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TENSOR PRODUCTS OF DIMENSION GROUPS AND K_0 OF UNIT-REGULAR RINGS

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We study direct limits of finite products of matrix algebras (i.e., locally matricial algebras), their ordered Grothendieck groups (K_0), and their tensor products. Given a dimension group G, a general problem is to determine whether G arises as K_0 of a unit-regular ring or even as K_0 of a locally matricial algebra. If G is countable, this is well known to be true. Here we provide positive answers in case (a) the cardinality of G is $\aleph_1$, or (b) G is an arbitrary infinite tensor product of the groups considered in (a), or (c) G is the group of all continuous real-valued functions on an arbitrary compact Hausdorff space. In cases (a) and (b), we show that G in fact appears as K_0 of a locally matricial algebra. Result (a) is the basis for an example due to de la Harpe and Skandalis of the failure of a determinantal property in a non-separable *AF* C^*-algebra [**18**, Section 3].

Crucial to results (b) and (c) is an analysis of states on tensor products of dimension groups. In particular, we prove that the extremal states on such tensor products are exactly the pure tensors of extremal states on the factors. As a consequence, the extreme boundary of the state space of a tensor product of dimension groups is homeomorphic to the Cartesian product of the extreme boundaries of the state spaces of the factors.

In Section 1, result (a) is derived. The basic construction of infinite (possibly uncountable) tensor products of dimension groups is presented in the second section. In Section 3, the analogous tensor products of locally matricial algebras are discussed, and we show that K_0 preserves tensor products in this context. Result (b) is then an immediate consequence. The fourth section is concerned with the tensor product factorization of extremal states. The final section contains a proof of result (c). We first express an arbitrary compact Hausdorff space as a subspace of a Cartesian product of compact metric spaces; the tensor product results are then applied, and the desired unit-regular ring is obtained as a quotient of a completion.

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All rings and algebras in this paper are unital, as are all ring and algebra maps, and all modules. We refer the reader to [**9**, **10**, **14**] for basic definitions concerning unit-regular rings, partially ordered K_0, and dimension groups.

1. K_0 of locally matricial algebras.

Definition. Let F be a field. A *matricial F-algebra* is any F-algebra that is isomorphic (as an F-algebra) to a finite direct product of full matrix algebras over F. A *locally matricial F-algebra* is any F-algebra that is a direct union of matricial F-subalgebras. Equivalently, an F-algebra is locally matricial if and only if it is isomorphic to a direct limit of matricial F-algebras (in the category of F-algebras and F-algebra maps). An *ultramatricial F-algebra* is any F-algebra that is isomorphic (as an F-algebra) to a direct limit of a countable sequence of matricial F-algebras and F-algebra maps. Equivalently, an F-algebra is ultramatricial if and only if it is locally matricial and countable-dimensional.

Definition. A *unit-regular ring* is a ring R with the property that for each $x \in R$, there exists a unit (i.e., an invertible element) $u \in R$ such that $xux = x$.

For example, all semisimple artinian rings are unit-regular [**4**, Corollary to Theorem 1]. Since unit-regularity is preserved in direct limits, it follows that all locally matricial algebras are unit-regular.

Given a ring R, we make $K_0(R)$ into a pre-ordered abelian group with positive cone $K_0(R)^+$ equal to the collection of all stable isomorphism classes $[A]$ of finitely generated projective right R-modules A, as in [**12**, **9**]. If R is unit-regular, then $K_0(R)$ is actually partially ordered [**12**, Propositions 2.1, 2.2; **9**, Proposition 15.2].

We shall need the concepts of order-units and interpolation groups, which may be found in [**9**, **10**, **14**]. For the concepts of dimension groups, positive homomorphisms, and the category of pre-ordered abelian groups, see [**10**].

Arbitrary direct limits exist in the category of pre-ordered abelian groups [**9**, p. 208; **10**, Proposition 18.6], and the functor K_0 from the category of rings to the category of pre-ordered abelian groups preserves direct limits [**9**, Proposition 15.11; **10**, Proposition 18.7]. Note that any direct limit of dimension groups is a dimension group. Since K_0 of any semisimple artinian ring is isomorphic to a finite direct product of copies of **Z**, with the product ordering [**9**, Lemma 15.22], it is a dimension group. Hence, K_0 of any locally matricial algebra is a dimension group.

For the concepts of normalized positive homomorphisms and the category of pre-ordered abelian groups with order-unit, see [**10**]. We view K_0 as a functor from the category of rings to the category of pre-ordered abelian groups with order-unit, so that K_0 sends a ring R to $(K_0(R), [R])$.

In this setting, K_0 again preserves direct limits [**9**, Proposition 15.11; **10**, Proposition 18.7].

Now K_0 of any unit-regular ring is an interpolation group with order-unit [**17**, p. 197; **14**, Proposition II.10.3], and as we have seen, K_0 of any locally matricial algebra is a dimension group with order-unit. Two basic problems are to determine which interpolation groups with order-unit appear as K_0 of unit-regular rings, and which dimension groups with order-unit appear as K_0 of locally matricial algebras. For countable dimension groups, the problem was solved by Elliott [**5**] and Effros-Handelman-Shen [**3**], as follows.

THEOREM 1.1. *Let F be a field. If (G, u) is a countable dimension group with order-unit, then there exists an ultramatricial F-algebra R such that*

$$(K_0(R), [R]) \cong (G, u).$$

Proof. According to [**3**, Theorem 2.2; **10**, Corollary 21.9], (G, u) is isomorphic to the direct limit of a sequence

$$(G_1, u_1) \rightarrow (G_2, u_2) \rightarrow \dots$$

in the category of pre-ordered abelian groups with order-unit, where each G_i is a finite direct product of copies of $\mathbf{Z}$, with the product ordering. The existence of an ultramatricial F-algebra R with $(K_0(R), [R])$ isomorphic to (G, u) then follows from [**5**, Theorem 5.5; **9**, Theorem 15.24].

With a few modifications in the techniques used to prove Theorem 1.1, this result can be extended to dimension groups of cardinality $\aleph_1$, as follows.

LEMMA 1.2. *Let F be a field, let R be a matricial F-algebra, and let S be a unit-regular F-algebra.*

(a) *If $f : (K_0(R), [R]) \rightarrow (K_0(S), [S])$ is a normalized positive homomorphism, then there exists an F-algebra map $\varphi : R \rightarrow S$ such that $K_0(\varphi) = f$.*

(b) *Let $\varphi, \psi : R \rightarrow S$ be F-algebra maps. Then $K_0(\varphi) = K_0(\psi)$ if and only if there exists an inner automorphism θ of S such that $\varphi = \theta\psi$.*

Proof. See [**9**, Lemma 15.23].

A simple patching argument based on Lemma 1.2 yields the following generalization of part (a) of the lemma.

LEMMA 1.3. *Let F be a field, let R be an ultramatricial F-algebra, and let S be a unit-regular F-algebra. Given any normalized positive homomorphism f from $(K_0(R), [R])$ to $(K_0(S), [S])$, there exists an F-algebra map $\varphi : R \rightarrow S$ such that $K_0(\varphi) = f$.*

Proof. See [**19**, Lemma 3].

LEMMA 1.4. *If G is a dimension group and X is a countable subset of G, then G contains a countable subgroup H such that $H \supseteq X$ and H is a dimension group (under the ordering inherited from G).*

Proof. Let H_1 be the subgroup of G generated by X. Then choose countable subgroups $H_1 \subseteq H_2 \subseteq \dots$ of G such that every suitable quadruple of elements in H_n may be interpolated in H_{n+1}, and such that every element of H_n is a difference of positive elements of H_{n+1}. The union of these H_n is the required subgroup H.

THEOREM 1.5. *Let F be a field, and let (G, u) be a dimension group with order-unit. If* $\mathrm{card}(G) \leqq \aleph_1$, *then there exists a locally matricial F-algebra R such that*

$$(K_0(R), [R]) \cong (G, u).$$

Proof. If G is countable, Theorem 1.1. applies. Hence, we may assume that G has cardinality exactly $\aleph_1$. Let Ω denote the first uncountable ordinal. Then $\mathrm{card}(\Omega) = \mathrm{card}(G)$, and so we may index G by Ω, say

$$G = \{x_\alpha \mid \alpha < \Omega\}.$$

Since each ordinal less than Ω is countable, we may use Lemma 1.4 to construct countable subgroups $\{G_\alpha \mid \alpha < \Omega\}$ in G such that $u \in G_1$ and $x_\alpha \in G_\alpha$ for all $\alpha < \Omega$, each G_α is a dimension group, and $G_\alpha \subseteq G_\beta$ whenever $\alpha \leqq \beta < \Omega$. Note that $\cup G_\alpha = G$.

For all $\alpha \leqq \beta < \Omega$, let $f_{\beta\alpha} : G_\alpha \to G_\beta$ be the inclusion map. The objects (G_α, u) together with the morphisms $f_{\beta\alpha}$ form a direct system in the category of pre-ordered abelian groups with order-unit, and the direct limit of this system is isomorphic to (G, u). We shall construct a corresponding direct system of ultramatricial F-algebras, and form R as the direct limit of that system.

By Theorem 1.1, there exist ultramatricial F-algebras R_α for each $\alpha < \Omega$ and isomorphisms

$$g_\alpha : (K_0(R_\alpha), [R_\alpha]) \to (G_\alpha, u)$$

in the category of pre-ordered abelian groups with order-unit. We construct F-algebra maps

$$\varphi_{\beta\alpha} : R_\alpha \to R_\beta \quad \text{for all } \alpha \leqq \beta < \Omega$$

such that $g_\beta K_0(\varphi_{\beta\alpha}) = f_{\beta\alpha} g_\alpha$ whenever $\alpha \leqq \beta < \Omega$ and $\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \varphi_{\gamma\alpha}$ whenever $\alpha \leqq \beta \leqq \gamma < \Omega$. To start, let φ_{11} be the identity map on R_1.

Now let $1 < \gamma < \Omega$, and assume that $\varphi_{\beta\alpha}$ has been constructed for all $\alpha \leqq \beta < \gamma$. Set $\varphi_{\gamma\gamma}$ equal to the identity map on R_γ. Let T be the direct limit of the direct system

$$\{R_\alpha, \varphi_{\beta\alpha} \mid \alpha \leqq \beta < \gamma\}$$

(in the category of F-algebras), and for all $\alpha < \gamma$ let $\psi_\alpha : R_\alpha \to T$ be the natural map. Then $(K_0(T), [T])$, together with the maps $K_0(\psi_\alpha)$, is a direct limit for the direct system

$$\{ (K_0(R_\alpha), [R_\alpha]), K_0(\varphi_{\beta\alpha}) \mid \alpha \leqq \beta < \gamma\}$$

in the category of pre-ordered abelian groups with order-unit. Note that since γ is countable and each R_α is ultramatricial, T must be ultramatricial.

For each $\alpha < \gamma$, the map $f_{\gamma\alpha}g_\alpha$ is a normalized positive homomorphism from $(K_0(R_\alpha), [R_\alpha])$ to (G_γ, u). Observe that whenever $\alpha \leqq \beta < \gamma$, then

$$f_{\gamma\beta}g_\beta K_0(\varphi_{\beta\alpha}) = f_{\gamma\beta}f_{\beta\alpha}g_\alpha = f_{\gamma\alpha}g_\alpha.$$

Hence, there exists a unique normalized positive homomorphism

$$h : (K_0(t), [T]) \to (G_\gamma, u)$$

such that

$$hK_0(\psi_\alpha) = f_{\gamma\alpha}g_\alpha \quad \text{for all } \alpha < \gamma.$$

By Lemma 1.3, there is an F-algebra map $\varphi : T \to R_\gamma$ such that $K_0(\varphi) = g_\gamma^{-1}h$. For all $\alpha < \gamma$, set

$$\varphi_{\gamma\alpha} = \varphi\psi_\alpha : R_\alpha \to R_\gamma,$$

and observe that

$$g_\gamma K_0(\varphi_{\gamma\alpha}) = g_\gamma K_0(\varphi)K_0(\psi_\alpha) = g_\gamma g_\gamma^{-1}hK_0(\psi_\alpha) = f_{\gamma\alpha}g_\alpha.$$

In addition, for all $\alpha \leqq \beta < \gamma$ we have

$$\varphi_{\gamma\beta}\varphi_{\beta\alpha} = \varphi\psi_\beta\varphi_{\beta\alpha} = \varphi\psi_\alpha = \varphi_{\gamma\alpha}.$$

This completes the inductive step of the construction.

Let R be the direct limit of the direct system

$$\{R_\alpha, \varphi_{\beta\alpha} \mid \alpha \leqq \beta < \Omega\}$$

(in the category of F-algebras), and for all $\alpha < \Omega$ let $\psi_\alpha : R_\alpha \to R$ be the natural map. Since each R_α is ultramatricial, R is locally matricial. Using again the fact that K_0 preserves direct limits, we find that $(K_0(R), [R])$ is isomorphic to the direct limit of the direct system

$$\{ (K_0(R_\alpha), [R_\alpha]), K_0(\varphi_{\beta\alpha}) \mid \alpha \leqq \beta < \Omega\}$$

(in the category of pre-ordered abelian groups with order-unit). As the family of isomorphisms g_α provides an isomorphism of this direct system onto the direct system

$$\{ (G_\alpha, u), f_{\beta\alpha} \mid \alpha \leqq \beta < \Omega\},$$

we conclude that $(K_0(R), [R]) \cong (G, u)$.

Lacking an analogue of Lemma 1.2 (b) for F-algebra maps between locally matricial F-algebras, we cannot extend the argument of Theorem 1.5 to higher cardinalities. However, we do conjecture that all dimension groups with order-unit are isomorphic to $(K_0(R), [R])$'s for locally matricial F-algebras R.

2. Tensor products of dimension groups. In this section, we introduce finite and infinite tensor products of pre-ordered abelian groups and prove that tensor products of dimension groups are dimension groups. The proofs are mostly routine.

Definition. Let $G_1, \ldots, G_n$ be pre-ordered abelian groups, and let G be the abelian group $G_1 \otimes \ldots \otimes G_n$. We make G into a pre-ordered abelian group by defining the positive cone G^+ to be the collection of all sums of elements from the set

$$\{x_1 \otimes \ldots \otimes x_n \mid x_i \in G_i^+ \text{ for all } i = 1, \ldots, n\}.$$

If $G_1, \ldots, G_n$ are all partially ordered, then so is G, as follows.

PROPOSITION 2.1. *Any tensor product of partially ordered abelian groups is a partially ordered abelian group.*

Proof. By induction, the problem reduces to the case of two partially ordered abelian groups G and H. To see that $G \otimes H$ is partially ordered, it suffices to show that whenever $x_1, \ldots, x_n$ are strictly positive elements of G and $y_1, \ldots, y_n$ are strictly positive elements of H, then

$$(x_1 \otimes y_1) + \ldots + (x_n \otimes y_n) \neq 0.$$

Let G' be the subgroup of G generated by $x, \ldots, x_n$, and note that the element

$$u = x_i + \ldots + x_n$$

is an order-unit in G'. Since $u > 0$, there exists a state s on (G', u), by [**12**, Corollary 3.3; **9**, Corollary 18.2]. As

$$s(x_1) + \ldots + s(x_n) = s(u) = 1,$$

we must have $s(x_j) > 0$ for at least one j. Renumber the x_i so that $s(x_i) > 0$ for $i = 1, \ldots, k$ while $s(x_i) = 0$ for $i = k + 1, \ldots, n$. Since $\mathbf{R}$ is divisible, s extends to a homomorphism $g : G \to \mathbf{R}$ (not necessarily positive).

Similarly, there is a homomorphism $h : H \to \mathbf{R}$ such that $h(y_i) \geqq 0$ for $i = 1, \ldots, k$ and $h(y_j) > 0$ for at least one $j \in \{1, \ldots, k\}$. There exists a homomorphism $f : G \otimes H \to \mathbf{R}$ such that

$$f(x \otimes y) = g(x)h(y) \quad \text{for all } x \in G \text{ and } y \in H.$$

Observing that

$$f((x_1 \otimes y_1) + \ldots + (x_n \otimes y_n))$$
$$= s(x_1)h(y_1) + \ldots + s(x_k)h(y_k) > 0,$$

we conclude that $(x_1 \otimes y_1) + \ldots + (x_n \otimes y_n) \neq 0$, as desired.

Note that tensor products of positive homomorphisms are positive homomorphisms. Namely, if $f_i : G_i \to H_i$ is a positive homomorphism between pre-ordered abelian groups, for each $i = 1, \ldots, n$, then the homomorphism

$$f_1 \otimes \ldots \otimes f_n : G_1 \otimes \ldots \otimes G_n \to H_1 \otimes \ldots \otimes H_n$$

is positive.

LEMMA 2.2. *Let $\{G_i, g_{ji}\}$ and $\{H_k, h_{mk}\}$ be direct systems of pre-ordered abelian groups and positive homomorphisms. Then $\{G_i \otimes H_k, g_{ji} \otimes h_{mk}\}$ is a direct system of pre-ordered abelian groups and positive homomorphisms, and the natural map*

$$\varinjlim(G_i \otimes H_k) \to (\varinjlim G_i) \otimes (\varinjlim H_k)$$

is an isomorphism of pre-ordered abelian groups.

Definition. A *simplicial group* is any partially ordered abelian group that is isomorphic (as a partially ordered abelian group) to $\mathbf{Z}^n$ (with the product ordering) for some nonnegative integer n. A *simplicial basis* for a simplicial group G is any basis $\{x_1, \ldots, x_n\}$ for G as a free abelian group such that

$$G^+ = \mathbf{Z}^+ x_1 + \ldots + \mathbf{Z}^+ x_n.$$

(The empty set is considered to be a simplicial basis for the simplicial group $\{0\}$.)

PROPOSITION 2.3. *If $G_1, \ldots, G_n$ are dimension groups, then $G_1 \otimes \ldots \otimes G_n$ is a dimension group.*

Proof. By [**3**, Theorem 2.2; **10**, Theorem 21.7] every dimension group is isomorphic to a direct limit of simplicial groups (in the category of pre-ordered abelian groups). Thus, by Lemma 2.2 and induction on n, we may assume that $n = 2$ and that G_1 and G_2 are simplicial. As the tensor product of simplicial bases for G_1 and G_2 provides a simplicial basis for $G_1 \otimes G_2$, we are done.

LEMMA 2.4. *Let $G_1, \ldots, G_n$ be pre-ordered abelian groups.*
(a) *If $0 \leqq x_i \leqq y_i$ in G_i for each $i = 1, \ldots, n$, then*

$$0 \leqq x_1 \otimes \ldots \otimes x_n \leqq y_1 \otimes \ldots \otimes y_n$$

in $G_1 \otimes \ldots \otimes G_n$.

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(b) *If u_i is an order-unit in G_i for each $i = 1, \ldots, n$, then $u_1 \otimes \ldots \otimes u_n$ is an order-unit in $G_1 \otimes \ldots \otimes G_n$.*

Proof. (a) We have $x_1 \otimes \ldots \otimes x_n \geqq 0$ by definition of $(\otimes G_i)^+$. Since

$$[y_1 \otimes \ldots \otimes y_n] - [x_1 \otimes \ldots \otimes x_n]$$

$$= \sum_{i=1}^{n} [x_1 \otimes \ldots \otimes x_{i-1} \otimes (y_i - x_i) \otimes y_{i+1} \otimes \ldots \otimes y_n]$$

and each of the terms in the summation lies in $(\otimes G_i)^+$, the remaining inequality follows.

(b) We must show that any element x in $\otimes G_i$ is bounded above by a positive multiple of the element $u = u_1 \otimes \ldots \otimes u_n$. Now x is a sum of pure tensors, and it suffices to prove that each of these pure tensors is bounded above by a positive multiple of u. Thus we may assume that $x = x_1 \otimes \ldots \otimes x_n$ for some elements $x_i \in G_i$.

As each G_i has an order-unit, it is directed, and so each x_i is a difference of positive elements of G_i. Hence, x is a sum of terms of the form $\pm(y_1 \otimes \ldots \otimes y_n)$, where each $y_i \in G_i^+$, and we need only show that each of these terms is bounded above by a positive multiple of u. Since

$$-(y_1 \otimes \ldots \otimes y_n) \leqq 0 \leqq u$$

for all $y_i \in G_i^+$, we may thus assume that $x = x_1 \otimes \ldots \otimes x_n$ with each $x_i \in G_i^+$.

Each $x_i \leqq k_i u_i$ for some $k_i \in \mathbf{N}$. Let $k = \max\{k_1, \ldots, k_n\}$, so that each $x_i \leqq k u_i$. Using (a), we conclude that

$$x = x_1 \otimes \ldots \otimes x_n \leqq (ku_1) \otimes \ldots \otimes (ku_n) = k^n u.$$

Definition. Let $\{ (G_i, u_i) \mid i \in I\}$ be a nonempty family of pre-ordered abelian groups with order-unit, and let $\mathcal{A}$ be the family of all nonempty finite subsets of I. Then $\{\mathcal{A}, \subseteq\}$ is a directed set. For all $A \in \mathcal{A}$, set

$$G_A = \bigotimes_{i \in A} G_i \quad \text{and} \quad u_A = \bigotimes_{i \in A} u_i.$$

Then G_A is a pre-ordered abelian group, and u_A is an order-unit in G_A by Lemma 2.4. For all $A \subseteq B$ in $\mathcal{A}$, define a homomorphism $g_{BA} : G_A \to G_B$ such that

$$g_{BA}\left(\bigotimes_{i \in A} x_i\right) = \bigotimes_{j \in B} y_j \quad \text{with}$$

$$y_j = \begin{cases} x_j & (\text{if } j \in A) \\ u_j & (\text{if } j \in B - A) \end{cases}$$

for all pure tensors $\otimes x_i$ in G_A. Then g_{BA} is a normalized positive homomorphism from (G_A, u_A) to (G_B, u_B). The system

$$\{ (G_A, u_A), g_{BA} \mid A \subseteq B \text{ in } \mathscr{A}\}$$

is a direct system in the category of pre-ordered abelian groups with order-unit, and we define the direct limit of this system to be the tensor product of the family $\{ (G_i, u_i) \mid i \in I\}$. Thus

$$\bigotimes_{i \in I}(G_i, u_i) = \varinjlim \{ (G_A, u_A) \mid A \in \mathscr{A}\},$$

for short.

Set

$$(G, u) = \bigotimes_{i \in I}(G_i, u_i),$$

and for all $A \in \mathscr{A}$ let

$$q_A : (G_A, u_A) \to (G, u)$$

be the natural map. Then

$$G = \bigcup_{A \in \mathscr{A}} q_A(G_A).$$

Given $A \in \mathscr{A}$ and a pure tensor $x = \otimes x_i$ in G_A, note that $g_{BA}(x)$ is a pure tensor in G_B for any $B \in \mathscr{A}$ that contains A, and the new factors in $g_{BA}(x)$ are all of the form u_j for $j \in B - A$. Hence, we may view $q_A(x)$ as a "pure infinite tensor", that is, we write

$$q_A(x) = \bigotimes_{i \in I} y_i \quad \text{with} \quad y_i = \begin{cases} x_i \text{ (if } i \in A) \\ u_i \text{ (if } i \in I - A). \end{cases}$$

Note that G is generated, as an abelian group, by these pure infinite tensors. These particular elements of G consist of all symbols of the form $\otimes_{i \in I} z_i$ where $z_i \in G_i$ for all $i \in I$ and $z_i = u_i$ for all but finitely many $i \in I$. In particular,

$$u = \bigotimes_{i \in I} u_i.$$

PROPOSITION 2.5. *If $\{ (G_i, u_i) \mid i \in I\}$ is a nonempty family of dimension groups with order-unit, then $\otimes_{i \in I}(G_i, u_i)$ is a dimension group with order-unit.*

Proof. By construction, $\otimes_{i \in I}(G_i, u_i)$ is a pre-ordered abelian group (G, u) with order-unit. For each nonempty finite subset A of I, the tensor product

$$G_A = \bigotimes_{i \in A} G_i$$

is a dimension group by Proposition 2.3. Since G is the direct limit of the G_A, it too is a dimension group.

3. Tensor products of locally matricial algebras. We now investigate K_0 of tensor products of locally matricial algebras over a fixed field F. As all tensor products of algebras will be taken over F, we just write $\otimes$ in place of $\otimes_F$. Given a nonempty family $\{R_i \mid i \in I\}$ of F-algebras, the tensor product $\otimes_{i \in I} R_i$ is of course the direct limit of the tensor products of finitely many of the R_i, indexed by the nonempty finite subsets of I. As an abelian group $\otimes_{i \in I} R_i$ is generated by all pure infinite tensors $\otimes_{i \in I} r_i$ where $r_i \in R_i$ for all $i \in I$ and $r_i = 1$ for all but finitely many $i \in I$. Since finite tensor products of matricial F-algebras are matricial, we see that all tensor products of locally matricial F-algebras are locally matricial. Our first observation, analogous to Lemma 2.1, is routine.

LEMMA 3.1. *Let $\{R_i, \varphi_{ji}\}$ and $\{S_k, \psi_{mk}\}$ be direct systems of algebras over a field F. Then $\{R_i \otimes S_k, \varphi_{ji} \otimes \psi_{mk}\}$ is a direct system of F-algebras, and the natural map*

$$\varinjlim(R_i \otimes S_k) \to (\varinjlim R_i) \otimes (\varinjlim S_k)$$

is an F-algebra isomorphism.

In order to relate $K_0(\otimes R_i)$ to $\otimes K_0(R_i)$, we consider tensor products of R_i-modules. First, let A be a nonempty finite subset of I, and for each $i \in A$ let P_i be a right R_i-module. The F-vector space $\otimes_{i \in A} P_i$ then becomes a right module over the algebra $\otimes_{i \in A} R_i$ in the obvious manner, so that

$$\left(\bigotimes_{i \in A} x_i\right)\left(\bigotimes_{i \in A} r_i\right) = \bigotimes_{i \in A}(x_i r_i)$$

for all pure tensors $\otimes x_i \in \otimes P_i$ and $\otimes r_i \in \otimes R_i$. Using the natural map from $\otimes_{i \in A} R_i$ to $\otimes_{i \in I} R_i$, the module $\otimes_{i \in A} P_i$ induces a right module over $\otimes_{i \in I} R_i$, namely the module

$$\left(\bigotimes_{i \in A} P_i\right) \otimes_{\left(\bigotimes_{i \in A} R_i\right)} \left(\bigotimes_{i \in I} R_i\right).$$

We denote this induced module by $\otimes_{i \in I} P_i$, where $P_i = R_i$ for all $i \in I - A$. This module is generated, as an abelian group, by all pure infinite tensors $\otimes_{i \in I} x_i$, where $x_i \in P_i$ for all $i \in I$ and $x_i = 1$ for all but finitely many $i \in I$.

LEMMA 3.2. *Let F be a field, and let $\{R_i \mid i \in I\}$ be a nonempty family of F-algebras. For each $i \in I$, let P_i be a finitely generated projective right R_i-module, and assume that $P_i = R_i$ for all but finitely many $i \in I$. Then $\otimes_{i \in I} P_i$ is a finitely generated projective right module over the algebra $\otimes_{i \in I} R_i$.*

Proof. Choose a nonempty finite subset $A \subseteq I$ such that $P_i = R_i$ for all

$i \in I - A$. Then $\otimes_{i \in I} P_i$ is induced from the module $\otimes_{i \in A} P_i$, and so it suffices to show that $\otimes_{i \in A} P_i$ is a finitely generated projective right module over $\otimes_{i \in A} R_i$. Hence, there is no loss of generality in assuming that I is finite, and then induction on cardinality reduces the problem to two-element index sets. Thus we may assume that $I = \{1, 2\}$.

For each $i \in I$, choose a right R_i-module Q_i such that $P_i \oplus Q_i$ is a free right R_i-module F_i of finite rank n_i. Note that $F_1 \otimes F_2$ is a free right $(R_1 \otimes R_2)$-module of rank $n_1 n_2$. Since

$$(P_1 \otimes P_2) \oplus (P_1 \otimes Q_2) \oplus (Q_1 \otimes P_2) \oplus (Q_1 \otimes Q_2) \cong F_1 \otimes F_2,$$

we conclude that $P_1 \otimes P_2$ is a finitely generated projective right module over $R_1 \otimes R_2$, as desired.

LEMMA 3.3. *Let F be a field, and let $\{R_i \mid i \in I\}$ be a nonempty family of F-algebras. Then there is a unique normalized positive homomorphism*

$$f : \bigotimes_{i \in I} (K_0(R_i), [R_i]) \rightarrow \left(K_0\left(\bigotimes_{i \in I} R_i\right), \left[\bigotimes_{i \in I} R_i\right]\right)$$

such that whenever P_i is a finitely generated projective right R_i-module for each $i \in I$, and $P_i = R_i$ for all but finitely many $i \in I$, then

$$f\left(\bigotimes_{i \in I} [P_i]\right) = \left[\bigotimes_{i \in I} P_i\right].$$

Proof. Because of the compatible definitions of $\otimes_{i \in I}(K_0(R_i), [R_i])$ and $\otimes_{i \in I} R_i$ as direct limits of finite tensor products, and because K_0 preserves direct limits, it suffices to construct the corresponding homomorphisms for all tensor products over finite subsets of I. Then induction on cardinality reduces the problem to two-element index sets. Thus we may assume that $I = \{1, 2\}$.

Set $R = R_1 \otimes R_2$. We claim that if P_i and Q_i are finitely generated projective right R_i-modules for each $i \in I$, and if $[P_i] = [Q_i]$ in $K_0(R_i)$ for each i, then

$$[P_1 \otimes P_2] = [Q_1 \otimes Q_2] \quad \text{in } K_0(R).$$

Each P_i is stably isomorphic to Q_i, and so there is a free right R_i-module F_i of finite rank such that

$$P_i \oplus F_i \cong Q_i \oplus F_i.$$

Now

$$(P_1 \otimes P_2) \oplus (F_1 \otimes P_2) \cong (P_1 \oplus F_1) \otimes P_2 \cong (Q_1 \oplus F_1) \otimes P_2$$
$$\cong (Q_1 \otimes P_2) \oplus (F_1 \otimes P_2),$$

whence $P_1 \otimes P_2$ and $Q_1 \otimes P_2$ are stably isomorphic. Similarly, $Q_1 \otimes P_2$

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and $Q_1 \otimes Q_2$ are stably isomorphic, and hence

$$[P_1 \otimes P_2] = [Q_1 \otimes P_2] = [Q_1 \otimes Q_2]$$

in $K_0(R)$, as claimed.

Thus there is a well-defined map

$$g : K_0(R_1)^+ \times K_0(R_2)^+ \to K_0(R)$$

such that

$$g([P_1], [P_2]) = [P_1 \otimes P_2]$$

for all finitely generated projective right R_i-modules P_i. Clearly g is biadditive, and so g extends to a biadditive map

$$g' : K_0(R_1) \times K_0(R_2) \to K_0(R).$$

Because g' is biadditive, it induces a group homomorphism

$$f : K_0(R_1) \otimes K_0(R_2) \to K_0(R)$$

such that

$$f([P_1] \otimes [P_2]) = [P_1 \otimes P_2]$$

for all finitely generated projective right R_i-modules P_i. It is obvious that we obtain a normalized positive homomorphism

$$f : (K_0(R_1), [R_1]) \otimes (K_0(R_2), [R_2]) \to (K_0(R), [R]),$$

and that f is unique.

Definition. We refer to the map f constructed in Lemma 3.3. as *the natural map from* $\bigotimes_{i \in I}(K_0(R_i), [R_i])$ *to* $(K_0(\bigotimes_{i \in I} R_i), [\bigotimes_{i \in I} R_i])$.

PROPOSITION 3.4. *Let F be a field, and let $\{R_i \mid i \in I\}$ be a nonempty family of locally matricial F-algebras. Then the natural map*

$$f : \bigotimes_{i \in I}(K_0(R_i), [R_i]) \to \left(K_0\left(\bigotimes_{i \in I} R_i\right), \left[\bigotimes_{i \in I} R_i\right]\right)$$

is an isomorphism of pre-ordered abelian groups with order-unit.

Proof. As in Lemma 3.3, it suffices to prove the case in which $I = \{1, 2\}$.

We may assume that R_1 is a direct limit of matricial F-algebras S_{1j}, and that R_2 is a direct limit of matricial F-algebras S_{2k}. Set

$$R = R_1 \otimes R_2 \quad \text{and} \quad S = \varinjlim(S_{1j} \otimes S_{2k}).$$

By Lemma 3.1, the natural map $\varphi : S \to R$ is an F-algebra isomorphism, and so φ induces an isomorphism

$$K_0(\varphi) : (K_0(S), [S]) \rightarrow (K_0(R), [R])$$

of pre-ordered abelian groups with order-unit. Since K_0 preserves direct limits, we also have a natural isomorphism

$$g : \varinjlim(K_0(S_{1j} \otimes S_{2k}), [S_{1j} \otimes S_{2k}]) \rightarrow (K_0(S), [S]),$$

as well as natural isomorphisms

$$h_1 : \varinjlim(K_0(S_{1j}), [S_{1j}]) \rightarrow (K_0(R_1), [R_1])$$

$$h_2 : \varinjlim(K_0(S_{2k}), [S_{2k}]) \rightarrow (K_0(R_2), [R_2]).$$

Tensoring h_1 with h_2 provides an isomorphism

$$h : \left(\varinjlim(K_0(S_{1j}), [S_{1j}])\right) \otimes \left(\varinjlim(K_0(S_{2k}), [S_{2k}])\right)$$

$$\rightarrow (K_0(R_1) [R_1]) \otimes (K_0(R_2), [R_2]).$$

Finally, Lemma 2.2 shows that the natural map

$$t : \varinjlim((K_0(S_{1j}), [S_{1j}]) \otimes (K_0(S_{2k}), [S_{2k}]))$$

$$\rightarrow \left(\varinjlim(K_0(S_{1j}), [S_{1j}])\right) \otimes \left(\varinjlim(K_0(S_{2k}), [S_{2k}])\right)$$

is an isomorphism.

For all j, k, let f_{jk} denote the natural map

$$(K_0(S_{1j}), [S_{1j}]) \otimes (K_0(S_{2k}), [S_{2k}]) \rightarrow (K_0(S_{1j} \otimes S_{2k}), [S_{1j} \otimes S_{2k}]).$$

Since these f_{jk} are compatible, they induce a normalized positive homomorphism

$$\bar{f} : \varinjlim((K_0(S_{1j}), [S_{1j}]) \otimes (K_0(S_{2k}), [S_{2k}]))$$

$$\rightarrow \varinjlim(K_0(S_{1j} \otimes S_{2k}), [S_{1j} \otimes S_{2k}]).$$

Observe that $fht = K_0(\varphi)g\bar{f}$. Thus to prove that f is an isomorphism (of pre-ordered abelian groups with order-unit), we need only show that $\bar{f}$ is an isomorphism, and for that it suffices to show that each f_{jk} is an isomorphism.

Therefore we may assume, without loss of generality, that R_1 and R_2 are matricial. Hence, we may identify R_1 with $S_1 \times \ldots \times S_m$ and R_2 with

$T_1 \times \ldots \times T_n$, where the S_j and the T_k are full matrix algebras over F. Since K_0 preserves finite direct products [**9**, Proposition 15.13], another diagram chase reduces the problem to proving that each of the natural maps

$$(K_0(S_j), [S_j]) \otimes (K_0(T_k), [T_k]) \to (K_0(S_j \otimes T_k), [S_j \otimes T_k])$$

is an isomorphism.

Thus there is no loss of generality in assuming that $R_1 = M_p(F)$ and $R_2 = M_q(F)$ for some positive integers p and q. Let e_1 and e_2 be rank one idempotent matrices in R_1 and R_2. There is an isomorphism of $R_1 \otimes R_2$ onto $M_{pq}(F)$, under which the element $e = e_1 \otimes e_2$ corresponds to a rank one idempotent matrix in $M_{pq}(F)$. By [**9**, Lemma 15.22], the groups $K_0(R_i)$ and $K_0(R_1 \otimes R_2)$ are all infinite cyclic, with generators $[e_iR_i]$ and $[e(R_1 \otimes R_2)]$, while also

$$K_0(R_i)^+ = \mathbf{Z}^+[e_iR_i] \quad \text{and} \quad K_0(R_1 \otimes R_2)^+ = \mathbf{Z}^+[e(R_1 \otimes R_2)].$$

Observing that

$$f([e_1R_1] \otimes [e_2R_2]) = [(e_1R_1) \otimes (e_2R_2)] = [e(R_1 \otimes R_2)],$$

we conclude that f is an isomorphism of pre-ordered abelian groups. As f is already normalized, the proof is complete.

THEOREM 3.5. *Let F be a field, and let $\{(G_i, u_i) \mid i \in I\}$ be a nonempty family of dimension groups with order-unit. If* $\text{card}(G_i) \leqq \aleph_1$ *for all $i \in I$, then there exists a locally matricial F-algebra R such that*

$$(K_0(R), [R]) \cong \bigotimes_{i \in I} (G_i, u_i).$$

Proof. Use Theorem 1.5 and Proposition 3.4.

4. State spaces of tensor products of dimension groups. Here we show that any extremal state on a tensor product of dimension groups is a pure tensor of extremal states on the factors. The concepts of states, state spaces, extreme points, and extreme boundaries may be found in [**9**, **14**]. We use $\partial_e S$ to denote the extreme boundary of a convex set S.

The state space $S(G, u)$ of a pre-ordered abelian group (G, u) with order-unit is viewed as a subset of the real vector space $\mathbf{R}^G$ of all real-valued functions on G, and $\mathbf{R}^G$ is assumed to have the product topology. Then $S(G, u)$ is a compact convex subset of $\mathbf{R}^G$ [**9**, Proposition 17.11]. If G is an interpolation group, then $S(G, u)$ is a Choquet simplex [**14**, Theorem I.2.5].

Definition. Let $\{(G_i, u_i) \mid i \in I\}$ be a nonempty family of pre-ordered abelian groups with order-unit, and set

$$(G, u) = \bigotimes_{i \in I} (G_i, u_i).$$

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Given states $s_i \in S(G_i, u_i)$ for all $i \in I$, there is a unique homomorphism $s : G \to \mathbf{R}$ such that

$$s\left(\bigotimes_{i \in I} x_i\right) = \prod_{i \in I} s_i(x_i)$$

for all pure tensors $\otimes x_i$ in G, and we observe that s is a state on (G, u). We refer to s as the *tensor product* of the states s_i, denoted

$$s = \bigotimes_{i \in I} s_i.$$

LEMMA 4.1. *Let (G_1, u_1) and (G_2, u_2) be pre-ordered abelian groups with order-unit, and set*

$$(G, u) = (G_1, u_1) \otimes (G_2, u_2).$$

For $i = 1, 2$, let

$$q_i : (G_i, u_i) \to (G, u)$$

be the natural map. Let $s \in S(G, u)$, and set $s_i = sq_i$ for $i = 1, 2$. If s_1 is extremal, then $s = s_1 \otimes s_2$.

Proof. It suffices to show that $s(x_1 \otimes x_2) = s_1(x_1)s_2(x_2)$ for any pure tensor $x_1 \otimes x_2$ in G. Choose a positive integer m such that $x_2 \leqq mu_2$, and set

$$y_1 = (m + 1)u_2 \quad \text{and} \quad y_2 = y_1 - x_2.$$

Since u_2 is an order-unit in G_2, so is y_1. Also, since $y_2 \geqq u_2$, we see that y_2 is an order-unit in G_2. Now $x_2 = y_1 - y_2$, whence

$$x_1 \otimes x_2 = (x_1 \otimes y_1) - (x_1 \otimes y_2).$$

We need only show that

$$s(x_1 \otimes y_j) = s_1(x_1)s_2(y_j) \quad \text{for each } j = 1, 2.$$

Thus we may assume that x_2 is an order-unit in G_2.

As x_2 is an order-unit, we obtain $s_2(x_2) > 0$. Setting

$$t_1(a) = s(a \otimes x_2)/s_2(x_2) = s(a \otimes x_2)/s(u_1 \otimes x_2)$$

for all $a \in G_1$, we obtain a state t_1 in $S(G_1, u_1)$. Choose a positive integer n such that $x_2 \leqq nu_2$. For all $a \in G_1^+$, we have

$$a \otimes x_2 \leqq n(a \otimes u_2),$$

whence

$$t_1(a) \leqq ns(a \otimes u_2)/s_2(x_2) = ns_1(a)/s_2(x_2).$$

Thus $t_1 \leqq [n/s_2(x_2)]s_1$ on G_1. According to [**14**, Proposition I.2.4], t_1 must lie in the face generated by s_1 in $S(G_1, u_1)$. Since s_1 is extremal, the face it generates is just the singleton $\{s_1\}$, and so $t_1 = s_1$. Thus

$$s_1(x_1) = t_1(x_1) = s(x_1 \otimes x_2)/s_2(x_2),$$

and therefore

$$s(x_1 \otimes x_2) = s_1(x_1)s_2(x_2),$$

as desired.

PROPOSITION 4.2. *Let* $\{(G_i, u_i) \mid i \in I\}$ *be a nonempty family of pre-ordered abelian groups with order-unit. Set*

$$(G, u) = \bigotimes_{i \in I}(G_i, u_i),$$

and for all $i \in I$ *let*

$$q_i : (G_i, u_i) \to (G, u)$$

be the natural map. Let $s \in S(G, u)$, *and set* $s_i = sq_i$ *for all* $i \in I$. *If each* s_i *is extremal, then*

$$s = \bigotimes_{i \in I} s_i,$$

and s *is extremal.*

Proof. We first show that $s = \otimes s_i$. Thus consider any pure tensor $x = \otimes x_i$ in G. There exists a finite subset $J \subseteq I$ such that $x_i = u_i$ for all $i \in I - J$. Set

$$(G', u') = \bigotimes_{j \in J}(G_j, u_j) \quad \text{and} \quad (G'', u'') = \bigotimes_{i \in I-J}(G_i, u_i)$$

and identify (G, u) with $(G', u') \otimes (G'', u'')$. Using Lemma 4.1, we infer by induction on the cardinality of J that

$$s = \left(\bigotimes_{j \in J} s_j\right) \otimes t$$

for some state t in $S(G'', u'')$. Consequently,

$$s(x) = \left[\prod_{j \in J} s_j(x_j)\right]\left[t\left(\bigotimes_{i \in I-J} x_i\right)\right] = \left[\prod_{j \in J} s_j(x_j)\right][t(u'')]$$
$$= \prod_{j \in J} s_j(x_j) = \prod_{i \in I} s_i(x_i) = \left(\bigotimes_{i \in I} s_i\right)(x).$$

Therefore $s = \otimes s_i$.

It remains to show that s is extremal. Given any positive convex combination $s = \alpha s' + (1 - \alpha)s''$ in $S(G, u)$, we obtain positive convex combinations

$$s_i = \alpha(s'q_i) + (1 - \alpha)(s''q_i)$$

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in each $S(G_i, u_i)$. As each s_i is extremal,

$$s'q_i = s''q_i = s_i \quad \text{for all } i \in I.$$

Now each $s'q_i$ and each $s''q_i$ is extremal. Applying the result of the previous paragraph, we conclude that

$$s' = \bigotimes_{i \in I} s'q_i = \bigotimes_{i \in I} s_i = s,$$

and similarly $s'' = s$. Therefore s is extremal.

In proving a converse to Proposition 4.2 (namely, that if s is extremal then each s_i is extremal), we shall require that each G_i be an interpolation group. Our proof involves completing interpolation groups with respect to state-metrics, as in [**13**], to which we refer the reader for the construction of such completions and the associated terminology.

LEMMA 4.3. *Let (G_1, u_1) and (G_2, u_2) be pre-ordered abelian groups with order-unit. Set*

$$(G, u) = (G_1, u_1) \otimes (G_2, u_2),$$

and let

$$q_1 : (G_1, u_1) \rightarrow (G, u)$$

be the natural map. Let $s \in S(G, u)$, and set $s_1 = sq_1$. Let $\overline{G}_1$ denote the s_1-completion of G_1, and let $\overline{s}_1$ be the natural extension of s_1 to $\overline{G}_1$. Let $\varphi_1 : G_1 \rightarrow \overline{G}_1$ be the natural map, and let H_1 be the convex subgroup of $\overline{G}_1$ generated by $\varphi_1(u_1)$. Set

$$(H, v) = (H_1, \varphi_1(u_1)) \otimes (G_2, u_2),$$

and let j be the identity map on G_2. Then there exists a state t on (H, v) such that

$$t(\varphi_1 \otimes j) = s \quad \text{and} \quad t(x \otimes u_2) = \overline{s}_1(x) \quad \text{for all } x \in H_1.$$

Proof. For each $y \in G_2$, define a homomorphism $f_y : G_1 \rightarrow \mathbf{R}$ according to the rule

$$f_y(x) = s(x \otimes y).$$

We claim that f_y is uniformly continuous with respect to the s_1-metric.

Choose a positive integer m such that $-mu_2 \leqq y \leqq mu_2$. For all $a \in G_1^+$, we have

$$-m(a \otimes u_2) = a \otimes (-mu_2) \leqq a \otimes y \leqq a \otimes (mu_2) = m(a \otimes u_2),$$

and consequently

$$-ms_1(a) = -ms(a \otimes u_2) \leqq s(a \otimes y) \leqq ms(a \otimes u_2) = ms_1(a),$$

so that

$$|s(a \otimes y)| \leqq ms_1(a).$$

Now consider an arbitrary element $x \in G_1$. Whenever $x = a - b$ for some $a, b, \in G_1^+$, we have

$$|f_y(x)| = |s(a \otimes y) - s(b \otimes y)| \leqq |s(a \otimes y)| + |s(b \otimes y)|$$
$$\leqq ms_1(a) + ms_1(b) = ms_1(a + b).$$

Hence,

$$|f_y(x)| \leqq m|x|_{s_1}.$$

Thus f_y is uniformly continuous with respect to the s_1-metric, as claimed.

As a result, f_y extends uniquely to a continuous homomorphism $g_y : \bar{G}_1 \to \mathbf{R}$ such that $g_y\varphi_1 = f_y$. For any $y, z \in G_2$, observe that $f_{y+z} = f_y + f_z$, whence $g_{y+z} = g_y + g_z$ by the uniqueness of g_{y+z}. Moreover, for any $y \in G_2^+$, the map f_y is positive (because the maps $(-)\otimes y$ and s are positive), from which it follows that g_y is a positive homomorphism [**13**, Lemma 2.2].

Define a map $g : H_1 \times G_2 \to \mathbf{R}$ according to the rule $g(x, y) = g_y(x)$, and observe that g is biadditive. Then g induces a homomorphism $t : H \to \mathbf{R}$ such that

$$t(x \otimes y) = g_y(x) \quad \text{for all } x \in H_1 \text{ and } y \in G_2.$$

Whenever $x \in H_1^+$ and $y \in G_2^+$, we have $t(x \otimes y) \geqq 0$ because g_y is a positive homomorphism. Thus t is a positive homomorphism. For all $x \in G_1$ and $y \in G_2$, we compute that

$$t(\varphi_1 \otimes j)(x \otimes y) = t(\varphi_1(x) \otimes y) = g_y\varphi_1(x) = f_y(x) = s(x \otimes y).$$

Therefore $t(\varphi_1 \otimes j) = s$. In particular,

$$t(v) = t(\varphi_1(u_1) \otimes u_2) = t(\varphi_1 \otimes j)(u_1 \otimes u_2) = s(u) = 1,$$

and so t is a state on (H, v).

For all $x \in G_1$, observe that

$$f_{u_2}(x) = s(x \otimes u_2) = s_1(x).$$

Consequently,

$$\bar{s}_1\varphi_1 = s_1 = f_{u_2}$$

and so $\bar{s}_1 = g_{u_2}$. Thus we conclude that

$$t(x \otimes u_2) = g_{u_2}(x) = \bar{s}_1(x)$$

for all $x \in H_1$.

PROPOSITION 4.4. *Let (G_1, u_1) be an interpolation group with order-unit, and let (G_2, u_2) be a pre-ordered abelian group with order-unit. Set*

$$(G, u) = (G_1, u_1) \otimes (G_2, u_2),$$

and for $i = 1, 2$ *let*

$$q_i : (G_i, u_i) \rightarrow (G, u)$$

be the natural map. Let $s \in S(G, u)$, *and set* $s_1 = sq_1$. *If* s *is extremal, then* s_1 *is extremal.*

Proof. We continue the notation of Lemma 4.3, and we set $\varphi = \varphi_1 \otimes j$. By [**13**, Theorem 1.6], $\overline{G}_1$ is Dedekind complete, whence H_1 is Dedekind complete.

If s_1 is not extremal, then H_1 is not totally ordered [**13**, Theorem 2.3]. Consequently, there must exist a nontrivial characteristic element e in $B(H_1, \varphi_1(u_1))$ [**14**, Theorem I.4.3]. Set

$$f = \varphi_1(u_1) - e,$$

so that e and f are strictly positive characteristic elements of $(H_1, \varphi_1(u_1))$, and

$$e + f = \varphi_1(u_1).$$

Now $p_e \otimes j$ and $p_f \otimes j$ are positive homomorphisms from H to itself, and their sum is the identity map on H, whence

$$t(p_e \otimes j)\varphi + t(p_f \otimes j)\varphi = t\varphi = s.$$

Set $\alpha = \overline{s}_1(e)$ and $\beta = \overline{s}_1(f)$. Since $e > 0$ and $f > 0$, we obtain

$$\alpha = |e|_{\overline{s}_1} > 0 \quad \text{and} \quad \beta = |f|_{\overline{s}_1} > 0$$

from [**13**, Lemma 1.1 (d)]. In addition, $\alpha + \beta = \overline{s}_1\varphi_1(u_1) = 1$. Observe that

$$\begin{aligned} t(p_e \otimes j)\varphi(u) &= t(p_e \otimes j)(\varphi_1(u_1) \otimes u_2) = t(p_e\varphi_1(u_1) \otimes u_2) \\ &= t(e \otimes u_2) = \overline{s}_1(e) = \alpha, \end{aligned}$$

and similarly $t(p_f \otimes j)\varphi(u) = \beta$. Hence, the maps

$$t' = \alpha^{-1}t(p_e \otimes j)\varphi \quad \text{and} \quad t'' = \beta^{-1}t(p_f \otimes j)\varphi$$

are states on (G, u), and $\alpha t' + \beta t'' = s$.

As s is extremal, we must have $t' = t'' = s$. For all $x \in G_1$, we compute that

$$\begin{aligned} t'q_1(x) &= \alpha^{-1}t(p_e \otimes j)\varphi(x \otimes u_2) = \alpha^{-1}t(p_e\varphi_1(x) \otimes u_2) \\ &= \alpha^{-1}\overline{s}_1 p_e\varphi_1(x), \end{aligned}$$

and similarly $t''q_1(x) = \beta^{-1}\overline{s}_1 p_f\varphi_1(x)$. Thus

$$\alpha^{-1}\overline{s}_1 p_e\varphi_1 = \beta^{-1}\overline{s}_1 p_f\varphi_1.$$

Since $\bar{s}_1 p_e(a) \leqq \bar{s}_1(a)$ and $\bar{s}_1 p_f(a) \leqq \bar{s}_1(a)$ for all $a \in H_1^+$, [**13**, Lemma 2.1] shows that $\bar{s}_1 p_e$ and $\bar{s}_1 p_f$ are continuous with respect to the $\bar{s}_1$-metric. As $\varphi_1(G_1)$ is dense in H_1, we conclude that

$$\alpha^{-1}\bar{s}_1 p_e = \beta^{-1}\bar{s}_1 p_f.$$

However,

$$\alpha^{-1}\bar{s}_1 p_e(e) = \alpha^{-1}\bar{s}_1(e) = 1$$

while

$$\beta^{-1}\bar{s}_1 p_f(e) = 0,$$

and so we have a contradiction.

Therefore s_1 must be extremal.

THEOREM 4.5. *Let $\{ (G_i, u_i) \mid i \in I\}$ be a nonempty family of interpolation groups with order-unit. Set*

$$(G, u) = \bigotimes_{i \in I}(G_i, u_i),$$

and for all $i \in I$ let

$$q_i : (G_i, u_i) \rightarrow (G, u)$$

be the natural map. Let $s \in S(G, u)$, and set $s_i = sq_i$ for all $i \in I$. Then s is extremal if and only if each s_i is extremal, in which case

$$s = \bigotimes_{i \in I} s_i.$$

Proof. If each s_i is extremal, then s is extremal and $s = \otimes s_i$, by Proposition 4.2. Conversely, assume that s is extremal. For any $j \in I$, we may identify (G, u) with

$$(G_j, u_j) \otimes \left(\bigotimes_{i \in I - J}(G_i, u_i)\right).$$

Hence, it follows from Proposition 4.4. that s_j is extremal.

COROLLARY 4.6. *Let $\{ (G_i, u_i) \mid i \in I\}$ be a nonempty family of interpolation groups with order-unit, and set*

$$(G, u) = \bigotimes_{i \in I}(G_i, u_i).$$

Then $\partial_e S(G, u)$ is homeomorphic to

$$\prod_{i \in I} \partial_e S(G_i, u_i).$$

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Proof. For each $i \in I$, let $q_i : (G_i, u_i) \to (G, u)$ be the natural map. The induced map

$$S(q_i) : S(G, u) \to S(G_i, u_i)$$

is continuous, and by Theorem 4.5 it restricts to a map of $\partial_e S(G, u)$ to $\partial_e S(G_i, u_i)$. Together these restrictions induce a continuous map

$$f : \partial_e S(G, u) \to \prod_{i \in I} \partial_e S(G_i, u_i)$$

such that $f(s)_i = sq_i$ for all $s \in \partial_e S(G, u)$ and all $i \in I$.

If $s, t \in \partial_e S(G, u)$ with $f(s) = f(t)$, then Theorem 4.5 shows that

$$s = \bigotimes_{i \in I}(sq_i) = \bigotimes_{i \in I}(tq_i) = t.$$

Thus f is injective. Given $s_i \in \partial_e S(G_i, u_i)$ for all $i \in I$, we may form the state

$$s = \bigotimes_{i \in I} s_i \quad \text{in } S(G, u).$$

As $sq_i = s_i$ for all $i \in I$, we see by Theorem 4.5 that $s \in \partial_e S(G, u)$, and then

$$f(s) = (s_i \mid i \in I).$$

Thus f is surjective.

Now f is a bijection, and we may describe f^{-1} according to the rule

$$f^{-1}(s) = \bigotimes_{i \in I} s_i$$

for any $s = (s_i \mid i \in I)$ in $\prod \partial_e S(G_i, u_i)$. Consider a convergent net $s^k \to s$ in $\prod \partial_e S(G_i, u_i)$. For each $i \in I$, we have $s_i^k \to s_i$, so that

$$s_i^k(x_i) \to s_i(x_i) \quad \text{for all } x_i \in G_i.$$

Given any pure tensor $x = \otimes x_i$ in G, there is a finite subset $J \subseteq I$ such that $x_i = u_i$ for all $i \in I - J$. Hence,

$$f^{-1}(s^k)(x) = \prod_{j \in J} s_j^k(x_j) \to \prod_{j \in J} s_j(x_j) = f^{-1}(s)(x).$$

As every element of G is a finite sum of pure tensors, we conclude that

$$f^{-1}(s^k)(y) \to f^{-1}(s)(y)$$

for all $y \in G$, whence

$$f^{-1}(s^k) \to f^{-1}(s).$$

Therefore f^{-1} is continuous, and so f is a homeomorphism.

5. Pseudo-rank functions. Here we apply the results of the previous sections to the problem of realizing $C(X, \mathbf{R})$ as $K_0(R)$, where X is a compact Hausdorff space and R is a unit-regular algebra. As a consequence, the probability measure simplex $M_1^+(X)$ is realized as the simplex $\mathbf{P}(R)$ of pseudo-rank functions on R. When X is a direct product of compact metric spaces, we show that in fact $M_1^+(X)$ can be obtained as $\mathbf{P}(R)$ for R a central simple locally matricial algebra.

For the concept of pseudo-rank functions on a regular ring R, see [**9**]. We use $\mathbf{P}(R)$ to denote the set of all pseudo-rank functions on R. The set $\mathbf{P}(R)$ is viewed as a subset of the real vector space $\mathbf{R}^R$ of all real-valued functions on R, and $\mathbf{R}^R$ is assumed to have the product topology. Then $\mathbf{P}(R)$ is a compact convex subset of $\mathbf{R}^R$ [**6**, pp. 270, 273; **9**, Proposition 16.17], and $\mathbf{P}(R)$ is a Choquet simplex [**7**, Corollary 3.6; **9**, Theorem 17.5]. In addition, $\mathbf{P}(R)$ is affinely homeomorphic to the state space of $(K_0(R), [R])$ [**9**, Proposition 17.12].

Definition. Given a compact Hausdorff space X, we use $M_1^+(X)$ to denote the set of all probability measures on X.

By means of the Riesz Representation Theorem, $M_1^+(X)$ is identified with a subset of the dual of the real Banach space $C(X, \mathbf{R})$, and we assume that $M_1^+(X)$ has the weak* topology from $C(X, \mathbf{R})^*$. Then $M_1^+(X)$ is a Choquet simplex, and $\partial_e M_1^+(X)$ is homeomorphic to X [**1**, Corollary II.4.2]. Conversely, if K is any Choquet simplex for which $\partial_e K$ is compact, then K is affinely homeomorphic to $M_1^+(\partial_e K)$ [**1**, Corollary II.4.2]. Note that $M_1^+(X)$ is the state space of $(C(X, \mathbf{R}), 1)$.

THEOREM 5.1. *Let F be a field, and let X be any nonempty direct product of compact metric spaces. Then there exists a central simple locally matricial F-algebra R such that* $\mathbf{P}(R)$ *is affinely homeomorphic to* $M_1^+(X)$.

Proof. There is a nonempty family $\{X_i \mid i \in I\}$ of compact metric spaces such that

$$X = \prod_{i \in I} X_i.$$

For each $i \in I$, there exists a simple ultramatricial F-algebra R_i such that $\mathbf{P}(R_i)$ is affinely homeomorphic to $M_1^+(X_i)$ [**8**, Corollary 5.2; **9**, Corollary 17.24]. (Alternatively, use Lemma 1.4 and the separability of $C(X_i, \mathbf{R})$ to choose a dense countable subgroup G_i of $C(X_i, \mathbf{R})$ such that $1 \in G_i$ and G_i is a dimension group, and apply Theorem 1.1 to $(G_i, 1)$.) Thus $\partial_e \mathbf{P}(R_i)$ is homeomorphic to X_i.

We claim that the center of R_i is F. Any central element $x \in R_i$ is in the center of some matricial subalgebra of R_i. Hence, x is algebraic over F and its minimal polynomial is a product of linear terms. Since R_i is simple, x must be scalar.

Set

$$R = \bigotimes_{i \in I} R_i.$$

Since each R_i is a central simple locally matricial F-algebra, so is R. Now

$$(K_0(R), [R]) \cong \bigotimes_{i \in I} (K_0(R_i), [R_i])$$

by Proposition 3.4, and so

$$\begin{aligned} \partial_e \mathbf{P}(R) &\approx \partial_e S(K_0(R), [R]) \approx \prod_{i \in I} \partial_e S(K_0(R_i), [R_i]) \\ &\approx \prod_{i \in I} \partial_e \mathbf{P}(R_i) \approx \prod_{i \in I} X_i = X \end{aligned}$$

(where $\approx$ denotes homeomorphism), because of Corollary 4.6. As $\mathbf{P}(R)$ is a Choquet simplex, we conclude from [**1**, Corollary II.4.2] that $\mathbf{P}(R)$ is affinely homeomorphic to $M_1^+(X)$.

Definition. Let R be a regular ring such that $\mathbf{P}(R)$ is nonempty. For all $x \in R$, define

$$N^*(x) = \sup\{N(x) \mid N \in \mathbf{P}(R)\}.$$

The rule $d(x, y) = N^*(x - y)$ then defines a pseudo-metric d on R [**11**, Lemma 1.2], known as the N^*-*metric*. If d is actually a metric, and if R is complete with respect to d, then R is said to be N^*-*complete*. In general, the N^*-*completion* of R is the (Hausdorff) completion of R with respect to d. As observed in [**15**, Proposition 14], the N^*-completion of R is a regular ring; moreover, the N^*-completion of R is N^*-complete (with respect to its own N^*-metric) by [**2**, Corollary 1.14].

In [**2**, p. 246], it is shown that the partially ordered Banach space of affine continuous real-valued functions on an arbitrary metrizable Choquet simplex can be realized as $K_0(R)$ for R an N^*-complete unit-regular ring. Hence, any metrizable Choquet simplex appears as $\mathbf{P}(R)$ for such an R. We remove the metrizability assumption for Choquet simplices of the form $M_1^+(X)$.

THEOREM 5.2. *Let F be a field, and let X be any nonempty compact Hausdorff space. Then there exists an N^*-complete unit-regular F-algebra R such that*

$$(K_0(R), [R]) \cong (C(X, \mathbf{R}), 1)$$

and hence such that $\mathbf{P}(R)$ is affinely homeomorphic to $M_1^+(X)$.

Proof. Any compact Hausdorff space is homeomorphic to a subspace of a direct product of copies of the unit interval. Hence, we may assume that X is a subspace of some nonempty direct product Y of copies of $[0, 1]$. By Theorem 5.1, there exists a locally matricial F-algebra R_0 such that $\mathbf{P}(R_0)$ is affinely homeomorphic to $M_1^+(Y)$.

Let R_1 be the direct limit of the system

$$R_0 \xrightarrow{\varphi_0} M_2(R_0) \xrightarrow{\varphi_1} M_4(R_0) \xrightarrow{\varphi_2} M_8(R_0) \rightarrow \dots$$

of matrix algebras, where each φ_n is the block diagonal map. In view of [**9**, Proposition 16.20], each of the induced maps

$$\mathbf{P}(\varphi_n) : \mathbf{P}(M_{2^{n+1}}(R_0)) \rightarrow \mathbf{P}(M_{2^n}(R_0))$$

is an affine homeomorphism. Since $\mathbf{P}(-)$ converts direct limits to inverse limits [**9**, Proposition 16.21], we see that $\mathbf{P}(R_1)$ is affinely homeomorphic to $\mathbf{P}(R_0)$. Thus $\mathbf{P}(R_1)$ is affinely homeomorphic to $M_1^+(Y)$.

Since we may replace R_1 by $R_1/\ker(\mathbf{P}(R_1))$, there is no loss of generality in assuming that the kernel of $\mathbf{P}(R_1)$ is zero. (In the terminology of [**2**], R_1 is N^*-torsion-free.) In addition, R_1 is a locally matricial F-algebra, whence R_1 is unit-regular and $K_0(R_1)$ is a dimension group.

Now let R_2 be the N^*-completion of R_1. By [**2**, Theorem 1.13], R_2 is unit-regular, and the restriction map $\mathbf{P}(R_2) \rightarrow \mathbf{P}(R_1)$ is an affine homeomorphism. Thus $\mathbf{P}(R_2)$ is affinely homeomorphic to $M_1^+(Y)$, and so $\partial_e\mathbf{P}(R_2)$ is homeomorphic to Y. In addition, R_2 is N^*-complete [**2**, Corollary 1.14].

For each $n \in \mathbf{N}$, there is a set of $2^n \times 2^n$ matrix units in R_1, and hence there is a set of $2^n \times 2^n$ matrix units in every homomorphic image of R_2. Consequently, R_2 has no simple artinian homomorphic images. Thus

$$(K_0(R_2), [R_2]) \cong (\mathrm{Aff}(\mathbf{P}(R_2)), 1),$$

by [**11**, Corollary 4.12], where $\mathrm{Aff}(\mathbf{P}(R_2))$ denotes the partially ordered real Banach space of all affine continuous real-valued functions on $\mathbf{P}(R_2)$. Since $\partial_e\mathbf{P}(R_2)$ is homeomorphic to Y and so is compact, the restriction map

$$\mathrm{Aff}(\mathbf{P}(R_2)) \rightarrow C(\partial_e\mathbf{P}(R_2), \mathbf{R})$$

is an isomorphism, by [**1**, Proposition II.3.13]. Hence, there exists an isomorphism

$$f : (K_0(R_2), [R_2]) \rightarrow (C(Y, \mathbf{R}), 1).$$

Let H be the ideal of $C(Y, \mathbf{R})$ consisting of all functions in $C(Y, \mathbf{R})$ that vanish on X. Then $f^{-1}(H)$ is an ideal of $K_0(R_2)$. Set

$$J = \{x \in R_2 \mid [xR_2] \in f^{-1}(H)\}.$$

According to [**9**, Lemma 15.19], J is a two-sided ideal of R_2, and $f^{-1}(H)$ is generated (as a subgroup of $K_0(R_2)$) by the set $\{ [yR_2] \mid y \in J \}$.

Set $R = R_2/J$, and note that R is a unit-regular F-algebra. By [**16**, Proposition 7; **9**, Proposition 15.15], the quotient map $R_2 \to R$ induces an isomorphism

$$(K_0(R_2)/f^{-1}(H), [R_2] + f^{-1}(H)) \to (K_0(R), [R]).$$

Therefore

$$(K_0(R), [R]) \cong (C(Y, \mathbf{R})/H, 1 + H) \cong (C(X, \mathbf{R}), 1).$$

Now $S(K_0(R), [R])$ and $S(C(X, \mathbf{R}), 1)$ are affinely homeomorphic, and consequently $\mathbf{P}(R)$ and $M_1^+(X)$ are affinely homeomorphic.

Since evaluations at points of X are states on $(C(X, \mathbf{R}), 1)$, there are enough states on $(C(X, \mathbf{R}), 1)$ to separate points of $C(X, \mathbf{R})$. Hence, there are enough states on $(K_0(R), [R])$ to separate points of $K_0(R)$. In particular, for any nonzero element $x \in R$, there is a state s on $(K_0(R), [R])$ such that $s([xR]) > 0$, and so there is a pseudo-rank function $N \in \mathbf{P}(R)$ such that $N(x) > 0$. Thus $\ker(\mathbf{P}(R))$ is zero. Equivalently, J equals the kernel of a subset of $\mathbf{P}(R_2)$. Therefore R is N^*-complete, by [**11**, Theorem 1.13].

The referee has asked whether Theorem 5.1 extends to Choquet simplices which are products of metrizable Choquet simplices in any suitable sense, and whether Theorem 5.2 extends to Choquet simplices which are affinely homeomorphic to closed faces of suitable products of metrizable Choquet simplices. Cartesian products are not suitable, since a Cartesian product of Choquet simplices is not a Choquet simplex unless all but one of the factors is a singleton. We may define the tensor product of a family $\{K_i \mid i \in I\}$ of compact convex sets to be the state space of

$$\bigotimes_{i \in I} (\mathrm{Aff}(K_i), 1).$$

(There are alternative choices for a tensor product of compact convex sets, but for Choquet simplices they coincide with the definition just given [**20**, Theorem 2.2 and Corollary 2.6].) Theorem 5.1 could be extended to show that any tensor product of metrizable Choquet simplices is affinely homeomorphic to $\mathbf{P}(R)$ for some central simple locally matricial F-algebra R. Similarly, Theorem 5.2 could be extended to show that for any closed face K of a tensor product of metrizable Choquet simplices, there is an N^*-complete unit-regular F-algebra R for which

$$(K_0(R), [R]) \cong (\mathrm{Aff}(K), 1)$$

and $\mathbf{P}(R)$ is affinely homeomorphic to K. We have not developed Theorems 5.1 and 5.2 in this generality because there are no known characterizations either of tensor products of metrizable Choquet simplices or of closed faces of such tensor products. In particular, we do not know whether every Choquet simplex is affinely homeomorphic to a closed face of a tensor product of metrizable Choquet simplices.

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