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ON HEREDITARY AND COHEREDITARY MODULES

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1. Introduction. A recent paper by Goro Azumaya on M-projective and M-injective modules [**1**] suggests a generalization of the concept of hereditary rings to modules which is also capable of dualization. Section 2 is devoted to preliminaries on M-projective and M-injective modules.

In section 3, we introduce the notion of hereditary and cohereditary modules. An R-module is called hereditary if every R-submodule of it is projective. Cohereditary modules are defined dually. We characterize hereditary and cohereditary modules in terms of M-projectivity and M-injectivity. It is shown that the class of hereditary modules is closed under submodules and direct sums, and that the class of cohereditary modules is closed under homomorphic images and finite direct sums.

Section 4 consists of applications and some generalizations of section 3. Hereditary rings are characterized in terms of cofaithful hereditary modules. Semi-cohereditary rings are defined using Vamos' idea [**7**] of the dual notion of finitely generated. It is shown that these rings are precisely the V-rings in the sense of Carl Faith [**4**]. Throughout this paper, R will always denote an associative ring with 1. All modules are unital left R-modules. All maps will be R-homomorphisms.

2. Preliminaries. Let M be a fixed R-module.

Definition 2.1. A module Q is called M-projective if given any epimorphism $f : M \rightarrow N$ and any $g : Q \rightarrow N$, there exists $h : Q \rightarrow M$ such that $f \circ h = g$. M-injective modules are defined dually.

Definition 2.2. Let A, B be R-modules and $\psi : A \rightarrow B$ an epimorphism. ψ is called an M-epimorphism if there exists $h : A \rightarrow M$ such that $\ker \psi \cap \ker h = 0$. M-monomorphisms are defined dually.

THEOREM 2.3 [**1**]. *Let M, Q be R-modules. Then the following conditions are equivalent.*

(1) *Q is M-projective.*

(2) *Given any M-epimorphism $\psi : A \rightarrow B$, and any $g : Q \rightarrow B$, there exists $f : Q \rightarrow A$ such that $\psi \circ f = g$.*

(3) *Every M-epimorphism onto Q splits.*

The dual of this theorem characterizes M-injective modules.

Let M and Q be fixed R-modules.

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Definition 2.4. $C_p(M)$ = the class of all M-projective modules. $C_i(M)$ = the class of all M-injective modules. $C^p(Q)$ = the class of all modules N such that Q is N-projective. $C^i(Q)$ = the class of all modules N such that Q is N-injective.

Using Theorem 2.3 and its dual, Azumaya has proved the following

PROPOSITION 2.5 [**1**]. (1) *$C_p(M)$ is closed under the formation of direct sums and direct summands.*

(2) *$C_i(M)$ is closed under the formation of direct products and direct factors.*

(3) *$C^p(Q)$ is closed under submodules, homomorphic images and the formation of finite direct sums. If Q has a projective cover, $C^p(Q)$ is closed under the formation of arbitrary direct products.*

(4) *$C^i(Q)$ is closed under submodules, homomorphic images and direct sums.*

3. Hereditary and cohereditary modules.

Definition 3.1. A module M is called hereditary if every submodule of M is projective. Cohereditary modules are defined dually.

From [**3**, Theorem 5.4, p. 14], we note that a ring R is left hereditary if and only if every projective left R-module is hereditary or equivalently if every injective left R-module is cohereditary. It is obvious that a submodule (respectively, homomorphic image) of a hereditary (respectively, cohereditary) module is hereditary (respectively, cohereditary).

THEOREM 3.2. *The following conditions are equivalent for a projective module M.*

(1) *M is hereditary.*

(2) *Every quotient of an M-injective module is M-injective.*

(3) *Every quotient of an injective module is M-injective.*

Proof. (1) $\Rightarrow$ (2). Let A be M-injective and consider the following diagram with exact rows:

$$\begin{array}{ccccc} A & \xrightarrow{\pi} & B & \longrightarrow & 0 \\ & & f\uparrow & & \\ 0 & \longrightarrow & N & \xrightarrow{g} & M. \end{array}$$

Since N is projective, there exists $h : N \to A$ such that $\pi \circ h = f$. By the M-injectivity of A, we obtain $l : M \to A$ such that $l \circ g = h$. Then $k = \pi \circ l : M \to B$, gives $k \circ g = \pi \circ l \circ g = \pi \circ h = f$.

(2) $\Rightarrow$ (3). This is clear.

(3) $\Rightarrow$ (1). Let

$$0 \to N \xrightarrow{\tau} M \text{ be exact.}$$

By [**3**, Proposition 5.1, p. 12], to show that N is projective, it suffices to consider the following diagram with exact rows, and A injective:

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\tau"] \arrow[d, "f"] & M \\ A \arrow[r, "\pi"] & B \arrow[r] & 0. \end{tikzcd}$$

Using (3), we obtain $g : M \to B$ such that $g \circ \tau = f$. Since M is projective, there exists $h : M \to A$ such that $\pi \circ h = g$. Then $k = h \circ \tau : N \to A$, gives $\pi \circ k = g \circ \tau = f$.

THEOREM 3.2′. *The dual of Theorem* 3.2.

PROPOSITION 3.3. *Let* $\{M_i\}_{i \in I}$ *be a family of R-modules. Then,*

$$M = \sum_{i \in I} \oplus M_i$$

is hereditary if and only if each M_i *is hereditary.*

Proof. $\Rightarrow$: This is clear.

$\Leftarrow$: Let M_i be hereditary $(i \in I)$. Consider an epimorphism $\pi : Q \to Q'$ with Q injective. By Theorem 3.2, Q' is M_i-injective $(i \in I)$, and then by Proposition 2.5, (4) Q' is M-injective. Thus, M is hereditary.

PROPOSITION 3.3′. *Let* $\{M_i\}_{i=1}^n$ *be R-modules* $(n \in \mathbf{Z}^+)$. *Then*

$$M = \sum_{i=1}^n \oplus M_i$$

is cohereditary if and only if each M_i *is cohereditary.*

Proof. Use Theorem 3.2′ and Proposition 2.5 (3).

4. Applications.

Definition 4.1 [**2**]. An R-module M is called cofaithful if there exists a positive integer n, and a monomorphism $\theta : R \to M^n = M \oplus \ldots \oplus M$ (n copies).

PROPOSITION 4.2. *Let R be any ring. Then the following are equivalent.*

(1) *R is left hereditary.*

(2) *There exists a cofaithful hereditary R-module.*

Moreover, if R is left perfect, then the above conditions are equivalent to

(3) *There exists a faithful cohereditary R-module.*

Proof. (1) $\Rightarrow$ (2). Let $M = {}_RR$. Then M is cofaithful, hereditary.

(2) $\Rightarrow$ (1). Let M be a cofaithful, hereditary R-module. Then, we obtain an embedding $\theta : R \to M^n$, for some positive integer n. Since M is hereditary, by Proposition 3.3, M^n is hereditary, and hence R is left hereditary.

(1) $\Rightarrow$ (3). Let $M = E(R)$ = injective hull of ${}_RR$. Then M is faithful and cohereditary.

Next in (3) $\Rightarrow$ (1), let us assume that R is left perfect. Let Q be any faithful and cohereditary left R-module. We obtain an embedding $0 \rightarrow R \rightarrow \prod Q$, for some direct product of copies of Q, and then $\prod Q$ is cohereditary, since R is left perfect. By Proposition 2.5 (3), R is left hereditary.

Remarks. (i) If R is left noetherian, then a direct sum of cohereditary modules is cohereditary, and so a proof similar to that of Proposition 4.2 shows that R is left hereditary if and only if there exists a cofaithful cohereditary module.

(ii) If R is commutative artinian, then R is hereditary if and only if R is semi-simple. The following example shows that the commutativity is necessary.

Example. Let

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a \in \mathbf{Q}, b, c \in \mathbf{R} \right\}.$$

It can be shown that R is right hereditary, and is right artinian, but not left artinian [**5**, p. 72].

We next give some examples of hereditary and cohereditary modules.

Example 1. Let

$$R = \left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} | a \in \mathbf{Z}, b, c \in \mathbf{Q} \right\}.$$

Then, R is not left hereditary and

$${}_RM = \left\{ \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} | a \in \mathbf{Z} \right\}$$

is hereditary.

Example 2. Let R be any Boolean ring. Then every simple R-module is cohereditary [**6**].

Next, we generalize some of our earlier concepts.

Definition 4.4. A module M is called semi-hereditary if every finitely generated (f.g.) submodule of M is projective.

Using the "dual notion of f.g." introduced by Vamos [**7**], we make the following

Definition 4.4′ A module M is called semi-cohereditary if every co-finitely generated (co-f.g.) quotient of M is injective. (A module M is called co-f.g. if $E(M)$ = injective hull of $M \cong E(S_1) \oplus \ldots \oplus E(S_n)$, for some simple R-modules, S_i.)

Definition 4.5. A ring R is called left semi-cohereditary if ${}_RR$ is semi-cohereditary.

PROPOSITION 4.6. *The following conditions are equivalent.*

(1) *R is left semi-cohereditary.*
(2) *Every simple left R-module is injective.*
(3) *Every co-f.g. left R-module is injective.*

Proof. (1) $\Rightarrow$ (2). Let M be any simple R-module. We obtain an epimorphism $\pi : R \rightarrow M$. Clearly M is co-f.g. and hence is injective.

(2) $\Rightarrow$ (3). Let A be any co-f.g. left R-module. Then, socle of A injective, essential in $E(A)$ injective, implies socle of $A = E(A) = A$.

(3) $\Rightarrow$ (1). This is clear.

Remarks. (i) The equivalence of (1) and (2) shows that left semi-cohereditary rings are precisely the left V-rings [**4**, p. 130].

(ii) Let R be left self injective. Then the three conditions of Proposition 4.6 are equivalent to condition

(4) Every co-f.g. quotient of an injective R-module is injective.
This can be regarded as the dual of [**3**, Proposition 6.2, p. 15].

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