Canad. Math. Bull. Vol. **44** (2), 2001 pp. 140–149

# On Quantizing Nilpotent and Solvable Basic Algebras

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*Abstract.* We prove an algebraic "no-go theorem" to the effect that a nontrivial Poisson algebra cannot be realized as an associative algebra with the commutator bracket. Using it, we show that there is an obstruction to quantizing the Poisson algebra of polynomials generated by a nilpotent basic algebra on a symplectic manifold. This result generalizes Groenewold's famous theorem on the impossibility of quantizing the Poisson algebra of polynomials on $\mathbf{R}^{2n}$. Finally, we explicitly construct a polynomial quantization of a symplectic manifold with a solvable basic algebra, thereby showing that the obstruction in the nilpotent case does not extend to the solvable case.

## 1 Introduction

We continue our study of Groenewold-Van Hove obstructions to quantization. Let $M$ be a symplectic manifold, and suppose that $\mathfrak{b}$ is a finite-dimensional "basic algebra" of observables on $M$. Given a Lie subalgebra $\mathcal{O}$ of the Poisson algebra $C^\infty(M)$ containing $\mathfrak{b}$, we are interested in determining whether the pair $(\mathcal{O}, \mathfrak{b})$ can be "quantized". (See Section 2 for the precise definitions.) Already we know that such obstructions exist in many circumstances: In [8] we showed that there are no nontrivial quantizations of the pair $\big(P(\mathfrak{b}), \mathfrak{b}\big)$ on a compact symplectic manifold, where $P(\mathfrak{b})$ is the Poisson algebra of polynomials on $M$ generated by $\mathfrak{b}$. Furthermore, in [10] we proved that there are no nontrivial finite-dimensional quantizations of $(\mathcal{O}, \mathfrak{b})$ on a noncompact symplectic manifold, for any such subalgebra $\mathcal{O}$.

It remains to understand the case when $M$ is noncompact and the quantizations are infinite-dimensional, which is naturally the most interesting and difficult one. Here one has little control over either the types of basic algebras that can appear (in examples they range from nilpotent to simple), their representations, or the structure of the polynomial algebras they generate [7].

In this paper we consider the problem of quantizing $\big(P(\mathfrak{b}), \mathfrak{b}\big)$ when the basic algebra is nilpotent. Our main result is (Section 5):

***Theorem 1*** *Let $\mathfrak{b}$ be a nilpotent basic algebra on a connected symplectic manifold. Then there is no quantization of $\big(P(\mathfrak{b}), \mathfrak{b}\big)$.*

This in turn is a consequence of an algebraic "no-go theorem" to the effect that a nontrivial Poisson algebra cannot be realized as an associative algebra with the

Received by the editors March 25, 1999; revised March 6, 2000.

The first author was supported in part by NSF grant DMS 96-23083. The second author was supported by KBN, grant No. 2 P03A 031 17.

AMS subject classification: Primary: 81S99; secondary: 58F06.

commutator bracket. The latter result, which is of independent general interest, is presented in Section 3.

When $M = \mathbf{R}^{2n}$ and $\mathfrak{b}$ is the Heisenberg algebra $\mathrm{h}(2n)$, Theorem 1 provides an entirely new proof of the classical theorem of Groenewold [13], [6]:

***Corollary 2*** *There is no quantization of the pair* $\Big(P\big(\mathrm{h}(2n)\big), \mathrm{h}(2n)\Big)$.

We remark that this version of the no-go theorem for $\mathbf{R}^{2n}$ does not use the Stone-Von Neumann theorem.

A natural question is whether this obstruction to quantization when $\mathfrak{b}$ is nilpotent extends to the case when $\mathfrak{b}$ is solvable. We show that it does *not*; in Section 6 we explicitly construct a polynomial quantization of $T^*\mathbf{R}_+$ with the "affine" basic algebra $\mathrm{a}(1)$.

## 2 Background

Let $M$ be a connected symplectic manifold. A key ingredient in the quantization process is the choice of a *basic algebra of observables* in the Poisson algebra $C^\infty(M)$. This is a Lie subalgebra $\mathfrak{b}$ of $C^\infty(M)$ such that:

(B1) $\mathfrak{b}$ is finitely generated,
(B2) the Hamiltonian vector fields $X_b$, $b \in \mathfrak{b}$, are complete,
(B3) $\mathfrak{b}$ is transitive and separating, and
(B4) $\mathfrak{b}$ is a minimal Lie algebra satisfying these requirements.

A subset $\mathfrak{b} \subset C^\infty(M)$ is "transitive" if $\{X_b(m) \mid b \in \mathfrak{b}\}$ spans $T_mM$ at every point. It is "separating" provided its elements globally separate points of $M$.

Now fix a basic algebra $\mathfrak{b}$, and let $\mathcal{O}$ be any Lie subalgebra of $C^\infty(M)$ containing 1 and $\mathfrak{b}$. Then by a *quantization* of the pair $(\mathcal{O}, \mathfrak{b})$ we mean a linear map $\mathcal{Q}$ from $\mathcal{O}$ to the linear space $\mathrm{Op}(D)$ of symmetric operators which preserve a fixed dense domain $D$ in some separable Hilbert space $\mathcal{H}$, such that for all $f, g \in \mathcal{O}$,

(Q1) $\mathcal{Q}(\{f,g\}) = \frac{i}{\hbar}[\mathcal{Q}(f), \mathcal{Q}(g)]$,
(Q2) $\mathcal{Q}(1) = I$,
(Q3) if the Hamiltonian vector field $X_f$ of $f$ is complete, then $\mathcal{Q}(f)$ is essentially self-adjoint on $D$,
(Q4) $\mathcal{Q}(\mathfrak{b})$ is irreducible,
(Q5) $D$ contains a dense set of separately analytic vectors for some set of Lie generators of $\mathcal{Q}(\mathfrak{b})$, and
(Q6) $\mathcal{Q}$ represents $\mathfrak{b}$ faithfully.

Here $\{\cdot,\cdot\}$ is the Poisson bracket and $\hbar$ is Planck's reduced constant.

In this paper we are interested in "polynomial quantizations", *i.e.*, quantizations of $\big(P(\mathfrak{b}), \mathfrak{b}\big)$.

We refer the reader to [7] for an extensive discussion of these definitions. However, we wish to elaborate on (Q4). There we mean irreducible in the analytic sense, *viz.* the only bounded operators which strongly commute with all $\mathcal{Q}(b) \in \mathcal{Q}(\mathfrak{b})$ are scalar multiples of the identity. There is another notion of irreducibility which is useful for our purposes: We say that $\mathcal{Q}(\mathfrak{b})$ is *algebraically irreducible* provided the only operators in $\mathrm{Op}(D)$ which (weakly) commute with all $\mathcal{Q}(b) \in \mathcal{Q}(\mathfrak{b})$ are scalar multiples of the identity. It turns out that a quantization is automatically algebraically irreducible.

***Proposition 3*** *Let $\mathcal{Q}$ be a representation of a finite-dimensional Lie algebra $\mathfrak{b}$ by symmetric operators on an invariant dense domain $D$ in a separable Hilbert space $\mathcal{H}$. If $\mathcal{Q}$ satisfies* (Q4) *and* (Q5), *then $\mathcal{Q}(\mathfrak{b})$ is algebraically irreducible.*

**Proof** We need the following two technical results, which are proven in [6]. Denote the closure of an operator $R$ by $\bar{R}$.

***Lemma 1*** *Let $R$ be an essentially self-adjoint operator and $S$ a closable operator which have a common dense invariant domain $D$. Suppose that $D$ consists of analytic vectors for $R$, and that $R$ (weakly) commutes with $S$. Then $\exp(i\bar{R})$ (weakly) commutes with $\bar{S}$ on $D$.*

***Lemma 2*** *Let $S$ be a closable operator. If a bounded operator (weakly) commutes with $\bar{S}$ on $D(S)$, then they also commute on $D(\bar{S})$.*

By virtue of (Q5) and Corollary 1 and Theorem 3 of [4], we may assume that there is a dense space $D_\omega \subseteq D$ of separately analytic vectors for some basis $\mathcal{B} = \{B_1, \dots, B_K\}$ of $\mathcal{Q}(\mathfrak{b})$. Suppose $T \in \mathrm{Op}(D)$ (weakly) commutes with every $B_k$. According to [4, Prop. 1], $T$ leaves $D_\omega$ invariant. Now by [18, Section X.6, Cor. 2] each $B_k \restriction D_\omega$ is essentially self-adjoint; moreover, $T_\omega := T \restriction D_\omega$ is symmetric and hence closable. Upon taking $R = B_k \restriction D_\omega$ and $S = T_\omega$ in Lemma 1, it follows that $\exp(i\overline{B_k \restriction D_\omega}) = \exp(i\overline{B_k})$ and $\overline{T_\omega}$ commute on $D_\omega$. Lemma 2 then shows that $\exp(i\overline{B_k})$ and $\overline{T_\omega}$ commute on $D(\overline{T_\omega})$ for all $B_k \in \mathcal{B}$.

By (Q5) the representation $\mathcal{Q}$ of $\mathfrak{b}$ can be integrated to a unitary representation $\mathfrak{Q}$ of the corresponding connected, simply connected group $G$ on $\mathcal{H}$ [4, Cor. 1] which, according to (Q4), is irreducible. From the construction of coordinates of the second kind on $\mathfrak{Q}(G)$, the map $\mathbf{R}^K \to \mathfrak{Q}(G)$ given by

$$(t_1, \dots, t_K) \mapsto \exp(it_1\overline{B_1}) \cdots \exp(it_K\overline{B_K})$$

is a diffeomorphism of an open neighborhood of $0 \in \mathbf{R}^K$ onto an open neighborhood of $I \in \mathfrak{Q}(G)$. Since $\mathfrak{Q}(G)$ is connected, the subgroup generated by such a neighborhood is all of $\mathfrak{Q}(G)$. It follows that as $\overline{T_\omega}$ commutes with each $\exp(it_k\overline{B_k})$, it commutes with every element of $\mathfrak{Q}(G)$. The unbounded version of Schur's lemma [19, (15.12)] then implies that $\overline{T_\omega} = \lambda I$ for some constant $\lambda$ on $D(\overline{T_\omega}) = \mathcal{H}$. Since $\overline{T_\omega}$ is the smallest closed extension of $T_\omega$ and $T_\omega \subset T \subset \bar{T}$, we see that $\bar{T} = \lambda I$, whence $T$ itself is a constant multiple of the identity. ■

## 3 An Algebraic No-Go Theorem

We first derive an algebraic obstruction to quantization. The idea is to compare the algebraic structures of Poisson algebras on the one hand with associative algebras of operators with the commutator bracket on the other.

https://doi.org/10.4153/CMB-2001-018-x Published online by Cambridge University Press

***Theorem 4*** *Let $\mathcal{P}$ be a unital Poisson subalgebra of $C^\infty(M)$ or $C^\infty(M)_{\mathbf{C}}$. If as a Lie algebra $\mathcal{P}$ is not commutative, it cannot be realized as an associative algebra with the commutator bracket.*

**Proof** To the contrary, let us assume that there is a Lie algebra isomorphism $\mathcal{Q}\colon \mathcal{P} \to \mathcal{A}$ onto an associative algebra $\mathcal{A}$ with the commutator bracket. Let us take $m \in M$ and $f, g \in \mathcal{P}$ such that $\{f, g\}(m) \neq 0$. In particular, then, $X_g(m) \neq 0$. Replacing $g$ by $g - g(m)1$, we can assume that $g(m) = 0$. The Lie subalgebra $\mathcal{P}_m = \{h \in \mathcal{P} \mid X_h(m) = 0\}$ is clearly of finite codimension in $\mathcal{P}$. Let us put $L = \operatorname{ad}^{-1}(\mathcal{P}_m) = \bigl\{h \in \mathcal{P} \mid \{\mathcal{P}, h\} \subset \mathcal{P}_m\bigr\}$. Since $\mathcal{Q}(\mathcal{P}_m)$ is a finite-codimensional Lie subalgebra of $\mathcal{A}$, the Lie subalgebra $\operatorname{ad}^{-1}\bigl(\mathcal{Q}(\mathcal{P}_m)\bigr) = \mathcal{Q}(L)$ is simultaneously an associative subalgebra and hence there is a finite-codimensional two-sided associative ideal $J$ contained in $\mathcal{Q}(L)$ [11, Prop. 2.1]. But associative ideals are Lie ideals with respect to the commutator bracket! Hence $\mathcal{Q}^{-1}(J)$ is a finite-codimensional (say $(l-2)$-codimensional) Lie ideal of $\mathcal{P}$ contained in $L$. In particular, some linear combination of $g^2, g^3, \ldots, g^l$, say $\hat{g} = g^k + \sum_{i=k+1}^{l} a_i g^i$, $k \geq 2$, belongs to $\mathcal{Q}^{-1}(J)$. Then $\operatorname{ad}_f^{k-2} \hat{g} \in \mathcal{Q}^{-1}(J) \subset L$, where $\operatorname{ad}_f \hat{g} := \{f, \hat{g}\}$, and thus $\operatorname{ad}_f^{k-1} \hat{g} = \operatorname{ad}_f(\operatorname{ad}_f^{k-2} \hat{g}) \in \mathcal{P}_m$. But, as $g(m) = 0$, an easy calculation gives

$$X_{\mathrm{ad}_f^{k-1} \hat{g}}(m) = k!\,\{f, g\}^{k-1}(m) X_g(m) \neq 0,$$

a contradiction. ■

See [15] for complementary results regarding $P\bigl(\mathrm{h}(2n)\bigr)$ *vis-à-vis* the Weyl algebra.
In Section 5 we will use this result to prove the nonexistence of polynomial quantizations of $\bigl(P(\mathfrak{b}), \mathfrak{b}\bigr)$ when $\mathfrak{b}$ is nilpotent.

## 4 Nilpotent Basic Algebras

Let $\mathfrak{b}$ be a nilpotent basic algebra on a $2n$-dimensional connected symplectic manifold $M$. Since by (B1) $\mathfrak{b}$ is finitely generated and as every finitely generated nilpotent Lie algebra is finite-dimensional, [7, Prop. 2] shows that $M$ must be a coadjoint orbit in $\mathfrak{b}^*$. Now we have the "bundlization" results of Arnal *et al.* [1], Pedersen [17], Vergne [20], and Wildberger [21], which assert:

***Theorem 5*** *Let $\mathfrak{b}$ be a finite-dimensional nilpotent Lie algebra. For each $2n$-dimensional coadjoint orbit $O \subset \mathfrak{b}^*$, there exists a symplectomorphism* ("bundlization") *$\varphi_O\colon T^*\mathbf{R}^n \to O$. We may consider $b \in \mathfrak{b}$ as a (linear) function on $\mathfrak{b}^*$, and form $\Phi_O(b) = b|O \circ \varphi_O$. Then cotangent coordinates $(q_1, \ldots, q_n, p_1, \ldots, p_n)$ on $T^*\mathbf{R}^n$ may be chosen in such a way that $\Phi_O(b)$ has the form*

$$\phi_0 p_1 + \phi_1(q_1) p_2 + \cdots + \phi_{n-1}(q_1, \ldots, q_{n-1}) p_n + \phi_n(q_1, \ldots, q_n), \tag{1}$$

*where the $\phi_\alpha$ are polynomials.*

Thus we may assume that $M = T^*\mathbf{R}^n$ and that $\mathfrak{b}$ consists of elements of the form (1). See [12] for an analogous characterization of transitive nilpotent Lie algebras of vector fields.

The canonical example of a nilpotent basic algebra on $T^*\mathbf{R}^n$ is the Heisenberg algebra $\mathrm{h}(2n) = \mathrm{span}_{\mathbf{R}}\{1, q_\alpha, p_\alpha \mid \alpha = 1, \dots, n\}$. It is not difficult to see from (1) that, up to isomorphism, $\mathrm{h}(2)$ is the only nilpotent *basic* algebra on $T^*\mathbf{R}$. This is not true in higher dimensions, however:

$$\mathfrak{b} = \mathrm{span}_{\mathbf{R}}\{1, q_1, p_2, q_1p_2 + q_2, p_1\}$$

is a nilpotent basic algebra on $T^*\mathbf{R}^2$ which is not isomorphic to $\mathrm{h}(4)$. Regardless, all nilpotent basic algebras on $T^*\mathbf{R}^n$ enjoy the following property. We write $\mathbf{q} = (q_1, \dots, q_n)$, *etc.*

***Proposition 6*** *If $\mathfrak{b}$ is a nilpotent basic algebra on $T^*\mathbf{R}^n$, then as Poisson algebras $P(\mathfrak{b}) = \mathbf{R}[\mathbf{q}, \mathbf{p}]$.*

**Proof** That $P(\mathfrak{b}) \subseteq \mathbf{R}[\mathbf{q}, \mathbf{p}]$ is evident from Theorem 5. The opposite inclusion follows from an algorithm, developed in [17, Section 5.4], which constructs the $\{q_\alpha, p_\alpha \mid \alpha = 1, \dots, n\}$ as polynomial functions of elements of a basis of $\mathfrak{b}$. That $P(\mathfrak{b})$ and $\mathbf{R}[\mathbf{q}, \mathbf{p}]$ coincide as Lie algebras is due to the fact that the bundlization $\varphi_O$ is a symplectomorphism or, equivalently, that the coordinates $q_\alpha$, $p_\alpha$ are canonical. ■

We will establish a quantum analogue of this result in the next section.

Recall that the central ascending series for $\mathfrak{b}$ is

$$\{0\} = \mathfrak{b}^0 \subset \mathfrak{b}^1 \subset \cdots \subset \mathfrak{b}^\ell = \mathfrak{b}$$

for some positive integer $\ell$, where $\mathfrak{b}^{s+1} = \mathrm{ad}^{-1}(\mathfrak{b}^s)$. Then $\{\mathfrak{b}, \mathfrak{b}^s\} \subseteq \mathfrak{b}^{s-1}$. Also note that $\mathfrak{b}^1$ is the center of $\mathfrak{b}$ which, according to the transitivity condition in (B3), consists of constants. Choose a Jordan-Hölder basis $\{b_1, \dots, b_K\}$ of $\mathfrak{b}$. Then $\{b_i, b_j\} = \sum_{k=1}^K c_{ij}^k b_k$, where the structure constants $c_{ij}^k = 0$ whenever $k \geq \min\{i, j\}$. We take $b_1 = 1$. We call the smallest integer $N$ such that $b \in \mathfrak{b}^{N+1}$ the "nildegree" of $b \in \mathfrak{b}$. Then $\mathrm{nildeg}(b_i) \leq \mathrm{nildeg}(b_j)$ whenever $i < j$. The nildegree of a polynomial $f \in P(\mathfrak{b})$ is then the smallest integer $N$ such that

$$\big(\mathrm{ad}(b_{i_1}) \circ \cdots \circ \mathrm{ad}(b_{i_{N+1}})\big)\, f = 0$$

for all $i_1, \dots, i_{N+1} \in \{1, \dots, K\}$.

## 5 Proof of Theorem 1 and Related Results

Before proving Theorem 1 we establish several results which are useful in their own right.

Let the basic algebra $\mathfrak{b}$ be nilpotent. Fix a Lie subalgebra $\mathcal{O}$ of $P(\mathfrak{b})$ containing $\mathfrak{b}$. Suppose that $\mathcal{Q}\colon \mathcal{O} \to \mathrm{Op}(D)$ is a quantization of $(\mathcal{O}, \mathfrak{b})$ on some invariant dense domain $D$ in a Hilbert space.

***Proposition 7*** *$\mathcal{Q}$ is injective.*

**Proof** Let $L = \ker \mathcal{Q}$; then given $g \in L$, there is a $k$ such that $g \in \mathcal{O}^k$, where $\mathcal{O}^k$ is the subspace of $\mathcal{O}$ consisting of polynomials of nildegree at most $k$ in the elements of $\mathfrak{b}$. Consider the adjoint representation of $\mathfrak{b}$ on $\mathcal{O}^k \cap L$. (This makes sense as $L$ is a Lie ideal.) This is a nilrepresentation, so by Engel's theorem [16, Section X.2] there exists a nonzero element $f \in \mathcal{O}^k \cap L$ such that $\{f, b\} = 0$ for all $b \in \mathfrak{b}$. But then transitivity implies that $f$ is a constant, which contradicts (Q2). Thus $L = \{0\}$. ■

Thus condition (Q6) is actually redundant in the case of nilpotent basic algebras.

Let $\mathcal{A}$ be the associative algebra generated over $\mathbf{C}$ by $\{\mathcal{Q}(b) \mid b \in \mathfrak{b}\}$. The next result generalizes Proposition 6 to the quantum context.

***Proposition 8*** *$\mathcal{A}$ is isomorphic to a Weyl algebra.*[1]

**Proof** First we claim that the center of $\mathcal{A}$ is just $\mathbf{C}I$. Indeed, suppose $[A, \mathcal{Q}(b)] = 0$ for all $b \in \mathfrak{b}$. Since by construction every $A \in \mathcal{A}$ has an adjoint, we may decompose $A$ into its symmetric $A_s$ and skew-symmetric $A_a$ components. Algebraic irreducibility then implies that the symmetric operators $A_s$ and $iA_a$ are both scalar multiples of the identity.

Next let $\psi$ be the homomorphism of the universal enveloping algebra $U\big(\mathcal{Q}(\mathfrak{b}_{\mathbf{C}})\big)$ into $\mathcal{A}$ determined by the inclusion $\mathcal{Q}(\mathfrak{b}_{\mathbf{C}}) \hookrightarrow \mathcal{A}$. Then $J = \ker \psi$ is a two-sided ideal in $U\big(\mathcal{Q}(\mathfrak{b}_{\mathbf{C}})\big)$. Clearly, $\psi$ is an epimorphism and thus $U\big(\mathcal{Q}(\mathfrak{b}_{\mathbf{C}})\big)/J \approx \mathcal{A}$.

Since furthermore $\mathcal{Q}(\mathfrak{b}_{\mathbf{C}})$ is nilpotent, the desired result follows from [3, Theorem 4.7.9]. ■

By requiring $\mathcal{Q}$ to be complex linear, we may view it as a quantization of the complexification $\mathcal{O}_{\mathbf{C}}$. We next prove that $\mathcal{Q}$ maps $\mathcal{O}_{\mathbf{C}}$ into $\mathcal{A}$. That "polynomials quantize to polynomials" can be regarded as a generalized "Von Neumann rule," *cf.* [7].

***Proposition 9*** $\mathcal{Q}(\mathcal{O}_{\mathbf{C}}) \subseteq \mathcal{A}$.

**Proof** We argue inductively on the nildegree of $f \in \mathcal{O}$ that $\mathcal{Q}(f) \in \mathcal{A}$. In nildegree 0 this follows immediately from transitivity and (Q2). Now suppose it is also true for polynomials in $\mathcal{O}$ of nildegree $J \le N$, and let $f \in \mathcal{O}$ have nildegree $N + 1$. Then for each $b \in \mathfrak{b}$,

$$[\mathcal{Q}(f), \mathcal{Q}(b)] = -i\hbar\mathcal{Q}(\{f, b\}) \in \mathcal{A}$$

by (Q1) and the inductive hypothesis, since $\operatorname{nildeg}(\{f, b\}) < \operatorname{nildeg}(f)$. Thus the map

$$W \mapsto [\mathcal{Q}(f), W]$$

defines a derivation of the associative algebra $\mathcal{A}$. As it is well known that every derivation of a Weyl algebra is inner [3, Section 4.6.8], by Proposition 8 there is thus an $A \in \mathcal{A}$ such that $[\mathcal{Q}(f), W] = [A, W]$ for all $W \in \mathcal{A}$. Algebraic irreducibility then

[1]Recall that the Weyl algebra $W(2k)$ is the associative algebra over $\mathbf{C}$ generated by $\{z_\alpha, w_\beta \mid \alpha, \beta = 1, \dots, k\}$ and the relations $[z_\alpha, w_\beta] = -i\delta_{\alpha\beta}$, $[z_\alpha, z_\beta] = 0 = [w_\alpha, w_\beta]$.

https://doi.org/10.4153/CMB-2001-018-x Published online by Cambridge University Press

implies that the symmetric operator $\mathcal{Q}(f)$ and the symmetric component $A_s$ of $A$ differ by a constant multiple of $I$. Thus the inductive step is proved and so $\mathcal{Q}(\mathcal{O})$, and hence $\mathcal{Q}(\mathcal{O}_{\mathbf{C}})$, are contained in $\mathcal{A}$. ∎

We are finally ready to show that there is no quantization of $\big(P(\mathfrak{b}), \mathfrak{b}\big)$. Set $B_i = \mathcal{Q}(b_i)$. As $\mathcal{Q}(\mathfrak{b}_{\mathbf{C}})$ is nilpotent, we may likewise define the nildegree of the $B_i$ *etc.*[2] Since $\mathcal{Q}$ is faithful we have that $\operatorname{nildeg}(B_i) = \operatorname{nildeg}(b_i)$.

**Proof of Theorem 1** Suppose that $\mathcal{Q}\colon P(\mathfrak{b}) \to \operatorname{Op}(D)$ were a quantization of $\big(P(\mathfrak{b}), \mathfrak{b}\big)$. Let $\mathcal{P} = P(\mathfrak{b})_{\mathbf{C}}$. From Proposition 9 we know that $\mathcal{Q}(\mathcal{P}) \subseteq \mathcal{A}$, and from Proposition 7 we have that $\mathcal{Q}$ is injective. Thus if we can show that $\mathcal{Q}$ is surjective, then $\mathcal{Q}$ will be a Lie algebra isomorphism of $\mathcal{P}$ onto $\mathcal{A}$, thereby contradicting Theorem 4.

To this end, we shall prove inductively that

$(*_N)$ *If the monomial* $b_1^{r_1}\cdots b_K^{r_K} \in P(\mathfrak{b})$ *is of nildegree* $J$, $J \le N$, *then*

$$\mathcal{Q}(b_1^{r_1}\cdots b_K^{r_K}) = \mathcal{S}(B_1^{r_1}\cdots B_K^{r_K}) + \textit{polynomials of nildegree } < J,$$

where $\mathcal{S}$ denotes symmetrization over all factors.

We have already seen that condition $(*_0)$ holds. Now assume that $b_1^{r_1}\cdots b_K^{r_K}$ has nildegree $N+1$. By (Q1),

$$\begin{aligned}
[\mathcal{Q}(b_1^{r_1}\cdots b_K^{r_K}), B_j] &= -i\hbar\mathcal{Q}(\{b_1^{r_1}\cdots b_K^{r_K}, b_j\}) \\
&= -i\hbar\mathcal{Q}\Big(\sum_{l=1}^{K} r_l b_1^{r_1}\cdots b_l^{r_l-1}\{b_l, b_j\}\cdots b_K^{r_K}\Big) \\
&= -i\hbar\sum_{l,m=1}^{K} r_l c_{lj}^m \mathcal{Q}(b_1^{r_1}\cdots b_m^{r_m+1}\cdots b_l^{r_l-1}\cdots b_K^{r_K}) \\
&= -i\hbar\sum_{l,m=1}^{K} r_l c_{lj}^m \mathcal{S}(B_1^{r_1}\cdots B_m^{r_m+1}\cdots B_l^{r_l-1}\cdots B_K^{r_K}) \\
&\qquad + \text{polynomials of nildegree} < N
\end{aligned}$$

where the last equality follows from $(*_N)$, since

$$\begin{aligned}
\operatorname{nildeg}(c_{lj}^m b_1^{r_1}\cdots b_m^{r_m+1}\cdots b_l^{r_l-1}\cdots b_K^{r_K}) &\le \operatorname{nildeg}(\{b_1^{r_1}\cdots b_K^{r_K}, b_j\}) \\
&< \operatorname{nildeg}(b_1^{r_1}\cdots b_K^{r_K}).
\end{aligned}$$

Furthermore, direct computation yields

$$[\mathcal{S}(B_1^{r_1}\cdots B_K^{r_K}), B_j] = -i\hbar\sum_{l,m=1}^{K} r_l c_{lj}^m \mathcal{S}(B_1^{r_1}\cdots B_m^{r_m+1}\cdots B_l^{r_l-1}\cdots B_K^{r_K}).$$

[2]This is so even though $\mathcal{Q}$ need not be a nilrepresentation.

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Consequently for each $j = 1, \ldots, K$,

$$[\mathcal{Q}(b_1^{r_1} \cdots b_K^{r_K}) - \mathcal{S}(B_1^{r_1} \cdots B_K^{r_K}), B_j] = \text{polynomials of nildegree} < N.$$

This implies that the polynomial $\mathcal{Q}(b_1^{r_1} \cdots b_K^{r_K}) - \mathcal{S}(B_1^{r_1} \cdots B_K^{r_K}) \in \mathcal{A}$ has nildegree at most $N$, and $(*_{N+1})$ follows.

Applying $(*_N)$ recursively, we see that as the $\mathcal{S}(B_1^{r_1} \cdots B_K^{r_K})$ form a basis for $\mathcal{A}$, $\mathcal{Q}$ maps onto $\mathcal{A}$. ∎

Even though one cannot quantize *all* of $P(\mathfrak{b})$, it is possible to quantize 'sufficiently small' Lie subalgebras thereof (see, *e.g.* [6]). We emphasize that Propositions 7–9 are valid in this context. It is an open problem to determine the maximal quantizable Lie subalgebras of $P(\mathfrak{b})$.

## 6 Solvable Basic Algebras

We have shown that there is an obstruction to quantizing symplectic manifolds with nilpotent basic algebras. It is also known that there is an obstruction to quantizing $T^*S^1$ with the Euclidean basic algebra $\mathrm{e}(2)$, which is solvable [9]. Thus it is natural to wonder if the nilpotent no-go theorem extends to the solvable case. It turn out that it does *not*: We now show that there is a polynomial quantization of $T^*\mathbf{R}_+ = \{(q, p) \in \mathbf{R}^2 \mid q > 0\}$ with the "affine" basic algebra

$$\mathrm{a}(1) = \mathrm{span}_{\mathbf{R}}\{pq, q^2\}.$$

Upon writing $x = pq$, $y = q^2$, the bracket relation becomes $\{x, y\} = 2y$. Thus $\mathrm{a}(1)$ is the simplest example of a solvable algebra which is not nilpotent. The corresponding polynomial algebra $P = \mathbf{R}[x, y]$ is free, and has the crucial feature that for each $k \geq 0$, the subspaces $P_k$ are *ad*-invariant, *i.e.*,

$$\{P_1, P_k\} \subset P_k. \tag{2}$$

(Here $P_k$ denotes the subspace of homogeneous polynomials of degree $k$ in $x$ and $y$, and $P^k = \bigoplus_{l=0}^{k} P_l$. Note that $P_1 = \mathrm{a}(1)$). Because of this $\{P_k, P_l\} \subset P_{k+l-1}$, whence each $P_{(k)} = \bigoplus_{l \geq k} P_l$ is a Lie ideal. We thus have the semidirect sum decomposition

$$P = P^1 \ltimes P_{(2)}. \tag{3}$$

Now on to quantization. In view of (3), we can obtain a quantization $\mathcal{Q}$ of $P$ simply by finding an appropriate representation of $P^1 = \mathbf{R} \oplus P_1$ and setting $\mathcal{Q}(P_{(2)}) = \{0\}$!

The connected, simply connected covering group of $\mathrm{a}(1)$ is $\mathrm{A}(1)_+ = \mathbf{R} \rtimes \mathbf{R}_+$ with the composition law

$$(\nu, \lambda)(\beta, \delta) = (\nu + \lambda^2\beta, \lambda\delta).$$

($\mathrm{A}(1)_+$ is isomorphic to the group of orientation-preserving affine transformations of the line, whence the terminology.) Since $\mathrm{A}(1)_+$ is a semidirect product we can

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generate its unitary representations by induction. Following the recipe in [2, Section 17.1] we obtain two one-parameter families of unitary representations $U_\pm$ of $\mathrm{A}(1)_+$ on $L^2(\mathbf{R}_+, dq/q)$ given by

$$\big(U_\pm(\nu,\lambda)\psi\big)(q) = e^{\pm i\mu\nu q^2}\psi(\lambda q)$$

with $\mu > 0$. We identify the parameter $\mu$ with $\hbar^{-1}$. According to Theorems 4 and 5 in [2, Section 17.1] the remaining two representations (one for each choice of sign) are irreducible and inequivalent; moreover, up to equivalence these are the only nontrivial irreducible ones.

Let $D \subset L^2(\mathbf{R}_+, dq/q)$ be the linear span of the functions $\sqrt{q}\,h_k(q)$, where the $h_k$ are the Hermite functions. Writing $\pi_\pm = -i\hbar dU_\pm$ we get the representation(s) of $\mathrm{a}(1)$ on the dense subspace $D$:

$$\pi_\pm(pq) = -i\hbar q\frac{d}{dq}, \quad \pi_\pm(q^2) = \pm q^2.$$

Extend these to $P^1$ by taking $\pi_\pm(1) = I$, and set $\mathcal{Q}_\pm = \pi_\pm \oplus 0$ (*cf.* (3)). Clearly (Q1)–(Q3) hold, by construction (Q4) is satisfied, and $\mathcal{Q}_\pm \restriction \mathrm{a}(1) = \pi_\pm$ is faithful. Finally, it is straightforward to verify that $D$ consists of analytic vectors for both $\pi_\pm(pq)$ and $\pi_\pm(q^2)$. Thus $\mathcal{Q}_\pm$ are the required quantization(s) of $(P, P_1)$.

***Remarks.***

1. The + quantization of $\mathrm{a}(1)$ is exactly what one obtains by geometrically quantizing $T^*\mathbf{R}_+$ in the vertical polarization. Carrying this out, we get $\mathcal{H} = L^2(\mathbf{R}_+, dq)$ and

$$pq \mapsto -i\hbar\Big(q\frac{d}{dq} + \frac{1}{2}\Big), \quad q^2 \mapsto q^2.$$

   The + quantization is unitarily equivalent to this via the transformation $L^2(\mathbf{R}_+, dq/q) \to L^2(\mathbf{R}_+, dq)$ which takes $f(q) \mapsto f(q)/\sqrt{q}$.
2. Note that $\mathrm{a}(1) \subset \mathrm{sp}(2,\mathbf{R})$. In fact, the + quantization is equivalent to the restrictions to $\mathrm{a}(1)$ of the metaplectic representations of $\mathrm{sp}(2,\mathbf{R})$ on both $L^2_{\mathrm{even}}(\mathbf{R}, dq)$ and $L^2_{\mathrm{odd}}(\mathbf{R}, dq)$ [7, Section 5.1].
3. Since $\mathcal{Q}(P_{(2)}) = 0$, the quantization is somewhat 'trivial'. However, there are quantizations which are nonzero on $P_{(2)}$: for instance, set $\mathcal{Q}(x^k) = k\mathcal{Q}(x)$ for $k > 0$, $\mathcal{Q}(x^l y) = \mathcal{Q}(y)$, and $\mathcal{Q}(x^l y^m) = 0$ for $m > 1$.
4. Our quantization of $T^*\mathbf{R}_+$ should be contrasted with that given in [14, Section 4.5]. Also, we observe that this example is symplectomorphic to $\mathbf{R}^2$ with the basic algebra $\mathrm{span}_{\mathbf{R}}\{p, e^{2q}\}$.
5. This is not the first example of a polynomial quantization; in [5] a quantization of the entire Poisson algebra of the torus was constructed. However, the basic algebra in that example was *infinite*-dimensional.

What makes this example work? After comparing it with other examples, it is evident that this polynomial quantization exists because we cannot decrease degree *in P*

https://doi.org/10.4153/CMB-2001-018-x Published online by Cambridge University Press

by taking Poisson brackets. (That is, we have (2) as opposed to merely $\{P_1, P_k\} \subset P^k$.) Based on this observation, it seems reasonable to suspect that there is an obstruction to quantizing $\big(P(\mathfrak{b}), \mathfrak{b}\big)$ only if it is possible to lower degree in $P(\mathfrak{b})$ by taking Poisson brackets. We shall pursue this line of investigation elsewhere (*cf.* also [7]).

We thank M. Gerstenhaber, B. Kaneshige, and N. Wildberger for providing us with helpful comments and references.

## References

[1] D. Arnal, J. C. Cortet, P. Molin, and G. Pinczon, *Covariance and geometrical invariance in $*$-quantization*. J. Math. Phys. **24**(1983), 276–283.
[2] A. O. Barut and R. Rączka, *Theory of Group Representations and Applications.* World Scientific, Singapore, 2nd edition, 1986.
[3] J. Dixmier, *Enveloping Algebras.* North-Holland, Amsterdam, 1977.
[4] M. Flato and J. Simon, *Separate and joint analyticity in Lie groups representations.* J. Funct. Anal. **13**(1973), 268–276.
[5] M. J. Gotay, *On a full quantization of the torus*. In: Quantization, Coherent States and Complex Structures (eds. J.-P. Antoine *et al.*), Plenum, New York, 1995, 55–62.
[6] ———, *On the Groenewold-Van Hove problem for* $\mathbf{R}^{2n}$. J. Math. Phys. **40**(1999), 2107–2116.
[7] ———, *Obstructions to quantization.* In: Mechanics: From Theory to Computation (eds. Journal of Nonlinear Science Editors), Springer, New York, 2000, 171–216.
[8] M. J. Gotay, J. Grabowski and H. B. Grundling, *An obstruction to quantizing compact symplectic manifolds.* Proc. Amer. Math. Soc. **28**(2000), 237–243.
[9] M. J. Gotay and H. B. Grundling, *On quantizing* $T^*S^1$. Rep. Math. Phys. **40**(1997), 107–123.
[10] ———, *Nonexistence of finite-dimensional quantizations of a noncompact symplectic manifold*. In: Differential Geometry and Applications (eds. I. Kolář *et al.*), Masaryk Univ., Brno, 1999, 593–596.
[11] J. Grabowski, *The Lie structure of* $C^*$ *and Poisson algebras.* Studia Math. **81**(1985), 259–270.
[12] ———, *Remarks on nilpotent Lie algebras of vector fields.* J. Reine Angew. Math. **406**(1990), 1–4.
[13] H. J. Groenewold, *On the principles of elementary quantum mechanics.* Physica **12**(1946), 405–460.
[14] C. J. Isham, *Topological and global aspects of quantum theory*. In: Relativity, Groups, and Topology II (eds. B. S. DeWitt and R. Stora), North-Holland, Amsterdam, 1984, 1059–1290.
[15] A. Joseph, *Derivations of Lie brackets and canonical quantization.* Commun. Math. Phys. **17**(1970), 210–232.
[16] M. A. Naimark and A. I. Štern, *Theory of Group Representations.* Springer, New York, 1982.
[17] N. V. Pedersen, *Geometric quantization and the universal enveloping algebra of a nilpotent Lie group.* Trans. Amer. Math. Soc. **315**(1989), 511–563.
[18] M. Reed and B. Simon, *Functional Analysis II*. Academic Press, New York, 1975.
[19] A. Robert, *Introduction to the Representation Theory of Compact and Locally Compact Groups*. London Math. Soc. Lecture Note Ser. **80**, Cambridge Univ. Press, Cambridge, 1983.
[20] M. Vergne, *La structure de Poisson sur l'algèbre symmétrique d'une algèbre de Lie nilpotente*. Bull. Soc. Math. France **100**(1972), 301–335.
[21] N. Wildberger, *Quantization and Harmonic Analysis on Lie Groups.* Dissertation, Yale University, 1983.

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