E}(P_{\mathrm{nr}}) \cong \mathrm{Gal}(k(P_{\mathrm{nr}})/k(P_E))$$

induces by restriction an isomorphism

$$D_{K_{S,\mathrm{nr}}/E_\ell}(P_{\mathrm{nr}}) = D_{K_{S,\mathrm{nr}}/E}(P_{\mathrm{nr}}) \cap \mathrm{Gal}(K_{S,\mathrm{nr}}/E_\ell) \cong \mathrm{Gal}(k(P_{\mathrm{nr}})/k(P_{E_\ell}))$$

for every $\ell \in \mathbb{L}$. Hence $k(P_{E_\ell})$ is the fixed field in $k(P_{\mathrm{nr}})$ of the kernel of $\rho_\ell \circ r^{-1}$. The family $(\rho_\ell)_{\ell\in\mathbb{L}\smallsetminus I}$ is independent over $D_{K_{S,\mathrm{nr}}/E}(P_{\mathrm{nr}})$ by Lemma 3.5. Hence Remark 3.1 shows that $(k(P_{E_\ell}))_{\ell\in\mathbb{L}\smallsetminus I}$ is linearly disjoint over $k(P_E)$. Define $F := E_{\ell_1} \cdots E_{\ell_s}$. There is a diagram of residue fields as follows.



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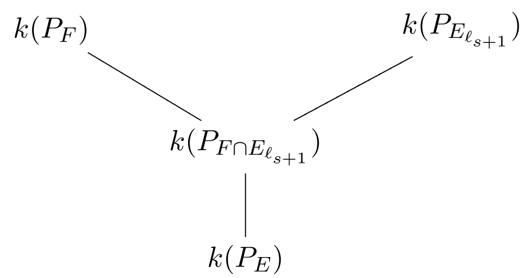
We have $k(P_F)=k(P_{E_{\ell_1}})\cdots k(P_{E_{\ell_s}})$, because

$$\begin{aligned}\mathrm{Gal}(k(P_{\mathrm{nr}})/k(P_{E_{\ell_1}})\cdots k(P_{E_{\ell_s}})) &= \bigcap_{i=1}^{s}\mathrm{Gal}(k(P_{\mathrm{nr}})/k(P_{E_{\ell_i}}))\\ &= r\Bigl(\bigcap_{i=1}^{s} D_{K_{S,\mathrm{nr}}/E_{\ell_i}}(P_{\mathrm{nr}})\Bigr)\\ &= r\Bigl(D_{K_{S,\mathrm{nr}}/E}(P_{\mathrm{nr}})\cap\bigcap_{i=1}^{s}\mathrm{Gal}(K_{S,\mathrm{nr}}/E_{\ell_i})\Bigr)\\ &= r(D_{K_{S,\mathrm{nr}}/E}(P_{\mathrm{nr}})\cap\mathrm{Gal}(K_{S,\mathrm{nr}}/F))\\ &= r(D_{K_{S,\mathrm{nr}}/F}(P_{\mathrm{nr}}))=\mathrm{Gal}(k(P_{\mathrm{nr}})/k(P_F)).\end{aligned}$$

Furthermore there is a diagram



and $k(P_F)\cap k(P_{E_{\ell_{s+1}}})=k(P_E)$ due to the fact that $(k(P_{E_\ell}))_{\ell\in\mathbb{L}\smallsetminus I}$ is linearly disjoint over $k(P_E)$. It follows that $k(P_{F\cap E_{\ell_{s+1}}})=k(P_E)$. Finally $k(P_E)=\kappa_E$, because P_E is a κ_E-rational point of $S^{(E)}$. This shows that the normalization of $S^{(E)}$ in $F\cap E_{\ell_{s+1}}$ has a κ_E-rational point and thus its function field $F\cap E_{\ell_{s+1}}$ must be regular over κ_E. □

Let $\ell\geqslant 5$ be a prime number. We denote by Σ_ℓ the set of isomorphism classes of groups which are either the cyclic group $\mathbb{Z}/\ell$, or the quotient of $\underline{H}(F)$ modulo its center, where F is a finite field of characteristic ℓ and $\underline{H}$ is a connected smooth algebraic group over F which is geometrically simple and simply connected. These are the simple groups of Lie type in characteristic ℓ. It is known (cf. [Ser10, Théorème 5]), that $\Sigma_\ell\cap\Sigma_{\ell'}=\emptyset$ for all primes $5\leqslant\ell<\ell'$. (As Serre points out in [Ser10], the proof of this theorem is essentially due to Artin [Art55]. It was completed in [KLST90].) In the following proof we shall strongly use this result.

End of proof of Theorem 3.4. Let E and I be as in Lemma 3.5. In order to finish up the proof of Theorem 3.4 it suffices to prove the following.

Claim. There is a finite subset $I'\subset\mathbb{L}$ containing I, such that $(E_\ell)_{\ell\in\mathbb{L}\smallsetminus I'}$ is linearly disjoint over E.

In fact, once this claim is proven, it follows that the family $(\rho_\ell)_{\ell\in\mathbb{L}\smallsetminus I'}$ is independent over $\mathrm{Gal}(K_{S,\mathrm{nr}}/E)$ by Remark 3.1, and Remark 3.2 implies that the whole family $(\rho_\ell)_{\ell\in\mathbb{L}}$ must be almost independent, as desired.

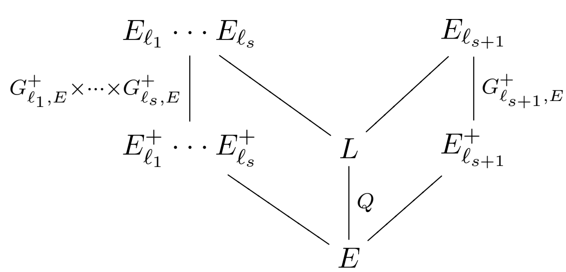
In [Ser10, Théorème 4] Serre proves: *There is a constant C such that for every prime number $\ell>C$ every finite simple subquotient of $GL_n(\mathbb{Z}_\ell)$ of order divisible by ℓ lies in Σ_ℓ.* This is a generalization of a well-known result of Nori (cf. [Nor87, Theorem B]).

Let us define $I' := I \cup \{2,3\} \cup \{\ell \in \mathbb{L} : \ell \leqslant C\}$. For $\ell \in \mathbb{L}$ every non-trivial quotient of $G_{E,\ell}^+$ has order divisible by ℓ: in fact, if $h : G_{E,\ell}^+ \to Q$ is an epimorphism onto a non-trivial group Q, then the image of some ℓ-Sylow subgroup of $G_{E,\ell}$ under h must be non-trivial. Hence, for every $\ell \in \mathbb{L} \smallsetminus I'$ every finite simple quotient of $G_{E,\ell}^+$ lies in Σ_ℓ.

We shall now prove the Claim. Let $s \geqslant 1$ and $\ell_1 < \cdots < \ell_{s+1}$ be elements of $\mathbb{L} \smallsetminus I'$. It suffices to show that $E_{\ell_1} \cdots E_{\ell_s} \cap E_{\ell_{s+1}} = E$, assuming by induction that the sequence $(E_{\ell_1}, \ldots, E_{\ell_s})$ is already linearly disjoint over E. This assumption implies

$$\mathrm{Gal}(E_{\ell_1} \cdots E_{\ell_s}/E) \cong G_{\ell_1,E} \times \cdots \times G_{\ell_s,E}$$

and $\mathrm{Gal}(E_{\ell_1} \cdots E_{\ell_s}/E_{\ell_1}^+ \cdots E_{\ell_s}^+) \cong G_{\ell_1,E}^+ \times \cdots \times G_{\ell_s,E}^+$. Suppose that $E_{\ell_1} \cdots E_{\ell_s} \cap E_{\ell_{s+1}} \neq E$. Then there would be an intermediate field L of that extension such that $Q := \mathrm{Gal}(L/E)$ is a finite simple group. We would have the following diagram of fields.



But L/κ_E is a regular extension (cf. Lemma 3.6), hence $\kappa_L = \kappa_E$. On the other hand $E_{\ell_i}^+/E$ is a constant field extension for every $i = 1, \ldots, s+1$ (cf. Lemma 3.5). It follows that $\mathrm{Gal}(LE_{\ell_{s+1}}^+/E_{\ell_{s+1}}^+) \cong Q$ and $\mathrm{Gal}(LE_{\ell_1}^+ \cdots E_{\ell_s}^+/E_{\ell_1}^+ \cdots E_{\ell_s}^+) \cong Q$. Hence Q is simultaneously a quotient group of $G_{\ell_1,E}^+ \times \cdots \times G_{\ell_s,E}^+$ and of $G_{\ell_{s+1},E}^+$. It follows that

$$Q \in (\Sigma_{\ell_1} \cup \cdots \cup \Sigma_{\ell_s}) \cap \Sigma_{\ell_{s+1}},$$

which contradicts Artin's theorem that $\Sigma_\ell \cap \Sigma_{\ell'} = \emptyset$ for all primes $5 \leqslant \ell < \ell'$. $\square$

4. Proof of the main theorem

Proof of Theorem 1.1. Let K be a finitely generated extension of $\mathbb{Q}$. Let X/K be a separated scheme of finite type. Let $T = (T_1, \ldots, T_r)$ be a transcendence base of $K/\mathbb{Q}$ and S_0 be the normalization of $\mathrm{Spec}(\mathbb{Q}[T])$ in K. Then S_0 is a normal $\mathbb{Q}$-variety with function field K. The spreading-out principles in [EGAIV3] (cf. in particular [EGAIV3, 8.8.2, 8.10.5 and 8.9.4]), allow us to construct a dense open subscheme $S \subset S_0$ and a flat separated morphism of finite type $f : \mathcal{X} \to S$ with generic fibre X.

We choose a closed point $P \in S$ and a closed point $P_{\mathrm{nr}} \in S_{\mathrm{nr}}$ over P and denote by $\overline{P}$: $\mathrm{Spec}(k(P_{\mathrm{nr}})) \to S_{\mathrm{nr}} \to S$ the corresponding geometric point of S. Note that $k(P_{\mathrm{nr}})$ is algebraically closed (cf. the second paragraph after Theorem 3.4). We define $\widetilde{k} := k(P_{\mathrm{nr}})$. Furthermore we denote by $\overline{\xi}$: $\mathrm{Spec}(\widetilde{K}) \to S$ the generic geometric point of S afforded by the choice of $\widetilde{K}$. We let $X_P := \mathcal{X} \times_S k(P)$, $X_{\overline{P}} = \mathcal{X} \times_S \mathrm{Spec}(k(P_{\mathrm{nr}}))$ and $X_{\overline{\xi}} = \mathcal{X} \times_S \mathrm{Spec}(\widetilde{K})$ be the corresponding fibres of $\mathcal{X}$. Note that $X_{\overline{\xi}} = X_{\widetilde{K}}$ and $X_{\overline{P}} = X_{P,\widetilde{k}}$.

Let $q \in \mathbb{N}$. From now on we shall consider two cases. For the first case we define $\rho_\ell := \rho_{\ell,X}^{(q)}$, $T_\ell := \mathrm{H}^q(X_{\overline{\xi}}, \mathbb{Z}_\ell)$, $V_\ell := \mathrm{H}^q(X_{\overline{\xi}}, \mathbb{Q}_\ell)$, $T_{\ell,P} := \mathrm{H}^q(X_{\overline{P}}, \mathbb{Z}_\ell)$, $V_{\ell,P} := \mathrm{H}^q(X_{\overline{P}}, \mathbb{Q}_\ell)$ and $\mathfrak{F}_\ell := \mathrm{R}^q f_*(\mathbb{Z}_\ell)$

for every $\ell \in \mathbb{L}$. For the second case we define $\rho_\ell := \rho^{(q)}_{\ell,X,\mathrm{c}}$, $T_\ell := \mathrm{H}^q_{\mathrm{c}}(X_{\overline{\xi}}, \mathbb{Z}_\ell)$, $V_\ell := \mathrm{H}^q_{\mathrm{c}}(X_{\overline{\xi}}, \mathbb{Q}_\ell)$, $T_{\ell,P} := \mathrm{H}^q_{\mathrm{c}}(X_{\overline{P}}, \mathbb{Z}_\ell)$, $V_{\ell,P} := \mathrm{H}^q_{\mathrm{c}}(X_{\overline{P}}, \mathbb{Q}_\ell)$ and $\mathfrak{F}_\ell := \mathrm{R}^q f_!(\mathbb{Z}_\ell)$ for every $\ell \in \mathbb{L}$. In both cases $\rho_{\ell,P}$ will stand for the representation of $\mathrm{Gal}(\widetilde{k}/k(P))$ on $V_{\ell,P}$.

All residue characteristics of S are zero. Hence there is a dense open subscheme $U \subset S$ such that for every $\ell \in \mathbb{L}$ the $\mathbb{Z}_\ell$-sheaves $\mathrm{R}^q f_*(\mathbb{Z}_\ell)|U$ and $\mathrm{R}^q f_!(\mathbb{Z}_\ell)|U$ are lisse and of formation compatible with any base change $U' \to U$ (cf. [Ill, Corollaire 2.6], [KL86, Théoreme 3.1.2] and [KL86, Théoreme 3.3.2]). Considering the cartesian diagrams

$$\begin{array}{ccc} X_{\overline{P}} & \longrightarrow & \tilde{k} \\ \downarrow & & \downarrow \\ f^{-1}(U) & \longrightarrow & U \end{array} \qquad \begin{array}{ccc} X_{\overline{\xi}} & \longrightarrow & \widetilde{K} \\ \downarrow & & \downarrow \\ f^{-1}(U) & \longrightarrow & U \end{array}$$

we can for every $\ell \in \mathbb{L}$ identify the stalks of $\mathfrak{F}_\ell$ by the following base change isomorphisms

$$\mathfrak{F}_{\ell,\overline{P}} \cong T_{\ell,P} \quad \text{and} \quad \mathfrak{F}_{\ell,\overline{\xi}} \cong T_\ell.$$

The fact that the $\mathbb{Z}_\ell$-sheaves $\mathfrak{F}_\ell|U$ are lisse implies that for every $\ell \in \mathbb{L}$ the representation ρ_ℓ factors through $\pi_1(U)$ and that there is a cospecialization isomorphism $\mathfrak{F}_{\ell,\overline{\xi}} \cong \mathfrak{F}_{\ell,\overline{P}}$. Putting these isomorphisms together and tensoring with $\mathbb{Q}_\ell$ we obtain a cospecialization isomorphism $sp_\ell : V_\ell \cong V_{\ell,P}$ for every $\ell \in \mathbb{L}$. In order to take the Tate twists into account let $\varepsilon_\ell : \mathrm{Gal}(\widetilde{K}/K) \to \mathbb{Q}_\ell^\times$ be the cyclotomic character of Gal_K and by $\varepsilon_{\ell,P} : \mathrm{Gal}(\widetilde{k}/k(P)) \to \mathbb{Q}_\ell^\times$ the cyclotomic character of $\mathrm{Gal}(\widetilde{k}/k(P))$. Let $d \in \mathbb{Z}$ and define $\rho_\ell(d) := \rho_\ell \otimes \varepsilon_\ell^{\otimes d}$ and $\rho_{\ell,P}(d) := \rho_\ell \otimes \varepsilon_{\ell,P}^{\otimes d}$. The cospecialization isomorphism sp_ℓ fits into a commutative diagram

$$\begin{array}{ccc} \mathrm{Gal}(K_{S,\mathrm{nr}}/K) & \xrightarrow{\rho_\ell(d)} & \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell) \\ \uparrow & & \\ D_{K_{S,\mathrm{nr}}/K}(P_{\mathrm{nr}}) & & \downarrow \\ \downarrow & & \\ \mathrm{Gal}(\widetilde{k}/k(P)) & \xrightarrow{\rho_{\ell,P}(d)} & \mathrm{Aut}_{\mathbb{Q}_\ell}(V_{\ell,P}) \end{array}$$

for every $\ell \in \mathbb{L}$.

There is a constant $b \in \mathbb{N}$ such that for every $\ell \in \mathbb{L}$ the inequality $\dim(V_\ell) \leqslant b$ holds true (cf. [Ill, Corollaire 1.3]). Furthermore, if we denote the torsion part of the finitely generated $\mathbb{Z}_\ell$-module T_ℓ by T'_ℓ, then T_ℓ/T'_ℓ injects into V_ℓ and the representation $\rho_\ell(d)$ factors through $\mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell/T'_\ell)$. Hence $\mathrm{im}(\rho_\ell(d))$ (and also $\mathrm{im}(\rho_{\ell,P}(d))$) is isomorphic to a closed subgroup of $\mathrm{GL}_b(\mathbb{Z}_\ell)$ for every $\ell \in \mathbb{L}$. Hence the families $(\rho_\ell(d))_{\ell\in\mathbb{L}}$ and $(\rho_{\ell,P}(d))_{\ell\in\mathbb{L}}$ of representations of $\pi_1(U)$ satisfy assumption a) of Theorem 3.4 (and condition (B) of [Ser10, p. 3]).

Now note that X_P is a separated scheme of finite type over the number field $k := k(P)$. For a place v of a number field we denote by p_v its residue characteristic. There is a finite extension k'/k and a finite set T of places of k' such that the following holds true.

(1) For every place v of k' with $v \notin T$ and every $\ell \in \mathbb{L} \smallsetminus \{p_v\}$ the representation $\rho_{\ell,P}(d)$ is unramified at v.

(2) For every $v \in T$, every extension $\hat{v}$ of v to $\widetilde{k}$ and every $\ell \in \mathbb{L} \smallsetminus \{p_v\}$ the image of the inertia group $I_{\hat{v}}$ under the representation $\rho_{\ell,P}(d)$ is a pro-ℓ group.

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This is shown for $d=0$ in [Ill, Théorème 4.3], and the case $d\neq 0$ follows as well, because the cyclotomic character $\varepsilon_{\ell,P}$ is unramified at every place v of k' with $p_v\neq\ell$. Because the family $(\rho_{\ell,P}(d))_{\ell\in\mathbb{L}}$ satisfies the condition (B) of [Ser10, p. 3] and conditions (1) and (2) and because k is a number field, Serre's theorem [Ser10, Théorème 1] implies that the family $(\rho_{\ell,P}(d))_{\ell\in\mathbb{L}}$ is almost independent. Now the above diagram shows that there is an open subgroup D' of $D_{K_{S,\mathrm{nr}}}(P_{\mathrm{un}})$ such that the restricted family $(\rho_\ell(d)|D')_{\ell\in\mathbb{L}}$ is independent, and our Theorem 3.4 implies that $(\rho_\ell(d))_{\ell\in\mathbb{L}}$ is almost independent as desired. □

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Appendix. Abelian varieties

The aim of this appendix is to give a more elementary direct proof of Corollary 1.2, based on our independence criterion (cf. Theorem 3.4) and on the corresponding results of Serre in the number field case. It avoids the use of étale cohomology.

Proof of Corollary 1.2. Let K be a finitely generated field of characteristic zero. Let A/K be an abelian variety. It is enough to show that the family $(\eta_{\ell,A})_{\ell\in\mathbb{L}}$ defined in the introduction is almost independent. Then Remark 3.1 implies the assertion. There is a normal $\mathbb{Q}$-variety S with function field K and an abelian scheme $f:\mathcal{A}\to S$ with generic fibre A.

Let P_{nr} be a closed point of S_{nr} and P the point of S below P_{nr}. Then the residue field $k(P_{\mathrm{nr}})$ is an algebraic closure of the number field $k(P)$. We define $\tilde{k}:=k(P_{\mathrm{nr}})$. Let $A_P:=\mathcal{A}\times_S \mathrm{Spec}(k(P))$ be the special fibre of $\mathcal{A}$ at P. Then A_P is an abelian variety over the number field $k(P)$.

Let n be an integer. The group scheme $\mathcal{A}[n]$ is finite and étale over S, because all residue characteristics of S are zero. Hence there is a finite extension E/K contained in $K_{S,\mathrm{nr}}$ such that $\mathcal{A}[n]\times_S S^{(E)}$ is a constant group scheme over $S^{(E)}$. In fact one can take $E=K(A[n])$. This implies that both evaluation maps

$$\mathcal{A}[n](S^{(E)})\to A[n](E)\quad\text{and}\quad \mathcal{A}[n](S^{(E)})\to A_P[n](\tilde{k})$$

are isomorphisms. In particular the action of Gal_K on $A[n](\tilde{K})$ factors through $\mathrm{Gal}(K_{S,\mathrm{nr}}/K)$ (and in fact through $\mathrm{Gal}(E/K)$). We obtain a composite isomorphism

$$A[n](\tilde{K})\cong\mathcal{A}[n](S^{(E)})\cong A_P[n](\tilde{k}).$$

Taking limits, we obtain for each $\ell \in \mathbb{L}$ an isomorphism

$$T_\ell(A) \cong T_\ell(A_P)$$

and the action of Gal_K on $T_\ell(A)$ factors through $\mathrm{Gal}(K_{S,\mathrm{nr}}/K)$. This isomorphism fits into a commutative diagram as shown below.

$$\begin{array}{ccc}
\mathrm{Gal}(K_{S,\mathrm{nr}}/K) & \xrightarrow{\eta_{\ell,A}} & \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell(A)) \\
\uparrow & & \Big\downarrow \\
D_{K_{S,\mathrm{nr}}/K}(P_{\mathrm{nr}}) & & \\
\downarrow & & \\
\mathrm{Gal}(k(P_{\mathrm{nr}})/k(P)) & \xrightarrow{\eta_{\ell,A_P}} & \mathrm{Aut}_{\mathbb{Z}_\ell}(T_\ell(A_P))
\end{array}$$

Recall that A_P is an abelian variety over the number field $k(P)$. Hence Serre's theorem (cf. [Ser10, § 3]) implies that the family $(\eta_{\ell,A_P})_{\ell\in\mathbb{L}}$ is almost independent. It follows that there is an open subgroup D' in $D_{K_{S,\mathrm{nr}}/K}(P_{\mathrm{nr}})$ such that the family $(\eta_{\ell,A})_{\ell\in\mathbb{L}}$ is independent over D'. Now, by our Theorem 3.4, the family $(\eta_{\ell,A})_{\ell\in\mathbb{L}}$ must be almost independent, as desired. □

References

Art55 E. Artin, *The orders of the classical simple groups*, Comm. Pure Appl. Math. **8** (1955), 455–472.

EGAII A. Grothendieck, *Éléments de géométrie algébrique (rédigé avec la cooperation de Jean Dieudonné): II. Étude globale élémentaire de quelques classes de morphismes*, Publ. Math. Inst. Hautes Études Sci. **8** (1961), 5–222.

EGAIV3 A. Grothendieck, *Éléments de géométrie algébrique (rédigé avec la cooperation de Jean Dieudonné): IV. Étude locale des schémas et des morphismes des schémas, Troisième partie*, Publ. Math. Inst. Hautes Études Sci. **28** (1966), 5–255.

FJ05 M. D. Fried and M. Jarden, *Field arithmetic*, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge 11*, 2nd revised and enlarged ed (Springer, Berlin, 2005).

Ill L. Illusie, *Constructibilité générique et uniformité enℓ*, Preprint.

KL81 N. Katz and S. Lang, *Finiteness theorems in geometric class field theory*, Enseign. Math. **27** (1981), 285–319.

KL86 N. Katz and G. Laumon, *Transformation de Fourier et majoration de sommes exponentielles*, Publ. Math. Inst. Hautes Études Sci. **62** (1986), 361–418. Erratum: *Publ. Math. Inst. Hautes Études Sci.* **69** (1989), 244.

KLST90 W. Kimmerle, R. Lyons, R. Sandling and D. Teague, *Composition factors from the group ring and Artin's theorem on orders of simple groups*, Proc. Lond. Math. Soc. **60** (1990), 89–122.

Mil80 J. Milne, *Étale cohomology* (Princeton University Press, New Jersey, 1980).

Nor87 M. Nori, *On subgroups of $GL_n(\mathbb{F}_p)$*, Invent. Math. (1987), 257–275.

Ser94 J.-P. Serre, *Propriétés conjecturales des groupes de Galois motiviques et des représentations ℓ-adiques*, Proc. Sympos. Pure Math. (1994), 377–400.

Ser00 J.-P. Serre, *Lettre à Ken Ribet du 7/3/1986*, in *Œuvres. Collected papers, IV, 1985–1998* (Springer, Berlin, 2000).

Ser10 J.-P. Serre, *Une critère d'indépendance pour une famille de représentations ℓ-adiques*, Preprint (2010), available at arXiv:1006.2442.

https://doi.org/10.1112/S0010437X12000711 Published online by Cambridge University Press

SGA4 M. Artin, A. Grothendieck and J.-L. Verdier, *Séminaire de Géomŕie Algébrique du Bois Marie 1963–64 - Théorie des topos et cohomologie étale des schémas (SGA 4)*, Lecture Notes in Mathematics, vol. 269, 270, 305 (Springer, 1972).

SGA7 A. Grothendieck, *Séminaire de Géométrie Algébrique du Bois Marie 1967–69 - Groupes de monodromy en géométrie algébrique (SGA 7), vol. 1*, Lecture Notes in Mathematics, vol. 288 (Springer, Berlin, 1972).

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