Instability, Chaos and Predictability
in Celestial Mechanics and
Stellar Dynamics

ISBN 1-56072-054-9

ON GRAVOTHERMAL INSTABILITY OF ANISOTROPIC SELF-GRAVITATING GAS SPHERES: SINGULAR EQUILIBRIUM SOLUTION

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ABSTRACT

A reinvestigation of the linear perturbation theory is presented, which examines the hydrostatic readjustment of an isolated self-gravitating gas sphere to a redistribution of energy. The here presented model describes a stellar system by the common equations of gas in hydrostatic equilibrium but with the effect of the anisotropic velocity distribution on the pressure gradient. We take as equilibrium models the singular isothermal solution with and without anisotropy. The total variation of the Boltzmann entropy resulting from a perturbation of the system caused by a redistribution of heat (i.e. r.m.s. kinetic energy of the stars) is calculated for anisotropic solutions to first order as well as to second order for the isotropic equilibrium. The extremized eigenfunctions which represent the entropy and anisotropy perturbation functions, are determined analytically. They exhibit gravothermal behaviour in the central region where heat is removed. It is also found that the anisotropy readjusts non-thermally in the sense that the system departs from isotropy although the total entropy increases.

1. INTRODUCTION

Momentum models for the evolution of stellar systems (often called "gaseous models") have been used frequently as a convenient representation of the stellar component in numerical astrophysical simulations as e.g. galaxy formation and evolution (Larson 1969,1970,1974,1975,1976; Burkert and Hensler 1987, 1988; Dunhuber et al. 1989) or the evolution of galactic nuclei (Langbein et al. 1990) as well as for isolated self-gravitating star clusters (see below). Hydrodynamic "gaseous" models with momentum equations up to second order and a zero heat flux closure (cf. e.g. Marochnik 1964) were also used to examine equilibria and stability of self-gravitating collisionless stellar systems (e.g. Kondrat'ev and Malkov 1986, 1987). Evans and Lynden-Bell (1989) showed that such "stellar hydrodynamic" equations can be solved by Green's function methods for Eddington's stellar systems with separable potentials in special coordinates based on principal velocity surfaces. These examples should illustrate that even for collisionless stellar systems it is rather common and convenient to use hydrodynamic models. However, it is not finally clear, to what extent or under which conditions such a hydrodynamic model holds as a description of real stellar systems.

The situation is different in the case of collisional stellar systems, which are the subject of the remainder of this paper. Collisions here denote the distant gravitative two-body encounters, which act on a stellardynamical relaxation timescale (see e.g. Chandrasekhar 1942), which is large compared to the dynamical timescale for most astrophysical star clusters. There is a fair amount of qualitative and quantitative comparisons between momentum and other models like the direct solution of the Fokker-Planck equation and direct N-body calculations. In particular isotropic one-mass (no stellar mass spectrum) momentum models have been adapted so as to yield sufficiently congruent results with Fokker-Planck models (compare results of Cohn 1980, Bettwieser and Sugimoto 1984, Heggie 1984, Heggie and Stephenson 1988, Cohn et al. 1989, Heggie and Ramamani 1989). There is also a quantitative comparison between an N-body calculation and a momentum model with heat flux closure (Bettwieser and Sugimoto 1985) which suffers from statistical noise due to a rather small N-body particle number (N = 1000), but its results give some evidence in favor of the gaseous models.

In general the stellar velocity distribution, however is not isotropic (different second order moments for different space coordinate directions) and depends also on the individual stellar mass as an additional independent variable. Anisotropic momentum models (called anisotropic "gaseous" models, because their equations at least up to second order still resemble very much normal hydrodynamical equations) for the secular evolution of star clusters were presented by e.g. Saito and Yoshizawa (1976) and Angeletti and Giannone (1977) (both based on Larson's (1970) hydrodynamical approach) and by Bettwieser and Spurzem (1986) with a generalized heat flux closure analogous to the isotropic case. Although the models of Larson and their decessors according to the analysis in Louis (1990) do not appropriately close the moment equations, there still remain different types of heat flux closures and one different closure in 5th order (Louis 1990). In the precollapse models their results are in very good agreement; but this needs not to be the case in other situations (e.g. post-collapse, multi-mass models); it is not yet clearly established (e.g. by comparison with other methods) how an appropriate anisotropic momentum model looks like.

A drastic change of the evolution of star clusters occurs with a stellar mass spectrum compared to the single mass case. The evolution is accelerated by the competing effects of mass segregation and the tendency towards equipartition; recently there have been published already rather sophisticated Fokker-Planck models including a mass spectrum as well as dissipative effects and binary formation and evolution (Murphy et al. 1990, Quinlan and Shapiro 1990). Since the detection of serious discrepancies between Fokker-Planck and gaseous multi-mass models (Bettwieser and Inagaki 1985) there has not been reported another attempt to improve the hydrodynamical models.

The general advantage of hydordynamical models is that any additional physical processes like e.g. binary formation and evolution or interaction with interstellar matter can be implemented in a straightforward way. Experiences with the numerical solution of the equations for hydrodynamical flows can be applied to the stellar dynamical problem. Therefore it is stressed that it is worth while to develop also the hydrodynamical models further and to study their consequences. Two basic approaches are possible: first to improve the quantitative comparisons between the numerical time-dependent solutions of the various methods, e.g. for anisotropic and multi-mass pre- and post-collapse evolution of star clusters. Another way would be to study the consequences of generalized thermodynamic concepts for self-gravitating gaseous spheres including the effects of anisotropy (anisotropy here and in the following always means a difference between the second-order moments, centered to the bulk mass motion, in different spatial directions; in other words, there is a direction dependent temperature). The results could be related to what one would expect in real stellar systems.

The latter approach led to the gravothermal instability picture (Antonov 1962, Lynden-Bell and Wood 1968, Hachisu and Sugimoto 1978 (henceforth HS), Nakada 1978, Hachisu et al. 1978) and was also applied for the examination of star clusters with rotation (Hachisu 1979) or with different mass groups (Yoshizawa et al. 1978, Inagaki and Wiyanto 1984, Wiyanto 1989). The concept is to study the consequence of the assumption that there is an H-functional which increases during the system's evolution. According to Yoshizawa et al. (1978) and references therein the usual Boltzmann entropy

$$S = - k \sum_i \int f_i \ln f_i d^3 v d^3 r \qquad (1)$$

(where k denotes Boltzmann's constant, i different mass groups, and f_i the single particle distribution function) is such a H-functional, provided the system's evolution is described by a Fokker-Planck equation of the type of Rosenbluth et al. (1957). A linear perturbation analysis is performed and those perturbation functions are searched by a variational procedure which extremize the entropy variation; if there are such functions for positive entropy variation the system is considered to be unstable. Such method is generalized here for the case of anisotropic hydrostatic equilibria as well as anisotropy perturbations even in the case of initially isotropic equilibrium solutions.

HS introduced the concept of inverse specific heat tensors; the core region of a self-gravitating system has a negative specific heat, since after the removal of heat (i.e. r.m.s. kinetic energy of the stars) the readjustment of hydrostatic equilibrium yields an increase of the central velocity dispersion (i.e. temperature). The halo, however, always has a positive specific heat. The total heat capacity of the core compared to the one of the halo decreases with increasing density contrast D. The unstable systems with D > 709 have the property, that the increase of temperature due to removal of heat in the centre is larger than the temperature increase due to the input of that same amount of heat into the halo. Thus an initially small temperature gradient will be enhanced by this readjustment and the gravothermal runaway starts. Such simple qualitative discussion demonstrates how the concepts of thermodynamics and specific heat tensors give physical insight into how the mechanism of gravothermal contraction and catastrophe works (for a review compare Sugimoto 1985).

Bettwieser and Sugimoto (1984, henceforth BS) presented such perturbation analysis for a singular isotropic and isothermal equilibrium solution (SIS). They found SIS to be gravothermally unstable, as is expected for the limiting case of a series of regular models with increasing density contrast. The advantage of taking SIS as equilibrium solution was the complete analytic tractability of the problem. In this paper the question how the linear perturbation analysis of gravothermal instability is changed by the possible anisotropy generation in comparison to the work of HS and BS is addressed. The method is analogous to HS and BS, i.e. to perturb hydrostatic equilibrium solutions to first order and to extremize thereafter the total variation of the Boltzmann entropy. Method and results are discussed in more detail in Spurzem (1991).

Within the next section the anisotropic hydrostatic equilibrium solutions are discussed and their first order perturbation. Section 3 outlines the extremization of the entropy functional and some results; the last section contains concluding remarks.

2. LINEAR PERTURBATION OF ANISOTROPIC HYDROSTATIC EQUILIBRIUM

Let M and R denote the total mass and radius of an isolated self-gravitating gas sphere. Then one defines appropriate normalized quantities

$$\phi = \frac{M_r}{M} ; \; x = \frac{r}{R} ; \; p, p_t = \frac{P, P_t}{GM^2/(4\pi R^4)} ; \; A = 2 - \frac{p_t}{p} ;$$

$$\psi = \frac{\rho}{M/(4\pi R^3)} ; \; \theta, \theta_t = \frac{\sigma^2, \sigma_t^2}{GM/R} , \tag{2}$$

where M_r, r, ρ, P, P_t, σ^2 and σ_t^2 denote the mass contained within a sphere of radius r, the mean mass density, the radial and tangential component of pressure and of the velocity dispersion, respectively, and A measures the degree of anisotropy of the velocity distribution of the particles of the gas. The equations of state are analogous to an ideal gas $p = \psi\theta$, $p_t = \psi\theta_t$. The quantity $\theta_{th} := (\theta + \theta_t)/3$ is introduced as thermodynamical temperature such that a caloric equation of state $\epsilon = 3\theta_{th}/2$ results for the energy density, where all quantities are normalized according to Eq.(2).

The Boltzmann entropy is (ignoring a "zero shift" due to constant summands):

$$S = \ln\left(\frac{\theta^{3/2}(1-A/2)}{\psi} \right) \tag{3}$$

We consider the equations of anisotropic hydrostatic equilibrium

$$\frac{\partial \ln p}{\partial \phi} = - \frac{\phi}{px^4} - \frac{A}{\psi x^3} ; \quad \frac{\partial \ln x}{\partial \phi} = \frac{1}{\psi x^3} \tag{4}$$

A solution of these equations is

$$x = \phi ; \; \psi = \frac{1}{2} ; \; \theta = \frac{1}{2-A} ; \; A(\phi) = \text{const.}; \tag{5}$$

which is referred to as the anisotropic singular isothermal solution (ASIS, Bettwieser and Spurzem 1986). BS discussed the gravothermal stability of the isotropic singular isothermal solution (SIS); such a solution occurs for example in the post-collapse evolution of globular clusters (Inagaki and Lynden-Bell 1983); it may be regarded as a model of a regular solution with an infinitely small homogeneous core;

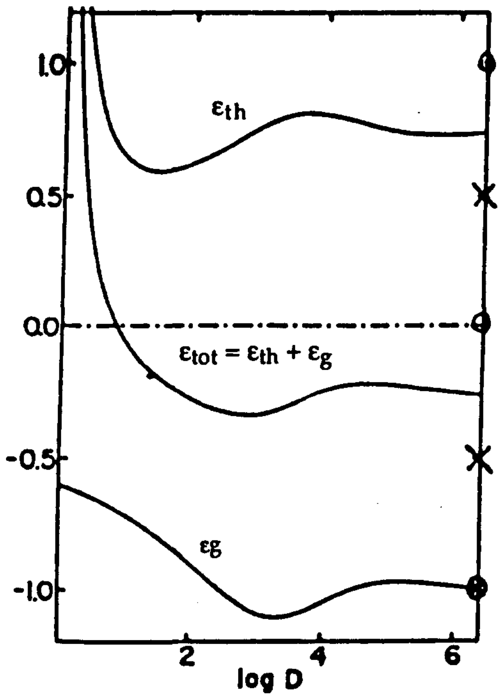


Figure 1: Series of isotropic hydrostatic equilibrium solutions from Hachisu and Sugimoto (1978); plotted are the dimensionless values of internal, gravitational and total energy versus the density contrast D between centre and outer boundary. The rightmost value of D belongs to solutions almost equivalent to a singular equilibrium solution (SIS) with infinite density contrast. Crosses mark the corresponding values for the anisotropic SIS ($A \to -\infty$) and circles for A = 1.

similarly ASIS could be regarded as such a solution with an anisotropic halo. Fig. 1 depicts how the ASIS solutions are related to the family of isothermal gas spheres; the values of the thermal (ε_{th}), gravitational (ε_g), and total energy (ε_{tot}) per unit mass are in our normalization

$$\varepsilon_{th} = \frac{1}{2}\left(1 + \frac{1}{2-A}\right); \quad \varepsilon_g = -1; \quad \varepsilon_{tot} = -\frac{1}{2}\left(1 - \frac{1}{2-A}\right) \qquad (6)$$

The linearly perturbed Eqs. (3) to (5) take the form

$$L_S \delta S = L_x \; \delta \ln x - L_A \delta A \; , \text{ with} \tag{7}$$

$$L_x = \frac{5}{3} \phi^2 \frac{d^2}{d\phi^2} + \frac{2}{3} (5+A) \phi \; \frac{d}{d\phi} - (2-A),$$

$$L_S = \frac{2}{3} \phi \frac{d}{d\phi} - \frac{2}{3} (2-A), \tag{8}$$

$$L_A = \frac{2}{3(2-A)} \phi \frac{d}{d\phi} + \frac{1}{3}$$

By multiplication with an integrating function $\chi = \phi^{2A/5}$ the operator L_x becomes of Sturm-Liouville type and can be inverted analytically by computing a Green's function $G_x(\phi,\phi')$ for the boundary conditions $\phi\delta \ln x = 0$ at $\phi = 1$ and $\phi = 0$; it follows

$$\delta \ln x = \int_0^1 G_x(\phi,\phi')\chi(\phi')(L_S(\phi')\delta S(\phi') + L_A(\phi')\delta A(\phi'))d\phi' \tag{9}$$

3. HYDROSTATIC READJUSTMENT

The entropy perturbation results from Eq.(3) as:

$$\delta S = \frac{3}{2} \delta \ln \Theta_{th} - \delta \ln \psi - \frac{A}{2(2-A)(3-A)} \delta A \tag{10}$$

The first order perturbation in total energy δE including thermal and potential energy is expressed as $\delta E = \int \delta q \; d\phi$ with

$$\delta q = \Theta_{th} \; \delta S + \frac{A}{2(2-A)(3-A)} \Theta_{th} \delta A + \delta \ln x \left(\frac{A\phi}{3x} - A\Theta_{th} - \frac{px^3}{3} \cdot \frac{dA}{d\phi} \right). \tag{11}$$

The quantity $\delta q(\phi)$ may be interpreted as the net amount of "heat" shifted to or from the zone between ϕ and $\phi + d\phi$. In case of isotropy the well known equation $\delta q = \Theta_{th} \delta S$ is recovered. The second term in Eq.(11) is due to the anisotropy dependent entropy and the $\delta \ln x$ terms are parts of terms related to pdV work and potential energy variation, which do cancel each other only in isotropic gaseous spheres.

With the condition that the system is enclosed in an adiabatic wall ($\delta E = 0$) and using the isotropic SIS solution the total variation of entropy $\delta^2\Sigma$ is correct to second order

$$\delta^2\Sigma = \int_0^1 d\phi\{-\frac{1}{3}(\delta S_{th})^2 + \frac{1}{3}\delta S_{th}\delta A + (\delta S_{th} - \frac{1}{2}\delta A).$$

$$(1+\frac{1}{3}\phi\frac{d}{d\phi})\delta \ln x - \frac{1}{6}\delta A\phi\frac{d}{d\phi}\delta \ln x\}. \tag{12}$$

Here the convenient quantity $\delta S_{th} = \delta S + A/(2-A)$ was introduced; the index "th" (= "thermal") is chosen in order to remind that δS_{th} contains what would be the entropy perturbations in absence of anisotropy perturbations. Let for brevity be $f = \delta S_{th}$ and $g = \delta A$; as subsidiary conditions are imposed $\delta E = 0$ (Lagrangian parameter μ) and the square integrability of f and g (Lagrangian parameters λ_1, λ_2). Thus a functional J(f,g) turns out:

$$J(f,g) = \delta^2\Sigma(f,g) - \lambda_1\{\int_0^1 f^2 d\phi - f_o^2\} - \lambda_2\{\int_0^1 g^2 d\phi - g_o^2\}$$

$$+ 2\mu\int_0^1 (f - \frac{1}{2}g)d\phi. \tag{13}$$

In order to compute those perturbation functions belonging to maximum entropy variation the variational problem $\Delta J(f,g) = 0$ can be solved analytically. For the extremizing functions the total entropy variation takes a very simple form:

$$\delta^2\Sigma = f_o^2\lambda_1 + g_o^2\lambda_2 \tag{14}$$

Note the analogy of this formula to the corresponding result of HS and BS for A = 0, $\delta A = 0$.

There are only certain combinations of the Lagrangian parameters which lead to physical (i.e. real) particular solutions, which fulfil the subsidiary conditions. Whereas μ can be chosen freely, there are only certain couples of λ_1, λ_2 allowed. The areas covered by such "good" λ_1, λ_2 values are depicted in a λ_1-λ_2-plane in Fig. 2 and labelled with "1" and "2". Area "1" contains the region with entirely real solutions, whereas in area "2" a solution containing two complex conjugate summands occurs. Area "1" corresponds in the isotropic calculation of BS to the range $1/3 > \lambda_1 > 7/29$, and area "2" to the range $7/29 > \lambda_1 > -1/5$. The singularities of our critical lines occur just at the critical values $\lambda_1 = -1/5, 7/29, 1/3$ of the isotropic BS problem. Note that due to the presence of anisotropy perturbations as an additional degree of freedom we have now a two-parameter family of solutions. Since

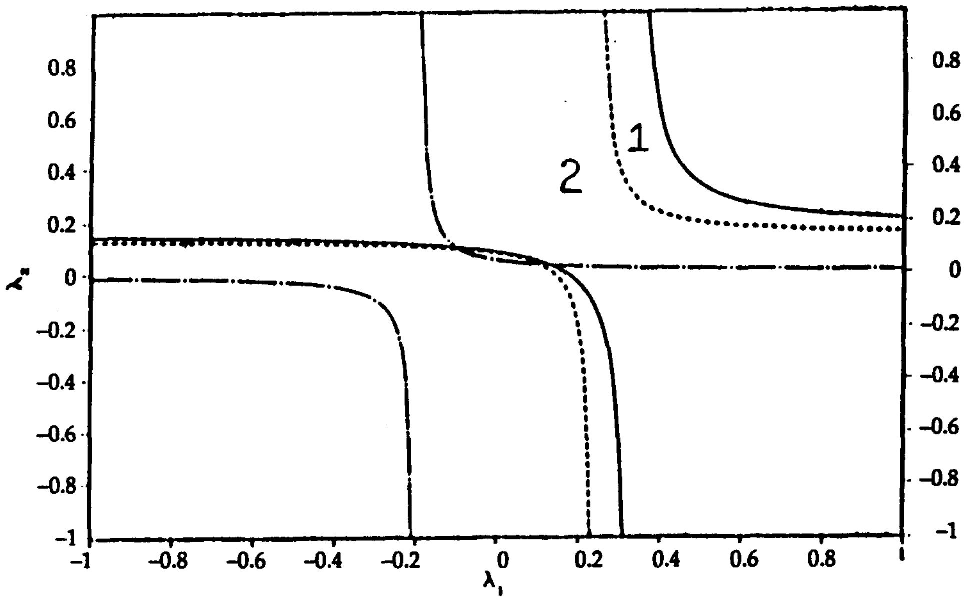


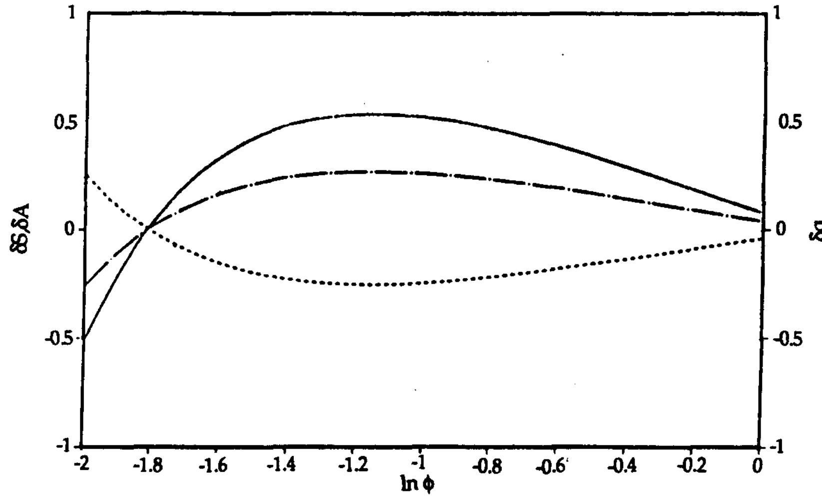
Figure 2: A = 0: Critical lines in the λ_1-λ_2 plane for the existence of particular solutions of the variational problem. The areas denoted by "1" and "2" within the critical lines allow for real ("1") and complex conjugate ("2") solutions of the variational problem; those parts lying in the first quadrant ($\lambda_1,\lambda_2 > 0$) belong to positive total entropy variation. Outside of those two regions no solution of the variational problem can be found.

both areas cover partly the first quadrant ($\lambda_1,\lambda_2 > 0$) one concludes from Eq.(14) that $\delta^2\Sigma > 0$ and thus SIS is gravothermally unstable.

As an example the particular solution for $\lambda_1 = \lambda_2 = 0.4$ is plotted in Figs. 3ab. In contrast to the isotropic analysis of BS, where they found an upper limit for the admitted λ-values, we have here an unbound spectrum of modes for λ_1, $\lambda_2 \to \infty$.

Characteristic results visible in the Figs. 3ab are summarized as follows:

i) the perturbations belonging to maximal entropy production have a contracting and an expanding region ($\delta \ln x < 0$ and v.v);



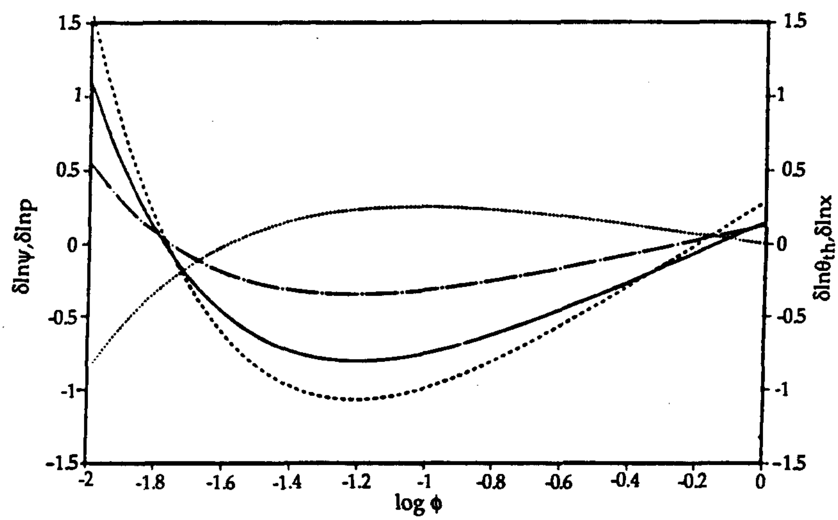


Figure 3ab: Second order perturbation analysis; plots of the perturbation functions belonging to maximum total entropy variation; isotropic equilibrium model; $\lambda_1 = \lambda_2 = 0.4$. Upper figure 3a: entropy δS(solid line), anisotropy δA (short dashes), heat exchange δq (dash-dotted line). Lower figure (1b): density $\delta \ln \psi$(solid line), radial pressure $\delta \ln p$ (short dashes), thermodynamic temperature $\delta \ln \Theta_{th}$ (dashed-dotted line), and radius $\delta \ln x$ (dotted line).

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note that by taking another sign of μ this behaviour can be exchanged; it does not alter, however the general feature that $\delta \ln x$ has a non-trivial zero point (compare Figure 3b);

ii) there is negative specific heat observed in the centre; heat and entropy are removed from the central part $\delta q < 0$, but the temperature increases thereof $\delta \ln \Theta_{th} > 0$; again the opposite, but also gravothermal result could be found by changing the sign of μ(with $\delta q > 0$, but $\delta \ln \Theta_{th} < 0$ in the centre).

iii) there is anisotropy generation with positive sign (i.e. $\sigma^2 > \sigma_t^2/2$) in the central regions and of negative sign outwards

This illustrates that an initially isotropic self-gravitating "gas" sphere can increase its total entropy in connection with anisotropy generation. This is a difference to the findings in the first order theory, where the global tendency of the anisotropy perturbation behaved in accord with thermodynamic expectations.

4. DISCUSSION AND CONCLUSIONS

The main result here is that the singular equilibrium solution is gravothermally unstable under inclusion of the effect of anisotropy perturbations. There can be found perturbation functions belonging to arbitrarily large entropy production. This is due to the lack of an equation restricting the anisotropy in hydrostatic equilibrium. However, in reality or in time-dependent numerical evolution calculations the anisotropy is determined by a separate second order moment equation.

Including anisotropy perturbations and anisotropic equilibrium models does not alter the general picture of gravothermal instability: if heat is transferred outwards, the negative specific heat in the centre of a self-gravitating system yields a temperature increase there (gravothermal contraction), or v.v. (gravothermal expansion). Initially isotropic models can increase their entropy further by generation of anisotropy. As the phenomenon of negative specific heat this result is from a thermodynamic viewpoint counterintuitive and can be explained by the presence of self-gravity.

At the present stage of the anisotropic perturbation analysis one should be very careful in conclusion about the behaviour of real stellar systems, because the singular equilibrium solution, which was taken here to keep the problem analytically tractable, is rather artificial, and also because this linear perturbation theory certainly will not describe evolutionary effects due to the full non-linear stellar dynamical evolution equations. What here is interesting to conclude is that the generation of anisotropy in stellar systems can be understood from thermodynamic principles; this

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gives some evidence in favour of a hydrodynamical (moment) description even of anisotropic stellar systems.

More realistic regular, but isotropic equilibrium solutions were analyzed in the isotropic case by HS. They had a discrete spectrum of modes, i.e. there exist only discrete values of λ_1, which lead to physical solutions of the variational problem, in contrast to the singular solution, which has a continuous spectrum (BS). Below the critical density constrat of $D_{crit} = 709$ there is no hydrostatic readjustment with positive total entropy variation at all. It would be interesting to calculate the hydrostatic readjustment of regular anisotropic equilibrium solutions. The general procedure should be along the way outlined in this paper, and the result could reveal whether the critical density contrast itself is a function of the equilibrium anisotropy profile. Possibly by this way the peculiarity of the number 709 can be removed; to get the correct limit of 709 for $A \to 0$, however, the second order perturbation theory should be applied also for the anisotropic equilibrium solutions, in order to get the correct limit for $A \to 0$.

I thank Dr. D.C.Heggie for enlightening and very helpful discussion. This work has been partly supported by the "Deutsche Forschungsgemeinschaft".

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