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ALGEBRAS GENERATED BY SYMMETRIC IDEMPOTENTS

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Let F be a field. If A is an F-algebra with involution that is generated (as a space) by symmetric idempotents, then A is a subdirect product of copies of F if and only if every idempotent in A is symmetric.

1. Introduction

This paper arose from the study of the questions raised by Herstein [2] concerning when the vector space generated by the symmetric idempotents in a simple ring with involution is equal to itself. If S is a simple ring and $C(S)$ the centroid of S, then $C(S)$ is a field and S is a $C(S)$-algebra. Let $E^*(S)$ be the $C(S)$-subspace generated by the non-zero symmetric idempotents. Chaung and Lee [1, Example 4] showed that $E^*(S)$ can be a ring and yet not be S itself. Observe that if $E^*(S)$ is a ring, then $E^*(S)$ is an algebra generated as a vector space by symmetric idempotents, the object of our investigation.

Let F be a field. In this paper we show that if A is an F-algebra with involution $*$ that is generated (as a space) by symmetric idempotents, then A is a subdirect product of copies of F if and only if every idempotent in A is symmetric.

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2. A commutivity condition

In this section we require only that A be an F-algebra generated by idempotents. If we ask when A is commutative, then we are led to

THEOREM 1. *Suppose F is a field and A is an F-algebra generated by idempotents. The following are equivalent:*

(i) A is commutative;

(ii) A has no non-zero nilpotent elements;

(iii) A is F-isomorphic to a subdirect product of copies of F.

Proof. Take A to be commutative. We let I denote the set of non-zero idempotents in A. Any non-zero element in A can be written in the form $\lambda_1 e_1 + \ldots + \lambda_n e_n$ where $0 \neq \lambda_i \in F$, $0 \neq e_i \in I$, e_i's distinct, and n is minimal. We call n the length of the element. If A has a non-zero nilpotent element, then choose one of minimal length among all such elements. Denote the element by w and express it as above. So for each $i = 1, \ldots, n$,

$$w - we_i = \sum \lambda_j (e_j - e_j e_i) ,$$

where $j = 1, \ldots, n$ and $j \neq i$, is an element of length less than n or $w - we_i$ is zero. But $w - we_i$ is nilpotent; so the latter must hold and $w = we_i$. Observe that

$$w^2 = (\lambda_1 + \ldots + \lambda_n)w$$

and then inductively we have

$$0 = w^k = (\lambda_1 + \ldots + \lambda_n)^{k-1} w ,$$

where k is the index of nilpotency of w. Consequently,

$$(\lambda_1 + \ldots + \lambda_n)^{k-1} = 0$$

or

$$\lambda_1 + \ldots + \lambda_n = 0 .$$

Let x_j be the product of the idempotents $e_j, e_{j+1}, \ldots, e_n$, $j = 1, \ldots, n$. Then

$$wx_1 = we_1x_2 = wx_2 = \ldots = we_n = w ,$$

but

$$\begin{aligned} wx_1 &= (\lambda_1 e_1 + \ldots + \lambda_n e_n)x_1 \\ &= \lambda_1 x_1 + \ldots + \lambda_n x_1 \\ &= (\lambda_1 + \ldots + \lambda_n)x_1 \\ &= 0 . \end{aligned}$$

So $w = 0$ and A has no non-zero nilpotent elements. Thus *(i)* implies *(ii)*.

One obtains *(iii)* from *(ii)* by recalling that in a ring without nilpotent elements the idempotents are central. So we may consider A to be commutative and without nilpotent elements. Using an F-algebra version of the Krull-McCoy Theorem, that a ring without nilpotent elements is isomorphic to a subdirect product of integral domains, we have that A is a subdirect product of F-algebras, A_i, i running over some index set Λ, where each A_i is without zero divisors. Each A_i, being an F-homomorphic image of A, must also be generated as an F-vector space by idempotents. Since $A_i \neq (0)$, it contains a non-zero idempotent. But since A_i is a ring without zero divisors, this idempotent is a unit element, say 1_i. In fact, since the idempotents in A must go into 0 or 1_i under the ith projection F-homomorphism, each element of A_i is of the form $1_i \cdot \lambda \in F$ and consequently A_i is a field which is isomorphic to F.

It is immediate that *(iii)* implies *(i)*.

A corollary to this theorem is of interest when A is noncommutative.

COROLLARY 1. *Suppose F be a field and A is an F-algebra generated by idempotents. All of the nilpotent elements in A are found in its commutator ideal.*

Proof. Let C be the commutator ideal of A. Then A/C is a

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commutative F-algebra generated by idempotents. If n is a nilpotent element in A, then $n + C$ is a nilpotent element in A/C. By Theorem 1 we must have $n + C = C$, or $n \in C$.

3. A *-version

We now suppose that A is an F-algebra with involution $*$ that is generated by symmetric idempotents and ask when A is commutative.

THEOREM 2. *Suppose F is a field and A is an F-algebra with involution generated by symmetric idempotents. The algebra A is commutative if and only if every idempotent in A is symmetric.*

Proof. Suppose every idempotent in A is symmetric. Then if e_1 and e_2 are idempotents in A, so is $e_1 + e_1e_2 - e_1e_2e_1$. Then we must have

$$\begin{aligned} e_1 + e_1e_2 - e_1e_2e_1 &= (e_1+e_1e_2-e_1e_2e_1)^* \\ &= e_1^* + e_2^*e_1^* - e_1^*e_2^*e_1^* \\ &= e_1 + e_2e_1 - e_1e_2e_1 . \end{aligned}$$

Consequently, $e_1e_2 = e_2e_1$ for any two symmetric idempotents in A. This is enough to show that A is commutative.

Now suppose that A is commutative. We let S denote the set of non-zero symmetric idempotents in A. Any non-zero element in A can be written in the form $\lambda_1e_1 + \ldots + \lambda_ne_n$ where $0 \neq \lambda_i \in F$, $0 \neq e_i \in S$, e_i's distinct, n minimal. We call n the length of the element. If A has an idempotent that is not symmetric, then choose one of minimal length among all such elements. Denote this element by e and express it as above. So for each $i = 1, \ldots, n$,

$$e - ee_i = \sum \lambda_j(e_j-e_je_i) ,$$

where $j = 1, \ldots, n$ and $j \neq i$, is an idempotent of length less than n, and hence $e - ee_i$ must be symmetric. So we know

$$e - ee_i = e^* - e^*e_i$$

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for each i . Multiplying by λ_i we have

$$\lambda_i e - e(\lambda_i e_i) = \lambda_i e^* - e^*(\lambda_i e_i) .$$

Summing over i from 1 to n we get

$$(\lambda_1 + \dots + \lambda_n)e - e = (\lambda_1 + \dots + \lambda_n)e^* - e^*e .$$

If $\lambda_1 + \dots + \lambda_n = 0$ then $e = e^*e$ which implies e is symmetric. If $\lambda_1 + \dots + \lambda_n = 1$, then $e^*e = e^*$. So $e = e^*e$. Thus we may assume below that

$$\lambda_1 + \dots + \lambda_n \neq 0, 1 .$$

Let x_i be the product of the idempotents $e_i, e_{i+1}, \dots, e_n$, $i = 1, \dots, n$. If we multiply

$$e = \lambda_1 e_1 + \dots + \lambda_n e_n$$

by x_1 , then we obtain

$$\begin{aligned} ex_1 &= \lambda_1 x_1 + \dots + \lambda_n x_1 \\ &= (\lambda_1 + \dots + \lambda_n)x_1 . \end{aligned}$$

After squaring and subtracting we have

$$\left[(\lambda_1 + \dots + \lambda_n)^2 - (\lambda_1 + \dots + \lambda_n)\right]x_1 = 0 .$$

This implies $x_1 = 0$. Now set $e - ee_i = s_i$, $i = 1, \dots, n$. Then

$$\begin{aligned} 0 &= ex_1 \\ &= ee_1 x_2 \\ &= e-s_1 x_2 \\ &= ex_2 - s_1 x_2 \\ &= ee_2 x_3 - s_1 x_2 \\ &= ex_3 - s_2 x_3 - s_1 x_2 \\ &\ \vdots \\ &= e - s_n - s_{n-1}x_n - s_{n-2}x_{n-1} - \dots - s_1 x_2 . \end{aligned}$$

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Since s_i and x_i, $i = 1, \ldots, n$, are symmetric elements, e is symmetric.

COROLLARY 2. *Suppose F is a field and A is an F-algebra with involution $*$ generated by symmetric idempotents. If e is an idempotent in A, then $e - e^*$ is an element in its commutator ideal.*

Proof. We again denote the commutator ideal of A by C. Define $C^* = \{c^*; c \in C\}$. Since $C = C^*$ we know A/C is a commutative F-algebra having involution which is generated by symmetric idempotents. If e is an idempotent in A, then $e + C$ is an idempotent in A/C. By Theorem 2 we know $e + C = e^* + C$, or $e - e^* \in C$.

If we combine Theorem 1 and Theorem 2, then we immediately obtain

THEOREM 3. *Suppose F is a field and A is an F-algebra with involution generated by symmetric idempotents. The following are equivalent:*

(i) *A is commutative;*

(ii) *every idempotent in A is symmetric;*

(iii) *A has no non-zero nilpotent elements;*

(iv) *A is F-isomorphic to a subdirect product of copies of F.*

References

[1] C.L. Chaung and P.H. Lee, "Idempotents in simple rings", *J. Algebra* 56 (1979), 510-515.

[2] I.N. Herstein, *Rings with involution* (University of Chicago Press, Chicago, 1979).

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