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TRANSITIVITY, LOWNESS, AND RANKS IN NSOP$_1$ THEORIES

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Abstract. We develop the theory of Kim-independence in the context of NSOP$_1$ theories satisfying the existence axiom. We show that, in such theories, Kim-independence is transitive and that $\mathop{\smile\hskip-0.9em\raise0.5ex\hbox{$\vert$}}^K$-Morley sequences witness Kim-dividing. As applications, we show that, under the assumption of existence, in a low NSOP$_1$ theory, Shelah strong types and Lascar strong types coincide and, additionally, we introduce a notion of rank for NSOP$_1$ theories.

§1. Introduction. This paper furthers the development of the theory of Kim-independence in the context of NSOP$_1$ theories satisfying the existence axiom. Building on earlier work in [4] and a suggestion of the second-named author [11], Kim-independence was introduced in [7], where it was shown to be a well-behaved notion of independence in NSOP$_1$ theories. This work established a strong analogy between the theory of non-forking independence in simple theories and Kim-independence in NSOP$_1$ theories, an analogy which subsequent works have only deepened [8, 9, 13]. Nonetheless, one major difference between the two notions of independence is that, unlike non-forking which makes sense over all sets, Kim-independence is only a sensible notion of independence *over models*: Kim-independence is defined in terms of formulas that divide with respect to a Morley sequence in a global invariant type, and such a sequence, in general, is only guaranteed to exist over a model. In [5], the second- and third-named author, together with Dobrowolski, focused on the context of NSOP$_1$ theories that satisfy *the existence* axiom. There, it was shown that Kim-independence may be defined over arbitrary sets and basic theorems of Kim-independence over models hold in this broader context.

The existence axiom states that every complete type has a global non-forking extension, i.e., every set is an extension base in the terminology of [3]. This is equivalent to the statement that, in every type, there is a (non-forking) Morley sequence and, hence, assuming existence, one may redefine Kim-independence in terms of the formulas that divide along Morley sequences of this kind. New technical challenges arise in this setting, but in [5] it was shown that Kim-independence satisfies Kim's lemma, symmetry, and the independence theorem for Lascar strong types. Moreover, all simple theories and all known examples in the growing list of NSOP$_1$ theories satisfy existence, and it is expected to hold in all NSOP$_1$ theories (see, e.g., [5, Fact 2.14]).

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Here we continue work on Kim-independence in NSOP_1 theories satisfying existence, in particular, exploring aspects of the theory that are too cumbersome or uninteresting over models. In Section 3, we show that Kim-independence is transitive in an NSOP_1 theory satisfying existence and that, moreover, Kim-dividing is witnessed by $\mathop{\smile\!\!\!\!\!\!|}^K$-Morley sequences. These results were first established over models for all NSOP_1 theories in [8] and our proofs largely follow the same strategy. Nonetheless, suitable replacements need to be found for notions that only make sense, in general, over models, like heirs and coheirs. We find that arguments involving these notions can often be replaced by an argument involving a tree-induction, as in the construction of tree Morley sequences in [8]. In Section 4, we apply these results to low NSOP_1 theories satisfying existence, showing that Shelah strong types and Lascar strong types coincide, generalizing a result of Buechler for simple theories [1] (see also [12, 17]). In Section 5, we introduce a notion of rank for NSOP_1 theories and establish some of its basic properties. Finally, in Section 6, we generalize the Kim–Pillay criterion for Kim-independence from [4, Theorem 6.1] and [8, Theorem 6.11] to give a criterion for NSOP_1 in theories satisfying existence, which, additionally, gives an abstract characterization of Kim-independence over arbitrary sets in this setting.

§2. Preliminaries.

In this paper, T will always be a complete theory with monster model $\mathbb{M}$. We will implicitly assume all models and sets of parameters are small, that is, of cardinality less than the degree of saturation and homogeneity of $\mathbb{M}$. If we discuss an I-indexed indiscernible sequence $(a_i)_{i\in I}$, we will implicitly assume I is linearly ordered by $<$ and, given $i \in I$, we will write $a_{<i}$ and $a_{\leq i}$ for the subsequences $(a_j)_{j<i}$ and $(a_j)_{j\leq i}$ respectively.

Definition 2.1. Suppose A is a set of parameters.

(1) We say that a formula $\varphi(x; a)$ *divides* over a set A if there is an A-indiscernible sequence $\langle a_i : i < \omega\rangle$ with $a_0 = a$ such that $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.
(2) A formula $\varphi(x; a)$ is said to *fork* over A if $\varphi(x; a) \vdash \bigvee_{i<k} \psi_i(x; c_i)$, for some $k < \omega$, with $\psi_i(x; c_i)$ dividing over A.
(3) We say a partial type divides (forks) over A if it implies a formula that divides (forks) over A.
(4) For tuples a and b, we write $a \mathop{\smile\!\!\!\!\!\!|}^d_A b$ or $a \mathop{\smile\!\!\!\!\!\!|}_A b$ to indicate that $\mathrm{tp}(a/Ab)$ does not divide over A or does not fork over A, respectively.
(5) A *Morley sequence* $(a_i)_{i\in I}$ over A is an infinite A-indiscernible sequence such that $a_i \mathop{\smile\!\!\!\!\!\!|}_A a_{<i}$ for all $i \in I$. If $p \in S(A)$, we say $(a_i)_{i\in I}$ is a Morley sequence *in* p if, additionally, $a_i \models p$ for all $i \in I$.

The following is one of the key definitions of this paper. It defines a context in which Kim-independence may be studied over arbitrary sets.

Definition 2.2. We define the *existence axiom* to be any one of the following equivalent conditions on T:

(1) For all parameter sets A, any type $p \in S(A)$ does not fork over A.
(2) For all parameter sets A, no consistent formula over A forks over A.

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(3) For all parameter sets A, every type $p \in S(A)$ has a global extension that does not fork over A.
(4) For all parameter sets A and any $p \in S(A)$, there is a Morley sequence in p.

If T satisfies the existence axiom, we will often abbreviate this by writing T is *with existence*. See, e.g., [5, Remark 2.6] for the equivalence of (1)–(4).

Under existence, we may define Kim-independence over arbitrary sets. The following definition was given in [5], but it was observed already in [7, Theorem 7.7] that this agrees with the original definition over models.

Definition 2.3. Suppose T satisfies the existence axiom.
(1) We say a formula $\varphi(x; a)$ *Kim-divides* over A if there is a sequence $\langle a_i : i < \omega \rangle$ which is a Morley sequence over A with $a_0 = a$ and $\{\varphi(x; a_i) : i < \omega\}$ inconsistent.
(2) A formula $\varphi(x; a)$ is said to *Kim-fork* over A if $\varphi(x; a) \vdash \bigvee_{i<k} \psi_i(x; c_i)$, where each $\psi_i(x; c_i)$ Kim-divides over A.
(3) We say a type Kim-divides (Kim-forks) over A if it implies a formula that Kim-divides (Kim-forks) over A.
(4) For tuples a and b, we write $a \mathop{\smile\!\!\!\!\!\!\;|}^K_A b$ to indicate that $\mathrm{tp}(a/Ab)$ does not Kim-divide over A.
(5) An $\mathop{\smile\!\!\!\!\!\!\;|}^K$-*Morley sequence* $(a_i)_{i \in I}$ over A is an infinite A-indiscernible sequence such that $a_i \mathop{\smile\!\!\!\!\!\!\;|}^K_A a_{<i}$ for all $i \in I$.

Remark 2.4. By Kim's lemma [12, Proposition 2.2.6], if T is simple, a formula Kim-divides over a set A if and only if it divides over A.

Definition 2.5 [6, Definition 2.2]. The formula $\varphi(x; y)$ has SOP$_1$ if there is a collection of tuples $(a_\eta)_{\eta \in 2^{<\omega}}$ so that:

- For all $\eta \in 2^\omega$, $\{\varphi(x; a_{\eta|\alpha}) : \alpha < \omega\}$ is consistent.
- For all $\eta \in 2^{<\omega}$, if ν extends $\eta \frown \langle 0 \rangle$, then $\{\varphi(x; a_\nu), \varphi(x; a_{\eta \frown 1})\}$ is inconsistent,

where $\trianglelefteq$ denotes the tree partial order on $2^{<\omega}$. We say T is SOP$_1$ if some formula has SOP$_1$ modulo T. T is NSOP$_1$ otherwise.

Definition 2.6. Suppose A is a set of parameters.
(1) We say that tuples a and b *have the same (Shelah) strong type over* A, written $a \equiv^S_A b$, if $E(a, b)$ holds (i.e., $E(a', b')$ holds for all corresponding finite subtuples a' and b' of a and b, respectively) for every A-definable equivalence relation $E(x, y)$ with finitely many classes.
(2) The group $\mathrm{Autf}(\mathbb{M}/A)$ of *Lascar strong automorphisms (of the monster) over* A is the subgroup of $\mathrm{Aut}(\mathbb{M}/A)$ generated by $\bigcup\{\mathrm{Aut}(\mathbb{M}/M) : A \subseteq M \prec \mathbb{M}\}$. We say a and b have *the same Lascar strong type over* A, written $a \equiv^L_A b$, if there is some $\sigma \in \mathrm{Autf}(\mathbb{M}/A)$ such that $\sigma(a) = b$. By a *Lascar strong type* over A, we mean an equivalence class of the relation $\equiv^L_A$.
(3) A type-definable equivalence relation E on α-tuples, for an ordinal α, is called *bounded* if it has small number of classes. We say a and b *have the same* KP-*strong type over* A, written $a \equiv^{\mathrm{KP}}_A b$, if $E(a, b)$ holds for all bounded type-definable equivalence relations over A.

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(4) We say that T is *G-compact over A* when $a \equiv^L_A b$ if and only if $a \equiv^{\mathrm{KP}}_A b$ for all (possibly infinite) tuples a, b. We say T is *G-compact* if it is G-compact over all finite sets A.

In [5], several basic facts about Kim-independence in NSOP_1 theories with existence were established. As we will make extensive use of them throughout the paper, we record them below.

FACT 2.7. *Assume T is $NSOP_1$ with existence and A is a set of parameters. Then the following properties hold.*

(1) *Extension: If $\pi(x)$ is a partial type over $B \supseteq A$ which does not Kim-divide over A, then there is a completion $p \in S(B)$ of π that does not Kim-divide over A. In particular, if $a \mathop{\smile\!\!\!\!\!\!|}^K_A b$ and c is arbitrary, there is some $a' \equiv_{Ab} a$ such that $a \mathop{\smile\!\!\!\!\!\!|}^K_A bc$* [5, *Proposition* 4.1].

(2) *Symmetry: $a \mathop{\smile\!\!\!\!\!\!|}^K_A b \iff b \mathop{\smile\!\!\!\!\!\!|}^K_A a$* [5, *Corollary* 4.9].

(3) *Kim's lemma for Morley sequences: The formula $\varphi(x; a)$ Kim-divides over A if and only if $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent for all Morley sequences $\langle a_i : i < \omega\rangle$ over A with $a_0 = a$* [5, *Theorem* 3.5].

(4) *Kim-forking = Kim-dividing: If a formula $\varphi(x; a)$ Kim-forks over A, then $\varphi(x; a)$ Kim-divides over A* [5, *Proposition* 4.1].

(5) *The chain condition: If $a \mathop{\smile\!\!\!\!\!\!|}^K_A b$ and $I = \langle b_i : i < \omega\rangle$ is a Morley sequence over A with $b_0 = b$, then there is $a' \equiv_{Ab} a$ such that I is Aa'-indiscernible and $a' \mathop{\smile\!\!\!\!\!\!|}^K_A I$ (this follows from* (3), *as in, e.g.,* [5, *Corollary* 5.15]).

(6) *The independence theorem for Lascar strong types: If $a_0 \equiv^L_A a_1$, $a_0 \mathop{\smile\!\!\!\!\!\!|}^K_A b$, $a_1 \mathop{\smile\!\!\!\!\!\!|}^K_A c$, and $b \mathop{\smile\!\!\!\!\!\!|}^K_A c$, then there is some a_* with $a_* \equiv^L_{Ab} a_0$, $a_* \equiv^L_{Ac} a_1$, and $a_* \mathop{\smile\!\!\!\!\!\!|}^K_A bc$* [5, *Theorem* 5.8].

(7) *T_A is G-compact for any small set A, where T_A is the theory of the monster model in the language with constants for the elements of A* [5, *Corollary* 5.9].

As these facts make up part of the standard tool box for reasoning about Kim-independence, we will often make implicit use of these properties. For example, Kim's lemma for Morley sequences, item (3) in the above list, is often used in this paper in the following way: if $I = \langle a_i : i < \omega\rangle$ is a Morley sequence over A with $a_0 = a$ which is Ab-indiscernible, then $a \mathop{\smile\!\!\!\!\!\!|}^K_A b$. To see this, by symmetry (Item (2)), it suffices to show that $b \mathop{\smile\!\!\!\!\!\!|}^K_A a$ which, by item (4), means that we need to show that there is no formula $\varphi(x; a) \in \mathrm{tp}(b/Aa)$ which Kim-divides over A. But if $\varphi(x; a)$ Kim-divides over A, then Kim's lemma implies that $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent. This set of formulas, however, is realized by b so there can be no such formula.

The following is local character of Kim-independence for NSOP_1 theories. The usual formulation of local character for non-forking independence in simple theories merely asserts that, for any type $p \in S(A)$, the set of B with $|B| \leq |T|$ such that p does not fork over B is *non-empty*, but it follows by base monotonicity, then, that p does not fork over C for any $B \subseteq C \subseteq A$. Because Kim-independence, in general, does not satisfy base monotonicity in NSOP_1 theories, the following is the appropriate analogue for this setting:

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FACT 2.8 [9, Theorem 3.9]. *Suppose T is NSOP$_1$ and $M \models T$ with $|M| \geq |T|$. Given any $p \in S(M)$ (in finitely many variables), the set X defined by*

$$X := \{N \prec M : |N| = |T| \text{ and } p \text{ does not Kim-divide over } N\}$$

satisfies the following:

(1) *X is closed: if $\langle N_i : i < |T|\rangle$ is a sequence of models in X with $N_i \subseteq N_j$ for all $i < j$, then $\bigcup_{i<|T|} N_i \in X$.*
(2) *X is unbounded: if $Y \subset M$ has cardinality $\leq |T|$, there is some $N \in X$ with $Y \subseteq N$.*

REMARK 2.9. It is an easy consequence of Fact 2.8 that if $M \models T$ is equal to the union of $\langle N_i : i < |T|^+\rangle$, an increasing and continuous (i.e., $N_\delta = \bigcup_{i<\delta} N_i$ for all limit δ) elementary chain of models of T of size $|T|$, then for any $p \in S(M)$, there is some $i < |T|^+$ such that p does not Kim-divide over N_i.

2.1. Trees. At several points in the paper, we will construct indiscernible sequences by an inductive construction of indiscernible trees. We recall the basic framework for these 'tree-inductions' from [7]. For an ordinal α, let the language $L_{s,\alpha}$ be $\langle \trianglelefteq, \wedge, <_{lex}, (P_\beta)_{\beta\leq\alpha}\rangle$. For us, a tree will mean a partial order $\trianglelefteq$ such that for all x, the elements $\{y : y \trianglelefteq x\}$ below x are linearly ordered (and not necessarily well-ordered) by $\trianglelefteq$ and such that for all x, y, x and y have an infimum, i.e., there is a $\trianglelefteq$-greatest element $z \trianglelefteq x, y$, which is called the meet of x and y. We may view a tree with α levels as an $L_{s,\alpha}$-structure by interpreting $\trianglelefteq$ as the tree partial order, $\wedge$ as the binary meet function, $<_{lex}$ as the lexicographic order, and P_β interpreted to define level β. The specific trees, and the interpretations of these symbols that turn them into $L_{s,\alpha}$-structures, that we will need in this paper are outlined precisely in Definition 2.12.

We now recall the modeling property. In what follows, we will write $\text{qftp}_{L'}(a)$ to denote the quantifier-free type of a in the language L' and write $\text{tp}_\Delta(b/A)$ to denote the Δ-type of b over A (i.e., the set of positive and negative instances of formulas in Δ with parameters from A satisfied by b). Although the subscript is used in two conflicting ways, it will be clear from context which is intended.

DEFINITION 2.10. Suppose I is an L'-structure, where L' is some language.

(1) We say $(a_i : i \in I)$ is a set of *I-indexed indiscernibles over A* if whenever $(s_0, \ldots, s_{n-1})$, $(t_0, \ldots, t_{n-1})$ are tuples from I with

$$\text{qftp}_{L'}(s_0, \ldots, s_{n-1}) = \text{qftp}_{L'}(t_0, \ldots, t_{n-1}),$$

then we have

$$\text{tp}(a_{s_0}, \ldots, a_{s_{n-1}}/A) = \text{tp}(a_{t_0}, \ldots, a_{t_{n-1}}/A).$$

(2) In the case that $L' = L_{s,\alpha}$ for some α, we say that an I-indexed indiscernible is *s-indiscernible*. As the only $L_{s,\alpha}$-structures we will consider will be trees, we will often refer to I-indexed indiscernibles in this case as *s-indiscernible trees*.
(3) We say that I-indexed indiscernibles have the *modeling property* if, given any $(a_i : i \in I)$ from $\mathbb{M}$ and any A, there is an I-indexed indiscernible $(b_i : i \in I)$ over A in $\mathbb{M}$ *locally based* on $(a_i : i \in I)$ over A. That is, given any finite set

of formulas Δ from $L(A)$ and a finite tuple $(t_0, \dots, t_{n-1})$ from I, there is a tuple $(s_0, \dots, s_{n-1})$ from I so that

$$\mathrm{qftp}_{L'}(t_0, \dots, t_{n-1}) = \mathrm{qftp}_{L'}(s_0, \dots, s_{n-1})$$

and also

$$\mathrm{tp}_\Delta(b_{t_0}, \dots, b_{t_{n-1}}) = \mathrm{tp}_\Delta(a_{s_0}, \dots, a_{s_{n-1}}).$$

FACT 2.11 [14, Theorem 4.3]. *Let I_s denote the $L_{s,\omega}$-structure*

$$I_s = (\omega^{<\omega}, \trianglelefteq, <_{lex}, \wedge, (P_\alpha)_{\alpha<\omega})$$

with all symbols being given their intended interpretations and each P_α naming the elements of the tree at level α. Then I_s-indexed indiscernibles have the modeling property.

Our trees will be understood to be an $L_{s,\alpha}$-structure for some appropriate α. As in [7], we introduce a distinguished class of trees $\mathcal{T}_\alpha$.

DEFINITION 2.12. Suppose α is an ordinal. We define $\mathcal{T}_\alpha$ to be the set of functions f such that:

- $\mathrm{dom}(f)$ is an end-segment of α of the form $[\beta, \alpha)$ for β equal to 0 or a successor ordinal. If α is a successor, we allow $\beta = \alpha$, i.e., $\mathrm{dom}(f) = \emptyset$.
- $\mathrm{ran}(f) \subseteq \omega$.
- Finite support: The set $\{\gamma \in \mathrm{dom}(f) : f(\gamma) \neq 0\}$ is finite.

We interpret $\mathcal{T}_\alpha$ as an $L_{s,\alpha}$-structure by defining:

- $f \trianglelefteq g$ if and only if $f \subseteq g$. Write $f \perp g$ if $\neg(f \trianglelefteq g)$ and $\neg(g \trianglelefteq f)$.
- $f \wedge g = f|_{[\beta,\alpha)} = g|_{[\beta,\alpha)}$ where $\beta = \min\{\gamma : f|_{[\gamma,\alpha)} = g|_{[\gamma,\alpha)}\}$, if non-empty (note that β will not be a limit, by finite support). Define $f \wedge g$ to be the empty function if this set is empty (note that this cannot occur if α is a limit).
- $f <_{lex} g$ if and only if $f \triangleleft g$ or, $f \perp g$ with $\mathrm{dom}(f \wedge g) = [\gamma + 1, \alpha)$ and $f(\gamma) < g(\gamma)$.
- For all $\beta \leq \alpha$, $P_\beta = \{f \in \mathcal{T}_\alpha : \mathrm{dom}(f) = [\beta, \alpha)\}$. Note that P_0 are the leaves of the tree (i.e., the *top* level) and P_α is empty for α limit.

Fact 2.11 and compactness can be used to show that $\mathcal{T}_\alpha$-indexed indiscernibles have the modeling property as well [7, Corollary 5.6].

DEFINITION 2.13. Suppose α is an ordinal.

(1) (Restriction) If $v \subseteq \alpha$, the *restriction of $\mathcal{T}_\alpha$ to the set of levels v* is the $L_{s,\alpha}$-substructure of $\mathcal{T}_\alpha$ with the following underlying set:

$$\mathcal{T}_\alpha \upharpoonright v = \{\eta \in \mathcal{T}_\alpha : \min(\mathrm{dom}(\eta)) \in v \text{ and } \beta \in \mathrm{dom}(\eta) \setminus v \implies \eta(\beta) = 0\}.$$

(2) (Concatenation) If $\eta \in \mathcal{T}_\alpha$, $\mathrm{dom}(\eta) = [\beta + 1, \alpha)$ for β non-limit, and $i < \omega$, let $\eta \frown \langle i \rangle$ denote the function $\eta \cup \{(\beta, i)\}$. We define $\langle i \rangle \frown \eta \in \mathcal{T}_{\alpha+1}$ to be $\eta \cup \{(\alpha, i)\}$. We write $\langle i \rangle$ for $\emptyset \frown \langle i \rangle$.

(3) (Canonical inclusions) If $\alpha < \beta$, we define the map $\iota_{\alpha\beta} : \mathcal{T}_\alpha \to \mathcal{T}_\beta$ by $\iota_{\alpha\beta}(f) := f \cup \{(\gamma, 0) : \gamma \in \beta \setminus \alpha\}$.

(4) (The all 0's path) If $\beta < \alpha$, then ζ_β denotes the function with $\mathrm{dom}(\zeta_\beta) = [\beta, \alpha)$ and $\zeta_\beta(\gamma) = 0$ for all $\gamma \in [\beta, \alpha)$. This defines an element of $\mathcal{T}_\alpha$ if and only if $\beta \in \{\gamma \in \alpha \mid \gamma \text{ is not limit}\} =: [\alpha]$.
(5) (Tuple notation) Given $\nu \in \mathcal{T}_\alpha$, we write $a_{\trianglerighteq \nu}$ for the tuple enumerating $\{a_\xi : \nu \trianglelefteq \xi \in \mathcal{T}_\alpha\}$.

In previous works on Kim-independence over arbitrary sets, there was a gap concerning the construction of Morley trees (and a parallel gap in the theory over models), first discovered by Jan Dobrowolski and Mark Kamsma. Namely, there is no reason *a priori* for an s-indiscernible tree locally based on a weakly spread out tree (see Definition 2.17) to be weakly spread out, which is needed to continue the induction. Over models this has a very easy fix: one can check that it is possible to choose a global M-invariant type $q \supseteq \mathrm{tp}((a_\eta)_{\eta \in \mathcal{T}_\alpha}/M)$ such that, if $(a'_\eta)_{\eta \in \mathcal{T}_\alpha} \models q$ then $(a'_\eta)_{\eta \in \mathcal{T}_\alpha}$ is $\mathbb{M}$-indiscernible. Morley sequences in such types allow the argument to work without change (the details for this case will appear elsewhere). But over sets, a lengthier argument is required to patch the proofs. The relevant notion for the modification is that of a *mutually s-indiscernible sequence*. We prove in Lemma 2.15 that, given an s-indiscernible tree, there is a Morley sequence starting with this tree which is mutually s-indiscernible, and then we show in Lemma 2.16 that this notion is preserved upon passage to an s-indiscernible tree.

Definition 2.14. We say a sequence $\langle (a_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \kappa \rangle$ is *mutually s-indiscernible over A* if, for all $i < \kappa$, $(a_{\eta,i})_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over $A\{a_{\eta,j} : \eta \in \mathcal{T}_\alpha, j \neq i\}$.

Lemma 2.15. *Assume A is an extension base. Given a tree $(a_\eta)_{\eta \in \mathcal{T}_\alpha}$ that is s-indiscernible over A, there is a sequence $I = \langle (a_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \omega \rangle$ such that $(a_{\eta,0})_{\eta \in \mathcal{T}_\alpha} = (a_\eta)_{\eta \in \mathcal{T}_\alpha}$, I is a Morley sequence over A, and I is mutually s-indiscernible over A.*

Proof. Let κ be sufficiently large with respect to $|A|$. By induction on $\gamma < \kappa$, we will choose $(a_{\eta,\gamma})_{\eta \in \mathcal{T}_\alpha}$ such that, taking I_γ to be the sequence $\langle (a_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \gamma \rangle$, we have that I_γ starts with $(a_\eta)_{\eta \in \mathcal{T}_\alpha}$, is mutually s-indiscernible over A, and satisfies

$$(a_{\eta,i})_{\eta \in \mathcal{T}_\alpha} \mathop{\smile\!\!\!\!\!\!|}_A (a_{\eta,j})_{\eta \in \mathcal{T}_\alpha, j<i}$$

for all $i < \gamma$. The sequence I_1 is already specified and trivially satisfies the requirements.

Assume we are given $(a_{\eta,i})_{\eta \in \mathcal{T}_\alpha}$ for all $i < \gamma$ and set $I_\gamma = \langle (a_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \gamma \rangle$. Apply extension to get some $(b_\eta)_{\eta \in \mathcal{T}_\alpha} \equiv_A (a_\eta)_{\eta \in \mathcal{T}_\alpha}$ such that

$$(b_\eta)_{\eta \in \mathcal{T}_\alpha} \mathop{\smile\!\!\!\!\!\!|}_A I_\gamma.$$

By the modeling property, we can take $(b'_\eta)_{\eta \in \mathcal{T}_\alpha}$ to be locally based on $(b_\eta)_{\eta \in \mathcal{T}_\alpha}$ and s-indiscernible over AI_γ, then we we still have $(b'_\eta)_{\eta \in \mathcal{T}_\alpha} \equiv_A (a_\eta)_{\eta \in \mathcal{T}_\alpha}$, as $(a_\eta)_{\eta \in \mathcal{T}_\alpha}$ was assumed to be s-indiscernible over A, and local basedness and strong finite character of non-forking imply

$$(b'_\eta)_{\eta \in \mathcal{T}_\alpha} \mathop{\smile\!\!\!\!\!\!|}_A I_\gamma.$$

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Now by induction on $i < \gamma$, we will choose $(a'_{\eta,i})_{\eta \in \mathcal{T}_\alpha}$ satisfying the following conditions:

(1) $(a'_{\eta,i})_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over

$$A \cup \{a'_{\eta,j} : \eta \in \mathcal{T}_\alpha, j < i\} \cup \{a_{\eta,k} : \eta \in \mathcal{T}_\alpha, k > i\} \cup \{b'_\eta : \eta \in \mathcal{T}_\alpha\}.$$

(2) $(b'_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over

$$A \cup \{a'_{\eta,j} : \eta \in \mathcal{T}_\alpha, j \leq i\} \cup \{a_{\eta,k} : \eta \in \mathcal{T}_\alpha, k > i\}.$$

(3) $(a'_{\eta,j})_{\eta \in \mathcal{T}_\alpha, j \leq i}(a_{\eta,k})_{\eta \in \mathcal{T}_\alpha, k > i} \equiv_A (a_{\eta,j})_{\eta \in \mathcal{T}_\alpha, j \leq i}(a_{\eta,k})_{\eta \in \mathcal{T}_\alpha, k > i}$.

(4) The following independence holds:

$$(b'_\eta)_{\eta \in \mathcal{T}_\alpha} \mathop{\smile\!\!\!\!\!\!|}_A (a'_{\eta,j})_{\eta \in \mathcal{T}_\alpha, j \leq i}(a_{\eta,k})_{\eta \in \mathcal{T}_\alpha, k > i.}$$

Fix $i < \gamma$ and suppose we have chosen $(a'_{\eta,j})_{\eta \in \mathcal{T}_\alpha}$ for all $j < i$. Pick $(a'_{\eta,i})_{\eta \in \mathcal{T}_\alpha}$ s-indiscernible over $A \cup \{a'_{\eta,j} : \eta \in \mathcal{T}_\alpha, j < i\} \cup \{a_{\eta,k} : \eta \in \mathcal{T}_\alpha, k > i\} \cup \{b'_\eta : \eta \in \mathcal{T}_\alpha\}$ and locally based on $(a_{\eta,i})_{\eta \in \mathcal{T}_\alpha}$. Then (1) is satisfied and (2) is easy to check using local basedness and the inductive assumption. We assumed I_γ was mutually s-indiscernible over A and hence by (3) of the inductive hypothesis, we know that $(a_{\eta,i})_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over

$$A \cup \{a'_{\eta,j} : \eta \in \mathcal{T}_\alpha, j < i\} \cup \{a_{\eta,k} : \eta \in \mathcal{T}_\alpha, k > i\},$$

and therefore $(a'_{\eta,i})_{\eta \in \mathcal{T}_\alpha}$ has the same type over this set, which establishes (3). Finally, (4) follows by local basedness, (3), and the invariance of non-forking independence. More explicitly, suppose there is a finite tuple b from $(b'_\eta)_{\eta \in \mathcal{T}_\alpha}$, a finite tuple a from $\{a'_{\eta,j} : j < i\} \cup \{a_{\eta,k} : k > i\}$, and a finite tuple $\overline{\eta}$ from $\mathcal{T}_\alpha$ such that

$$\models \varphi(b; a'_{\overline{\eta},i}, a),$$

where $\varphi(x; y, z) \in L(A)$ is a formula such that $\varphi(x; a'_{\overline{\eta},i}, a)$ forks over A. Local basedness entails that there is $\overline{\nu}$ with $\mathrm{qftp}_{L_{s,\alpha}}(\overline{\nu}) = \mathrm{qftp}_{L_{s,\alpha}}(\overline{\eta})$ such that

$$\models \varphi(b; a_{\overline{\nu},i}, a).$$

But by mutual s-indiscernibility, $a_{\overline{\nu},i} \equiv_{Aa} a_{\overline{\eta},i}$ and, by (3), $a_{\overline{\eta},i} \equiv_{Aa} a'_{\overline{\eta},i}$ and hence $\varphi(x; a_{\overline{\nu},i}, a)$ forks over A as well. This contradicts the inductive hypothesis that

$$(b'_\eta)_{\eta \in \mathcal{T}_\alpha} \mathop{\smile\!\!\!\!\!\!|}_A (a'_{\eta,j})_{\eta \in \mathcal{T}_\alpha, j < i}(a_{\eta,k})_{\eta \in \mathcal{T}_\alpha, k \geq i}.$$

This shows that our choice of $(a'_{\eta,i})_{\eta \in \mathcal{T}_\alpha}$ satisfies the requirements.

Having constructed our sequence $I'_\gamma = \langle (a'_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \gamma \rangle$, we have $I'_\gamma \equiv_A I_\gamma$ by (3) and $(b'_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over I'_γ by (2). Moreover, each $(a_{\eta,i})_{\eta \in \mathcal{T}_\alpha}$ is indiscernible over $A(b'_\eta)_{\eta \in \mathcal{T}_\alpha}(a_{\eta,j})_{\eta \in \mathcal{T}_\alpha, j \neq i}$ by (1). Finally, by (4), we have $(b'_\eta)_{\eta \in \mathcal{T}_\alpha} \mathop{\smile\!\!\!\!\!\!|}_A I'_\gamma$. Choosing $(a_{\eta,\gamma})$ such that

$$I_\gamma (a_{\eta,\gamma})_{\eta \in \mathcal{T}_\alpha} \equiv_A I'_\gamma (b'_\eta)_{\eta \in \mathcal{T}_\alpha}$$

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we arrive at $I_{\gamma+1}$. There is nothing to do at limits, so we have succeeded in constructing our sequence I_κ. Applying Erdős–Rado to I_κ, then, we obtain the desired sequence I. ⊣

LEMMA 2.16. *Suppose $(a_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$ is a tree of tuples such that $I = \langle a_{\trianglerighteq\langle i\rangle} : i<\omega\rangle$ is mutually s-indiscernible and Morley over A. Then if $(a'_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$ is s-indiscernible and locally based on $(a_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$ over A and $I' = \langle a'_{\trianglerighteq\langle i\rangle} : i<\omega\rangle$, then $I' \equiv_A I$ and thus I' is also mutually s-indiscernible and Morley over A.*

PROOF. Suppose $\overline{\eta}$ and $\overline{\nu}$ are tuples from $\mathcal{T}_{\alpha+1}\setminus\{\emptyset\}$ with $\text{qftp}_{L_{s,\alpha+1}}(\overline{\eta}) = \text{qftp}_{L_{s,\alpha+1}}(\overline{\nu})$. After possibly reordering the tuples, there are $i_0<\cdots<i_{k-1}$ and $j_0<\cdots<j_{k-1}$ such that $\overline{\eta}=(\overline{\eta}_0,\dots,\overline{\eta}_{k-1})$ and $\overline{\nu}=(\overline{\nu}_0,\dots,\overline{\nu}_{k-1})$ where each $\overline{\eta}_l$ comes from the tree $\trianglerighteq\langle i_l\rangle$ and $\overline{\nu}_l$ comes from the tree $\trianglerighteq\langle j_l\rangle$ for $l<k$. Then, in particular, $\text{qftp}_{L_{s,\alpha+1}}(\overline{\eta}_l) = \text{qftp}_{L_{s,\alpha+1}}(\overline{\nu}_l)$ for all $l<k$. Additionally, for all $l<k$, let $\overline{\eta}'_l$ be the element of the tree $\trianglerighteq\langle j_l\rangle$ corresponding to $\overline{\eta}_l$ (i.e., replace each node $\langle i_l\rangle^\frown\xi$ enumerated in $\overline{\eta}_l$ with $\langle j_l\rangle^\frown\xi$). Because I is an A-indiscernible sequence, we have

$$(a_{\overline{\eta}_0},\dots,a_{\overline{\eta}_{k-1}}) \equiv_A (a_{\overline{\eta}'_0},\dots,a_{\overline{\eta}'_{k-1}}).$$

Additionally, in the tree $\trianglerighteq\langle j_l\rangle$ (naturally viewed as an $L_{s,\alpha}$-structure), we have $\text{qftp}_{L_{s,\alpha}}(\overline{\eta}'_l) = \text{qftp}_{L_{s,\alpha}}(\overline{\eta}_l)$ for all $l<k$. Thus, mutual s-indiscernibility entails

$$(a_{\overline{\eta}'_0},\dots,a_{\overline{\eta}'_{k-1}}) \equiv_A (a_{\overline{\nu}_0},\dots,a_{\overline{\nu}_{k-1}}).$$

Thus we have shown that $a_{\overline{\eta}} \equiv_A a_{\overline{\nu}}$. Therefore, it follows, by local basedness, that $I\equiv_A I'$ and the result follows. ⊣

DEFINITION 2.17. Suppose $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ is a tree of tuples in $\mathbb{M}$, and A is a set of parameters.

(1) We say $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ is *weakly spread out over A* if for all $\eta\in\mathcal{T}_\alpha$ with $\text{dom}(\eta) = [\beta+1,\alpha)$ for some $\beta\in[\alpha]$, the sequence of cones $(a_{\trianglerighteq\eta^\frown\langle i\rangle})_{i<\omega}$ is a Morley sequence in $\text{tp}(a_{\trianglerighteq\eta^\frown\langle 0\rangle}/A)$.
(2) Suppose $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ is a tree which is weakly spread out and s-indiscernible over A and for all pairs of finite subsets w,v of α with $|w|=|v|$,

$$(a_\eta)_{\eta\in\mathcal{T}_\alpha\restriction w} \equiv_A (a_\eta)_{\eta\in\mathcal{T}_\alpha\restriction v}$$

then we say $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ is a *weakly Morley tree* over A.
(3) A *weak tree Morley sequence* over A is a A-indiscernible sequence of the form $(a_{\zeta_\beta})_{\beta\in[\alpha]}$ for some weakly Morley tree $(a_\eta)_{\eta\in\mathcal{T}_\alpha}$ over A. More generally, we will say an A-indiscernible sequence I is a weak tree Morley sequence over A if it is EM-equivalent to a sequence of this form.

REMARK 2.18. If $I=\langle b_i : i<\omega\rangle$ is an A-indiscernible sequence and $I\equiv_A J$ for some weak tree Morley sequence J over A, then I is a weak tree Morley sequence over A. In particular, if I is a subsequence of J, by the A-indiscernibility of J, the sequence I is also weak tree Morley over A.

FACT 2.19. *Suppose T is NSOP$_1$ with existence and A is a set of parameters.*

(1) *If $a \mathop{\smile\!\!\!\!\!\!\;|}^K_A b$, there is an Ab-indiscernible sequence $I=\langle a_i : i<\omega\rangle$ over A with $a_0=a$ such that I is weak tree Morley over A* [5, *Lemma* 4.7].

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(2) *Kim's lemma for weak tree Morley sequences*: *the formula $\varphi(x; a_0)$ Kim-divides over A if and only if $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent for some weak tree Morley sequence $\langle a_i : i < \omega\rangle$ over A if and only if $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent for all weak tree Morley sequences $\langle a_i : i < \omega\rangle$ over A* [5, *Corollary* 4.8].

(3) *If $a \equiv^L_A b$ and $a \mathop{\smile\!\!\!\!\!\!|}^K_A b$, there is an $\mathop{\smile\!\!\!\!\!\!|}^K$-Morley sequence over A starting with (a, b) as its first two elements* (*follows from Fact* 2.7 *as in* [7, *Corollary* 6.6]).

2.2. Further properties of Kim-independence.

Fact 2.20 [5, Lemma 5.7]. *Suppose T is NSOP$_1$ with existence. If A is a set of parameters, c is an arbitrary tuple, and $a \mathop{\smile\!\!\!\!\!\!|}^K_A b$, then there is $a' \equiv^L_{Ab} a$ such that $a' \mathop{\smile\!\!\!\!\!\!|}^K_A bc$.*

The following lemma is easy and well-known, but, in the absence of a clear reference, we provide a proof:

Lemma 2.21. *Suppose T is NSOP$_1$ with existence, A is a set of parameters, and $p(x) \in S(A)$.*

(1) *Given any tuple of variables y, there is a partial type $\Gamma(x, y)$ over A such that $(a, b) \models \Gamma(x, y)$ if and only if $a \models p$ and $a \mathop{\smile\!\!\!\!\!\!|}^K_A b$.*

(2) *There is a partial type $\Delta(x_i : i < \omega)$ over A such that $I = \langle a_i : i < \omega\rangle \models \Delta$ if and only if I is an $\mathop{\smile\!\!\!\!\!\!|}^K$-Morley sequence over A in p.*

Proof. (1) By compactness, we may assume y is finite. Fix $c \models p$ and define $\Gamma(x, y)$ by

$$\Gamma(x, y) = p(x) \cup \{\neg\varphi(y; x) : \varphi(y; c) \text{ Kim-divides over } A\}.$$

By symmetry, invariance, and Kim-forking = Kim-dividing, this partial type is as desired.

(2) One can take Δ to be the partial type that asserts $\langle x_i : i < \omega\rangle$ is A-indiscernible, every $x_i \models p$, and $x_i \mathop{\smile\!\!\!\!\!\!|}^K_A x_{<i}$ (which is type-definable over A by (1)). ⊣

The following lemma is the analogue of the 'strong independence theorem' of [16, Theorem 2.3] for Lascar strong types.

Lemma 2.22. *Suppose T is NSOP$_1$ with existence. If A is a set of parameters, $a_0 \mathop{\smile\!\!\!\!\!\!|}^K_A b$, $a_1 \mathop{\smile\!\!\!\!\!\!|}^K_A c$, $b \mathop{\smile\!\!\!\!\!\!|}^K_A c$, and $a_0 \equiv^L_A a_1$, then there is a such that $a \equiv^L_{Ab} a_0$, $a \equiv^L_{Ac} a_1$ and, additionally, we have $a \mathop{\smile\!\!\!\!\!\!|}^K_A bc$, $b \mathop{\smile\!\!\!\!\!\!|}^K_A ac$, and $c \mathop{\smile\!\!\!\!\!\!|}^K_A ab$.*

Proof. By Fact 2.20, there is $c' \equiv^L_{Ab} c$ such that $c' \mathop{\smile\!\!\!\!\!\!|}^K_A bc$. Let $\sigma \in \text{Autf}(\mathbb{M}/Ab)$ be an automorphism such that $\sigma(c') = c$ and let $c_0 = \sigma(c)$. Then we have $c \mathop{\smile\!\!\!\!\!\!|}^K_A bc_0$ and $c_0 \equiv^L_{Ab} c$ and hence, in particular, $c_0 b \equiv^L_A cb$. By symmetry and a second application of Fact 2.20 once again, we find $b''c'' \equiv^L_{Ac} bc_0$ with $b''c'' \mathop{\smile\!\!\!\!\!\!|}^K_A bc$. Let $\tau \in \text{Autf}(\mathbb{M}/Ac)$ be a strong automorphism with $\tau(b''c'') = bc_0$ and define $b_1 = \tau(b)$. Then by construction, we have $b''c'' \equiv_A bc_0$ and $bc_0 \equiv^L_A bc$, it follows that $b''c'' \equiv^L_A bc$, and hence $\tau(b''c'') \equiv^L_A \tau(bc)$, which, after unraveling definitions, gives $bc_0 \equiv^L_A b_1 c$. Moreover, since $b''c'' \mathop{\smile\!\!\!\!\!\!|}^K_A bc$, we obtain $bc_0 \mathop{\smile\!\!\!\!\!\!|}^K_A b_1 c$ by invariance.

Let $b_0 = b$ and $c_1 = c$. By Fact 2.19(3), we can extend the sequence $\langle (b_i, c_i) : i < 2\rangle$ to a weak tree Morley sequence $I = \langle (b_i, c_i) : i \in \mathbb{Z}\rangle$ over A.

Choose a' such that $a_1c_1 \equiv^L_A a'c_0$. Then we have $a_0 \equiv^L_A a'$, as well as $a_0 \mathop{\smile\!\!\!\!\!\!\mid}^K_A b_0$, $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_A c_0$ by our assumptions. Additionally, since $b \mathop{\smile\!\!\!\!\!\!\mid}^K_A c$, $b_0 = b$ and $c \equiv_{Ab} c_0$, we have $b_0 \mathop{\smile\!\!\!\!\!\!\mid}^K_A c_0$. Therefore, by Fact 2.7(6), there is a_* with $a_* \equiv^L_{Ab_0} a_0$, $a_* \equiv^L_{Ac_0} a'$, with $a_* \mathop{\smile\!\!\!\!\!\!\mid}^K_A b_0c_0$.

Because I is a weak tree Morley sequence and $a_* \mathop{\smile\!\!\!\!\!\!\mid}^K_A b_0c_0$, by Kim's lemma, compactness, and an automorphism, there is $a_{**} \mathop{\smile\!\!\!\!\!\!\mid}^K_A I$ such that $a_{**}b_0c_0 \equiv^L_A a_*b_0c_0$ and such that I is Aa_{**}-indiscernible. Note that, by construction, $a_{**} \equiv^L_{Ab} a_0$, $a_{**} \equiv^L_{Ac} a_1$, and $a_{**} \mathop{\smile\!\!\!\!\!\!\mid}^K_A bc$.

Additionally, the sequence $\langle b_i : i \in \mathbb{Z}^{\leq 0}\rangle$ is a weak tree Morley sequence over A which is $Aa_{**}c$-indiscernible and containing $b_0 = b$, hence $a_{**}c \mathop{\smile\!\!\!\!\!\!\mid}^K b$, by Kim's lemma for weak tree Morley sequences. Similarly, the sequence $\langle c_i : i \in \mathbb{Z}^{\geq 1}\rangle$ is a weak tree Morley sequence over A containing $c = c_1$ which is $Aa_{**}b$-indiscernible, yielding $a_{**}b \mathop{\smile\!\!\!\!\!\!\mid}^K_A c$. By symmetry, we conclude. ⊣

§3. Transitivity and witnessing.

3.1. Preliminary lemmas. We begin by establishing some lemmas, allowing us to construct sequences that are $\mathop{\smile\!\!\!\!\!\!\mid}^K$-Morley over more than one base simultaneously. The broad structure of the argument will follow that of [8], which established transitivity over models for Kim-independence in NSOP$_1$ theories, however, all uses of coheirs and heirs will need to be replaced.

In particular, the following lemma does not follow the corresponding [8, Lemma 3.1], instead producing the desired sequence by a tree-induction.

Lemma 3.1. *Suppose T is NSOP$_1$ and satisfies the existence axiom. If $A \subseteq B$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^K_A B$, then there is a weak tree Morley sequence $\langle a_i : i < \omega\rangle$ over B with $a_0 = a$ such that $a_i \mathop{\smile\!\!\!\!\!\!\mid}^K_A Ba_{<i}$ for all $i < \omega$.*

Proof. By induction on α, we will construct trees $(a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ so that:

(1) $(a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible and weakly spread out over B.
(2) $a^\alpha_\eta \models \mathrm{tp}(a/B)$ for all $\eta \in \mathcal{T}_\alpha$.
(3) If α is a successor, $a^\alpha_\emptyset \mathop{\smile\!\!\!\!\!\!\mid}^K_A Ba^\alpha_{\rhd\emptyset}$.
(4) If $\alpha < \beta$, then $a^\beta_{\iota_{\alpha\beta}(\eta)} = a^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$.

For $\alpha = 0$, we put $a^0_\emptyset = a$, and for δ limit, we will define $(a^\delta_\eta)_{\eta \in \mathcal{T}_\delta}$ by setting $a^\delta_{\iota_{\alpha\delta}(\eta)} = a^\alpha_\eta$ for all $\alpha < \delta$ and $\eta \in \mathcal{T}_\alpha$ which, by (4) and induction, is well-defined and satisfies the requirements.

Now suppose we are given $(a^\beta_\eta)_{\eta \in \mathcal{T}_\beta}$ satisfying the requirements for all $\beta \leq \alpha$. Let $\langle (a^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \omega\rangle$ be a mutually s-indiscernible Morley sequence over B with $a^\alpha_{\eta,0} = a^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$, which exists by Lemma 2.15. Apply extension to find $a_* \equiv_B a$ so that $a_* \mathop{\smile\!\!\!\!\!\!\mid}^K_A B(a^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha, i<\omega}$. Define a tree $(b_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ by setting $b_\emptyset = a_*$ and $b_{\langle i\rangle \frown \eta} = a^\alpha_{\eta,i}$ for all $i < \omega$ and $\eta \in \mathcal{T}_\alpha$. We may define $(a^{\alpha+1}_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ to be a

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tree which is s-indiscernible over B and locally based on $(b_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ over B. By an automorphism, we may assume $a^{\alpha+1}_{\iota_{\alpha\alpha+1}(\eta)} = a^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$, hence conditions (2), and (4) are clearly satisfied. Moreover, by Lemma 2.16, we have $\langle (a_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \omega \rangle \equiv_B \langle a^{\alpha+1}_{\trianglerighteq \langle i \rangle} : i < \omega \rangle$ so $\langle a^{\alpha+1}_{\trianglerighteq \langle i \rangle} : i < \omega \rangle$ is a Morley sequence over B. Then by (4) and induction, it follows that $(a^{\alpha+1}_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ is s-indiscernible and spread out over B, which shows (1).

For (3), we just note that, by symmetry, if $a^{\alpha+1}_\emptyset \not\mathop{\smile}\limits^K_A Ba^{\alpha+1}_{\triangleright \emptyset}$, there is some formula $\varphi(x; a^{\alpha+1}_\emptyset) \in \text{tp}(Ba^{\alpha+1}_{\triangleright\emptyset}/Aa^{\alpha+1}_\emptyset)$ that Kim-divides over A. As the tree $(a^{\alpha+1}_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ is locally based on $(b_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$, it follows that some tuple from $Bb_{\triangleright\emptyset}$ also realizes $\varphi(x; a_*)$ and $a_* \equiv_B a^{\alpha+1}_\emptyset$, so $\varphi(x; a_*)$ Kim-divides over A as well, contradicting the choice of a_*. This contradiction establishes (3), completing the induction.

By considering $(a^\kappa_\eta)_{\eta \in \mathcal{T}_\kappa}$ for κ sufficiently large, we may apply Erdős–Rado, as in [7, Lemma 5.10], to find the desired sequence. ⊣

The proof of the next lemma follows [8, Lemma 3.2].

Lemma 3.2. *Suppose T is an NSOP$_1$ theory satisfying the existence axiom. If $a \mathop{\smile}\limits^K_A b$ and $c \mathop{\smile}\limits^K_A b$, then there is c' so that $c' \equiv_{Ab} c$, $ac' \mathop{\smile}\limits^K_A b$, and $a \mathop{\smile}\limits^K_{Ab} c'$.*

Proof. Define a partial type $\Gamma(x; b, a)$ over Aab as follows:

$$\Gamma(x; b, a) = \text{tp}(c/Ab) \cup \{\neg\varphi(x, a; b) : \varphi(x, y; b) \in L(Ab) \text{ Kim-divides over } A\}.$$

Claim 1. *If $\langle a_i : i < \omega \rangle$ is an Ab-indiscernible sequence satisfying $a_0 = a$ and $a_i \mathop{\smile}\limits^K_A ba_{<i}$ for all $i < \omega$, then $\bigcup_{i<\omega} \Gamma(x; b, a_i)$ is consistent.*

Proof of claim. By induction on $n < \omega$, we will find $c_n \equiv^L_A c$ such that $c_n \mathop{\smile}\limits^K_A ba_{<n}$ and $c_n \models \bigcup_{i<n} \Gamma(x; b, a_i)$. For $n = 0$, we can put $c_0 = c$, since $c \mathop{\smile}\limits^K_A b$ by assumption. Assume we have found c_n, and, by Fact 2.20, choose c' such that $c' \equiv^L_A c$ and $c' \mathop{\smile}\limits^K_A a_n$. Then $c' \equiv^L_A c \equiv^L_A c_n$ and, since $a_n \mathop{\smile}\limits^K_A ba_{<n}$, we may apply Lemma 2.22 to find $c_{n+1} \equiv^L_A c$ such that $c_{n+1} \models \text{tp}(c_n/Aba_{<n}) \cup \text{tp}(c'/Aa_n)$ and such that $c_{n+1} \mathop{\smile}\limits^K_A ba_{<n+1}$ and $a_n c_{n+1} \mathop{\smile}\limits^K_A ba_{<n}$, hence, in particular, $c_{n+1} \models \bigcup_{i<n+1} \Gamma(x; b, a_i)$. The claim follows by compactness. ⊣

Next we define a partial type $\Delta(x; b, a)$ as follows:

$$\Delta(x; b, a) = \Gamma(x; b, a) \cup \{\neg\psi(x; b, a) : \psi(x; b, a) \in L(Aab) \text{ Kim-divides over } Ab\}.$$

Claim 2. *The set of formulas $\Delta(x; b, a)$ is consistent.*

Proof of claim. Suppose not. Then because Kim-forking and Kim-dividing are the same in NSOP$_1$ with existence, there is some formula $\psi(x; b, a) \in L(Aab)$ such that

$$\Gamma(x; b, a) \vdash \psi(x; b, a)$$

and $\psi(x; b, a)$ Kim-divides over Ab. As $a \mathop{\smile}\limits^K_A b$, we know by Lemma 3.1 that there is a sequence $\langle a_i : i < \omega \rangle$ with $a_0 = a$ which is a weak tree Morley sequence over Ab and satisfies $a_i \mathop{\smile}\limits^K_A ba_{<i}$ for all $i < \omega$. Then by Claim 1, $\bigcup_{i<\omega} \Gamma(x; b, a_i)$ is

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consistent. However, we have

$$\bigcup_{i<\omega} \Gamma(x; b, a_i) \vdash \{\psi(x; b, a_i) : i < \omega\}$$

and $\{\psi(x; b, a_i) : i < \omega\}$ is inconsistent, because weak tree Morley sequences witness Kim-dividing. This contradiction proves the claim. $\dashv$

To conclude, we may take c' to be any realization of $\Delta(x; b, a)$. $\dashv$

The next proposition is a strengthening of Fact 2.19(1).

PROPOSITION 3.3. *Suppose T is an NSOP$_1$ theory satisfying the existence axiom. If $a \mathop{\smile\!\!\!\!\!\!\;|}^K_A b$, then there is a sequence $I = \langle a_i : i < \omega \rangle$ with $a_0 = a$ such that I is a weak tree Morley sequence over A and an $\mathop{\smile\!\!\!\!\!\!\;|}^K$-Morley sequence over Ab.*

PROOF. By induction on α, we will construct trees $(a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ satisfying the following:

1. For all $\eta \in \mathcal{T}_\alpha$, $a^\alpha_\eta \models \text{tp}(a/Ab)$.
2. The tree $(a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible over Ab and weakly spread out over A.
3. If α is a successor, then $a^\alpha_\emptyset \mathop{\smile\!\!\!\!\!\!\;|}^K_{Ab} a^\alpha_{\rhd \emptyset}$.
4. $(a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha} \mathop{\smile\!\!\!\!\!\!\;|}^K_A b$.
5. If $\alpha < \beta$, then $a^\beta_{\iota_{\alpha\beta}(\eta)} = a^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$.

Put $a^0_\emptyset = a$ and for δ limit, if we are given $(a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ for every $\alpha < \delta$, we can define $(a^\delta_\eta)_{\eta \in \mathcal{T}_\delta}$ by setting $a^\delta_{\iota_{\alpha\delta}(\eta)} = a^\alpha_\eta$ for all $\alpha < \delta$ and $\eta \in \mathcal{T}_\alpha$, which is well-defined by (5) and is easily seen to satisfy the requirements.

Suppose now we are given $(a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$. Let $J = \langle (a^\alpha_{\eta,i})_{\eta \in \mathcal{T}_\alpha} : i < \omega \rangle$ be a mutually s-indiscernible Morley sequence over A with $(a^\alpha_{\eta,0})_{\eta \in \mathcal{T}_\alpha} = (a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$, which exists by Lemma 2.15. By (4), symmetry, and the chain condition, Fact 2.7(5) we may assume J is Ab-indiscernible and $J \mathop{\smile\!\!\!\!\!\!\;|}^K_A b$. By Lemma 3.2, there is $a_* \equiv_{Ab} a$ such that $a_* \mathop{\smile\!\!\!\!\!\!\;|}^K_{Ab} J$ and $a_* J \mathop{\smile\!\!\!\!\!\!\;|}^K_A b$. After defining a tree $(c_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ by $c_\emptyset = a_*$ and $c_{\langle i \rangle \frown \eta} = a^\alpha_{\eta,i}$ for all $\eta \in \mathcal{T}_\alpha$, these conditions on a_* imply that $(c_\eta)_{\eta \in \mathcal{T}_\alpha}$ satisfy (3) and (4), respectively. Let $(a^{\alpha+1}_\eta)_{\eta \in \mathcal{T}_{\alpha+1}}$ be any tree s-indiscernible over Ab locally based on $(c_\eta)_{\eta \in \mathcal{T}_\alpha}$ over Ab. This still satisfies (2) by Lemma 2.16. Moreover, as $c_{\unrhd \langle i \rangle} \models \text{tp}((a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}/Ab)$ for all $i < \omega$, it follows from local basedness that $a^{\alpha+1}_{\unrhd \langle i \rangle} \models \text{tp}((a^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}/Ab)$ for all $i < \omega$ as well. Hence, by an automorphism over Ab, we may assume $a^{\alpha+1}_{0 \frown \eta} = a^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$, which ensures the constructed tree satisfies (5), and (1)–(4) are easy to verify.

Given $(a^\kappa_\eta)_{\eta \in \mathcal{T}_\kappa}$ for κ sufficiently large, we may, by Erdős–Rado (see, e.g., [7, Lemma 5.10]), obtain a weak Morley tree $(b_\eta)_{\eta \in \mathcal{T}_\omega}$ over A satisfying (1)–(4). Then the sequence $I = \langle a_i : i < \omega \rangle$ defined by $a_i = b_{\zeta_i}$ for all $i < \omega$ is a weak tree Morley sequence over A, as it is a path in a weak Morley tree over A, but by (3), we have $a_i \mathop{\smile\!\!\!\!\!\!\;|}^K_{Ab} a_{<i}$ for all i, so I is $\mathop{\smile\!\!\!\!\!\!\;|}^K$-Morley over Ab as well. $\dashv$

3.2. Transitivity and witnessing. The following theorem establishes the transitivity of Kim-independence in NSOP$_1$ theories with existence.

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THEOREM 3.4. *Suppose T is NSOP$_1$ with existence. Then if $A \subseteq B$, $a \mathop{\smile\!\!\!\!\!\!\mid}^K_A B$ and $a \mathop{\smile\!\!\!\!\!\!\mid}^K_B c$, then $a \mathop{\smile\!\!\!\!\!\!\mid}^K_A Bc$.*

PROOF. By Proposition 3.3 and the assumption that $a \mathop{\smile\!\!\!\!\!\!\mid}^K_A B$, there is a sequence $I = \langle a_i : i < \omega \rangle$ with $a_0 = a$ such that I is an $\mathop{\smile\!\!\!\!\!\!\mid}^K$-Morley sequence over B and a weak tree Morley sequence over A. As $c \mathop{\smile\!\!\!\!\!\!\mid}^K_B a$, by symmetry, and I is $\mathop{\smile\!\!\!\!\!\!\mid}^K$-Morley over B, there is $I' \equiv_{Ba} I$ such that I' is Bc-indiscernible. Because I' is also a weak tree Morley sequence over A, it follows by Kim's lemma that $Bc \mathop{\smile\!\!\!\!\!\!\mid}^K_A a$. By symmetry, we conclude. ⊣

PROPOSITION 3.5. *Assume T is NSOP$_1$ with existence. The following are equivalent.*

(1) *$a \mathop{\smile\!\!\!\!\!\!\mid}^K_A b$.*
(2) *There is a model $M \supseteq A$ such that $M \mathop{\smile\!\!\!\!\!\!\mid}^K_A ab$ (or $M \mathop{\smile\!\!\!\!\!\!\mid}_A ab$) and $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$.*
(3) *There is a model $M \supseteq A$ such that $M \mathop{\smile\!\!\!\!\!\!\mid}^K_A a$ (or $M \mathop{\smile\!\!\!\!\!\!\mid}_A a$) and $a \mathop{\smile\!\!\!\!\!\!\mid}^K_M b$.*

PROOF. (1)⇒(2) Since $a \mathop{\smile\!\!\!\!\!\!\mid}^K_A b$, there is a Morley sequence $I = \langle a_i : i < \omega \rangle$ over A with $a_0 = a$ such that I is Ab-indiscernible. By [5, Lemma 2.17], there is a model N containing A such that $N \mathop{\smile\!\!\!\!\!\!\mid}_A I$ and I is a coheir sequence over N. By compactness and extension we can clearly assume the length of I is arbitrarily large, and $N \mathop{\smile\!\!\!\!\!\!\mid}_A Ib$. Hence by the pigeonhole principle, there is an infinite subsequence J of I such that all the tuples in J have the same type over Nb. Thus, for $a' \in J$, we have $a' \mathop{\smile\!\!\!\!\!\!\mid}^K_N b$ and $N \mathop{\smile\!\!\!\!\!\!\mid}_A a'b$. Hence $M = f(N)$ is a desired model, where f is an Ab-automorphism sending a' to a.

(2)⇒(3) Clear.

(3)⇒(1) follows from transitivity and symmetry of $\mathop{\smile\!\!\!\!\!\!\mid}^K$. ⊣

PROPOSITION 3.6. *Suppose T satisfies the existence axiom. The following are equivalent for a cardinal $\kappa \geq |T|$:*

(1) *T is NSOP$_1$.*
(2) *There is no increasing continuous sequence $\langle A_i : i < \kappa^+ \rangle$ of parameter sets and finite tuple d such that $|A_i| \leq \kappa$ and $d \not\mathop{\smile\!\!\!\!\!\!\mid}^K_{A_i} A_{i+1}$ for all $i < \kappa^+$.*
(3) *There is no set A of parameters of size κ^+ and $p(x) \in S(A)$ with x a finite tuple of variables such that for some increasing and continuous sequence of sets $\langle A_i : i < \kappa^+ \rangle$ with union A, we have $|A_i| \leq \kappa$ and p Kim-divides over A_i for all $i < \kappa^+$.*

PROOF. (1) $\Longrightarrow$ (2) It suffices to show that, given any increasing continuous sequence $\langle A_i : i < \kappa^+ \rangle$ of parameter sets and tuple d such that $|A_i| \leq \kappa$ and $d \not\mathop{\smile\!\!\!\!\!\!\mid}^K_{A_i} A_{i+1}$ for all $i < \kappa^+$, there is a continuous increasing sequence of models $\langle M_i : i < \kappa^+ \rangle$ and a finite tuple d' such that $|M_i| \leq \kappa$ and $d' \not\mathop{\smile\!\!\!\!\!\!\mid}^K_{M_i} M_{i+1}$ for all $i < \kappa^+$. This follows from Fact 2.8, since the existence of such a sequence of models implies T has SOP$_1$. Moreover, after naming constants, we may assume $\kappa = |T|$.

So suppose we are given $\langle A_i : i < |T|^+ \rangle$, an increasing continuous sequence of sets of parameters with $|A_i| \leq |T|$ for all $i < |T|^+$. Let $A = \bigcup_{i<|T|^+} A_i$, and suppose further that we are given some tuple d such that $d \not\mathop{\smile\!\!\!\!\!\!\mid}^K_{A_i} A_{i+1}$ for all $i < |T|^+$.

By induction on $i < |T|^+$ we will build increasing and continuous sequences $\langle A'_i : i < |T|^+ \rangle$ and $\langle M_i : i < |T|^+ \rangle$ satisfying the following for all $i < |T|^+$:

(1) $A'_0 = A_0$ and $A'_{\leq i} \equiv A_{\leq i}$.
(2) $M_i \models T$ with $|M_i| = |T|$ and $A'_i \subseteq M_i$.
(3) $A'_{i+1} \perp^K_{A'_i} M_i$.

To begin, we define $A'_0 = A_0$ and take M_0 be any model containing A'_0 of size $|T|$. Given $A'_{\leq i}$ and $M_{\leq i}$ satisfying the requirements, we pick A''_{i+1} such that $A'_{\leq i} A''_{i+1} \equiv A_{\leq i+1}$. Then we apply extension, to obtain $A'_{i+1} \equiv_{A'_{\leq i}} A''_{i+1}$ such that $A'_{i+1} \perp^K_{A'_i} M_i$. Note that $A'_{\leq i+1} \equiv A_{\leq i+1}$. We define M_{i+1} to be any model containing $A'_{i+1} M_i$ of size $|T|$. This satisfies the requirements.

At limit δ, we define $A'_\delta = \bigcup_{i<\delta} A'_i$ and $M_\delta = \bigcup_{i<\delta} M_i$. This clearly satisfies (1) and (2) and (3) is trivial. Therefore this completes the construction.

Let $M = \bigcup_{i<|T|^+} M_i$. Choose d' such that $d \langle A_i : i < |T|^+ \rangle \equiv d' \langle A'_i : i < |T|^+ \rangle$, which is possible by (1). Then we have $d' \not\perp^K_{A'_i} A'_{i+1}$ for all $i < |T|^+$.

Towards contradiction, suppose that there is some $i < |T|^+$ with the property that $d' \perp^K_{M_i} M_{i+1}$. Then, in particular, we have $d' \perp^K_{M_i} A'_{i+1}$. Additionally, because $M_i \perp^K_{A'_i} A'_{i+1}$, we know, by symmetry and transitivity, that $A'_{i+1} \perp^K_{A'_i} d' M_i$. By symmetry once more, we get $d' \perp^K_{A'_i} A'_{i+1}$, a contradiction. This shows that $d' \not\perp^K_{M_i} M_{i+1}$ for all $i < |T|^+$, completing the proof of this direction.

(2) $\Longrightarrow$ (3) Suppose (3) fails, i.e., we are given A of size κ^+, $p \in S(A)$, and an increasing continuous sequence of sets $\langle A_i : i < \omega \rangle$ such that $|A_i| \leq \kappa$ and p Kim-divides over A_i for all $i < \kappa^+$. We will define an increasing continuous sequence of ordinals $\langle \alpha_i : i < \kappa^+ \rangle$ such that $\alpha_i \in \kappa^+$ and $p \upharpoonright A_{\alpha_{i+1}}$ Kim-divides over A_{α_i} for all $i < \kappa^+$. We set $\alpha_0 = 0$ and given $\langle \alpha_j : j \leq i \rangle$, we know that there is some formula $\varphi(x; a_{i+1}) \in p$ that Kim-divides over A_{α_i}, by our assumption on p. Let α_{i+1} be the least ordinal $< \kappa^+$ such that a_{i+1} is contained in $A_{\alpha_{i+1}}$. For limit i, if we are given $\langle \alpha_j : j < i \rangle$, we put $\alpha_i = \sup_{j<i} \alpha_j$. Then we define $\langle A_i : i < \kappa^+ \rangle$ by $A'_i = A_{\alpha_i}$ for all $i < \kappa$, and let $d \models p$ be any realization. By construction, we have $d \not\perp^K_{A'_i} A'_{i+1}$ for all $i < \kappa^+$, which witnesses the failure of (2).

(3) $\Longrightarrow$ (1) This was established in [9, Theorem 3.9]. ⊣

Remark 3.7. In [2, Proposition 4.6] it is shown that in every theory with TP$_2$, there is an increasing chain of sets $\langle D_i : i < |T|^+ \rangle$ and tuple d such that $|D_i| \leq |T|$ and $d \not\perp^K_{D_i} D_{i+1}$ for all $i < |T|^+$. Hence, for non-simple NSOP$_1$ theories, the condition of continuity in the statement of Proposition 3.6 is essential.

The following theorem will be referred to as 'witnessing.' It shows that $\perp^K$-Morley sequences are witnesses to Kim-dividing. Over models this was established in [8, Theorem 5.1], however for us it will be deduced as a corollary of Proposition 3.6.

Theorem 3.8. *Suppose T is NSOP$_1$ with existence and $I = \langle a_i : i < \omega \rangle$ is an $\perp^K$-Morley sequence over A. If $\varphi(x; a_0)$ Kim-divides over A, then $\{\varphi(x; a_i) : i < \omega\}$ is inconsistent.*

Proof. Suppose towards contradiction that $\varphi(x;a_0)$ Kim-divides over A and $I=\langle a_i : i<\omega\rangle$ is an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence over A such that $\{\varphi(x;a_i) : i<\omega\}$ is consistent. By naming A as constants, we may assume $|A|\leq|T|$. We may stretch I such that $I=\langle a_i : i<|T|^+\rangle$. Define $A_i = Aa_{<i}$. Then $\langle A_i : i<|T|^+\rangle$ is increasing and continuous and $|A_i|\leq|T|$. Let $d\models\{\varphi(x;a_i) : i<|T|^+\}$. We claim $d \not\mathop{\smile\!\!\!\!\!\mid}^K_{A_i} A_{i+1}$ for all $i<|T|^+$. If not, then for some $i<|T|^+$, we have $d\mathop{\smile\!\!\!\!\!\mid}^K_{A_i} A_{i+1}$, or, in other words $d\mathop{\smile\!\!\!\!\!\mid}^K_{Aa_{<i}} a_i$. Since I is an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence, we also have $a_i\mathop{\smile\!\!\!\!\!\mid}^K_A a_{<i}$, hence $da_{<i}\mathop{\smile\!\!\!\!\!\mid}^K_A a_i$, by transitivity (Theorem 3.4). This entails, in particular, that $d\mathop{\smile\!\!\!\!\!\mid}^K_A a_i$, which is a contradiction, since $\varphi(x;a_i)$ Kim-divides over A. This completes the proof. ⊣

§4. Low NSOP$_1$ theories. This section is dedicated to proving that Lascar and Shelah strong types coincide in any low NSOP$_1$ theory with existence. This generalizes the corresponding result of Buechler for low simple theories [1] (also independently discovered by Shami [17]).

Definition 4.1. We say that the theory T is *low* if, for every formula $\varphi(x;y)$, there is some $k<\omega$, such that if $I=\langle a_i : i<\omega\rangle$ is an indiscernible sequence and $\{\varphi(x;a_i) : i<\omega\}$ is inconsistent, then it is k-inconsistent.

In [1], the definition of lowness is given in terms of the finiteness of certain $D(p,\varphi)$ ranks, which we will not need here. However, as observed in [1], the above definition coincides with this definition in the case that T is simple.

Lemma 4.2. *Suppose T is NSOP$_1$ with existence. Assume we are given tuples $(a_i)_{i\leq n}$ and $L(A)$-formulas $(\varphi_i(x;y_i))_{i\leq n}$ such that $a_i\mathop{\smile\!\!\!\!\!\mid}^K_A a_{<i}$ for all $i\leq n$. Then the following are equivalent*:

(1) *The formula $\bigwedge_{i\leq n}\varphi_i(x;a_i)$ does not Kim-divide over A.*

(2) *For all A-indiscernible sequences $\langle\overline{a}_j : j<\omega\rangle = \langle(a_{j,0},\dots,a_{j,n}) : j<\omega\rangle$ with $(a_{0,0},\dots,a_{0,n})=(a_0,\dots,a_n)$ and $a_{j,i}\mathop{\smile\!\!\!\!\!\mid}^K_A \overline{a}_{<j}a_{j,<i}$ for all $j<\omega$ and $i\leq n$, the following set of formulas does not Kim-divide over A*:

$$\left\{\bigwedge_{i\leq n}\varphi_i(x;a_{j,i}) : j<\omega\right\}.$$

(3) *There is an A-indiscernible sequence $\langle\overline{a}_j : j<\omega\rangle = \langle(a_{j,0},\dots,a_{j,n}) : j<\omega\rangle$ with $(a_{0,0},\dots,a_{0,n})=(a_0,\dots,a_n)$ and $a_{j,i}\mathop{\smile\!\!\!\!\!\mid}^K_A \overline{a}_{<j}a_{j,<i}$ for all $j<\omega$, $i\leq n$ such that*

$$\left\{\bigwedge_{i\leq n}\varphi_i(x;a_{j,i}) : j<\omega\right\}$$

is consistent.

Proof. $(1)\implies(2)$ Suppose we are given $\langle\overline{a}_j : j<\omega\rangle$ as in (2) and let $c\models\bigwedge_{i\leq n}\varphi_i(x;a_i)$ with $c\mathop{\smile\!\!\!\!\!\mid}^K_A\overline{a}_0$. As $\langle\overline{a}_j : j<\omega\rangle$ is A-indiscernible, for each $j>0$, there is some $\sigma_j\in\mathrm{Autf}(\mathbb{M}/A)$ with $\sigma_j(\overline{a}_0)=\overline{a}_j$. Define $c_0=c$ and $c_j=\sigma_j(c)$

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for all $j > 0$. Then we have $c_j \equiv^L_A c_0$ and $c_j \models \bigwedge_{i \leq n} \varphi_i(x; a_{j,i})$ for all $j < \omega$. By inductively applying the independence theorem (with respect to the lexicographic order on $\omega \times n$), we obtain $c_* \models \{\varphi_i(x; a_{j,i}) : i \leq n, j < \omega\}$ with $c_* \mathop{\smile\!\!\!\!\!\mid}^K_A \overline{a}_{<\omega}$, which establishes (2).

(2) $\Longrightarrow$ (3) It suffices to show that there is an A-indiscernible sequence $\langle \overline{a}_j : j < \omega \rangle = \langle (a_{j,0}, \dots, a_{j,n}) : j < \omega \rangle$ with $(a_{0,0}, \dots, a_{0,n}) = (a_0, \dots, a_n)$ and such that $a_{j,i} \mathop{\smile\!\!\!\!\!\mid}^K_A \overline{a}_{<j} a_{j,<i}$ for all $j < \omega$, $i \leq n$.

First, we construct by induction a sequence $\langle (a'_{j,0}, \dots, a'_{j,n} : j < \omega \rangle = \langle \overline{a}'_j : j < \omega \rangle$ with $\overline{a}'_j \equiv_A (a_0, \dots, a_n)$ and $a'_{j,i} \mathop{\smile\!\!\!\!\!\mid}^K_A \overline{a}'_{<j} a_{j,<i}$ for all $j < \omega$, $i \leq n$. Given $\overline{a}'_{\leq k}$, we apply extension to find $a'_{k+1,0} \equiv_A a_0$ with $a'_{k+1,0} \mathop{\smile\!\!\!\!\!\mid}^K_A \overline{a}'_{\leq k}$. Given $\overline{a}'_{k+1,\leq i}$ for $i < n$, we find $b_{k+1,i+1}$ such that $a'_{k+1,\leq i} b_{k+1,i+1} \equiv_A a_{\leq i} a_{i+1}$. By invariance, this implies $b_{k+1,i+1} \mathop{\smile\!\!\!\!\!\mid}^K_A a'_{k+1,\leq i}$. Applying extension once more, we can find $a'_{k+1,i+1} \equiv_{A a'_{k+1,\leq i}} b_{k+1,i+1}$ such that $a'_{k+1,i+1} \mathop{\smile\!\!\!\!\!\mid}^K_A \overline{a}'_{\leq k} a'_{k+1,\leq i}$. This completes the construction of $\langle \overline{a}'_j : j < \omega \rangle$. By Ramsey, compactness, and an automorphism, we can extract an A-indiscernible sequence $\langle \overline{a}_j : j < \omega \rangle$ with $\overline{a}_0 = (a_0, \dots, a_n)$ as desired.

(3) $\Longrightarrow$ (1) Let $c \models \left\{ \bigwedge_{i \leq n} \varphi_i(x; a_{j,i}) : j < \omega \right\}$. Note that, for each $i \leq n$, the sequence $\langle a_{j,i} : j < \omega \rangle$ is an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence over A. Hence, by witnessing, Theorem 3.8, and the fact that $c \models \{\varphi_i(x; a_{j,i}) : j < \omega\}$, we see that $c \mathop{\smile\!\!\!\!\!\mid}^K_A a_i$ for each $i \leq n$.

By induction on $k \leq n$, we will choose $c_k \equiv^L_A c$ such that $c_k \models \{\varphi_i(x; a_i) : i \leq k\}$ and $c_k \mathop{\smile\!\!\!\!\!\mid}^K_A a_{\leq k}$. To begin we set $c_0 = c$. Given c_k for some $k < n$, we apply the independence theorem to find c_{k+1} with $c_{k+1} \equiv^L_{Aa_{\leq k}} c_k$, $c_{k+1} \equiv^L_{Aa_{k+1}} c$, and $c_{k+1} \mathop{\smile\!\!\!\!\!\mid}^K_A a_{\leq k+1}$. After n steps, we obtain $c_n \models \bigwedge_{i \leq n} \varphi_i(x; a_i)$ with $c_n \mathop{\smile\!\!\!\!\!\mid}^K_A a_{\leq n}$, which shows (1). ⊣

Remark 4.3. The independence conditions of (2) and (3) do not imply that the sequence $\langle (a_{i,0}, \dots, a_{i,n}) : i < \omega \rangle$ is $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley, due to the lack of base monotonicity. Consequently, this lemma strengthens witnessing, Theorem 3.8, for formulas of a certain form, showing that they Kim-divide along sequences that are themselves not necessarily $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequences.

Corollary 4.4. *Suppose T is NSOP$_1$ with existence. Assume that for each $i \leq n$, we are given a complete type $p(y_i) \in S(A)$ and an $L(A)$-formula $\varphi_i(x; y_i)$.*

1. *There is a partial type $R(y_0, \dots, y_n)$ over A containing $\bigcup_{i \leq n} p_i(y_i)$ such that $a'_0, \dots, a'_n \models R(y_0, \dots, y_n)$ if and only if $a'_i \mathop{\smile\!\!\!\!\!\mid}^K_A a'_{<i}$ for all $i \leq n$ and $\bigwedge_{i \leq n} \varphi_i(x; a'_i)$ does not Kim-divide over A.*
2. *If, additionally, T is low, then there is a formula $\gamma(y_0, \dots, y_n)$ over A such that, if $(a'_0, \dots, a'_n) \models p(y_0) \cup \dots \cup p(y_n)$ and $a'_i \mathop{\smile\!\!\!\!\!\mid}^K_A a'_{<i}$ for all $i \leq n$, then $\mathbb{M} \models \gamma(a'_0, \dots, a'_n)$ if and only if $\bigwedge_{i \leq n} \varphi_i(x; a'_i)$ does not Kim-divide over A.*

Proof. By Lemma 2.21(1), there is a partial type $\Lambda(z_{i,j} : i < \omega, j \leq n)$ over A which expresses the following:

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(a) $z_{i,j} \models p_j$ for all $i < \omega$ and $j \leq n$.
(b) $z_{i,j} \mathop{\smile\!\!\!\!\!|}^K_A \overline{z}_{<i} z_{i,<j}$ for all $i < \omega$ and $j \leq n$, where $\overline{z}_i = (z_{i,0}, \dots, z_{i,n})$.
(c) The sequence $\langle \overline{z}_i : i < \omega \rangle$ is A-indiscernible.

Let $\lambda'(y_0, \dots, y_n, \overline{z}_i : i < \omega)$ be the partial type given by $\{y_j = z_{0,j} : j \leq n\} \cup \Lambda(\overline{z}_i : i < \omega)$.

To show (1), consider the partial type $R_0(y_0, \dots, y_n, \overline{z}_i : i < \omega)$ which extends λ' and expresses additionally that $\{\bigwedge_{i \leq n} \varphi_i(x; z_{i,j}) : i < \omega\}$ is consistent. Then R may be defined by

$$R(y_0, \dots, y_n) \equiv (\exists \overline{z}_i : i < \omega) \bigwedge R_0(y_0, \dots, y_n, \overline{z}_i; i < \omega).$$

This R is clearly type definable and, by Lemma 4.2(3), R has the desired properties.

For (2), we know, by the lowness of T, that there is some $k < \omega$ such that, if, given any $\langle (a'_{i,j})_{j \leq n} : i < \omega \rangle$ is an A-indiscernible sequence and

$$\left\{ \bigwedge_{j \leq n} \varphi_i(x; a_{i,j}) : i < \omega \right\}$$

is inconsistent, then this set of formulas is k-inconsistent. Consider the partial type $R_1(y_0, \dots, y_n, \overline{z}_i : i < \omega)$ which extends λ' and expresses additionally that $\{\bigwedge_{j \leq n} \varphi_i(x; z_{i,j}) : i < \omega\}$ is k-inconsistent. Then we will define R' by

$$R'(y_0, \dots, y_n) \equiv (\exists \overline{z}_i : i < \omega) \bigwedge R_1(y_0, \dots, y_n, \overline{z}_i; i < \omega).$$

It follows that if $(a'_j)_{j \leq n} \models \bigcup_{j \leq n} p_j(y_j)$ and $a'_j \mathop{\smile\!\!\!\!\!|}^K_A a'_{<j}$, then $(a'_j)_{j \leq n} \models R'$ if and only if $\bigwedge_{j \leq n} \varphi_j(x; a'_j)$ Kim-divides over A, by Lemma 4.2(2). We showed in (1) that the complement of R' is type-definable (by R), and therefore, by compactness, we obtain the desired γ. ⊣

THEOREM 4.5. *If T is a low NSOP$_1$ theory with existence, then Lascar strong types are strong types. That is, for any (possibly infinite) tuples a, b and small set of parameters A, if $a \equiv^S_A b$ then $a \equiv^L_A b$.*

PROOF. Let A be any small set of parameters. As T_A is G-compact by Fact 2.7(7) and, trivially, T_A is low, it suffices to prove the theorem when a and b realize a type $p(x)$ over $\emptyset$, where x is a finite tuple of variables. Let $r(x, y)$ be a partial type, closed under conjunctions, expressing that $x \equiv^L y$, i.e., $r(x, y)$ defines the finest type-definable equivalence relation over $\emptyset$ with boundedly many classes. Fix $\varphi(x; y) \in r(x, y)$. Note that, for any $a \models p$, $\varphi(x; a)$ does not divide over $\emptyset$, because if $\langle a_i : i < \omega \rangle$ is an indiscernible sequence with $a_0 = a$, then $a_0 \equiv^L a_i$ for all $i < \omega$, hence $a_0 \models \bigcup_{i<\omega} r(x; a_i)$. In particular, $\varphi(x; a)$ does not Kim-divide over $\emptyset$.

Define a relation $R_\varphi(u, v)$ expressing the following:

(1) $u, v \models p$.
(2) There exists v' satisfying:
 (a) $v' \equiv^L v$.
 (b) $u \mathop{\smile\!\!\!\!\!|}^K v'$.
 (c) $\varphi(x; u) \wedge \varphi(x; v')$ does not Kim-divide over $\emptyset$.

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Clearly if $v \models p$ and $v \equiv^L v'$, then $v' \models p$. By Lemma 2.21 and Corollary 4.4(1), there is a partial type $\Gamma(z, w)$ over $\emptyset$ such that $(v', u) \models \Gamma(z, w)$ if and only if $v', u \models p$, $u \mathop{\smile\!\!\!\!\!\!|}^K v'$, and $\varphi(x; v') \wedge \varphi(x; u)$ does not Kim-divide over $\emptyset$. It follows that $R_\varphi(u, v)$ if and only if $(\exists v') \left[\bigwedge r(v, v') \wedge \Gamma(v', u)\right]$, which shows $R_\varphi(u, v)$ is type-definable.

In a similar fashion, we define a relation $S_\varphi(u, v)$ to hold when the following conditions are satisfied:

(1) $u, v \models p$.
(2) There is v' satisfying the following:
  (a) $v' \equiv^L v$.
  (b) $u \mathop{\smile\!\!\!\!\!\!|}^K v'$.
  (c) $\varphi(x; u) \wedge \varphi(x; v)$ Kim-divides over $\emptyset$.

The type-definability of S_φ follows from an identical argument, using Corollary 4.4(2) in the place of Corollary 4.4(1) (this is where we make use of our hypothesis that T is low).

From here, our proof follows the argument of [1], as presented in [12, Section 5.2]. First, we show the following:

Claim 1. *If $u, v \models p$, then $R_\varphi(u, v)$ if and only if $\neg S_\varphi(u, v)$.*

Proof of Claim 1. First, it is clear that it is impossible for both $\neg R_\varphi(u, v)$ and $\neg S_\varphi(u, v)$ to hold since, by extension for Lascar strong types (Fact 2.20) there is $v' \equiv^L v$ with $u \mathop{\smile\!\!\!\!\!\!|}^K v'$ and it must be the case that either $\varphi(x, u) \wedge \varphi(x; v')$ Kim-divides or $\varphi(x; u) \wedge \varphi(x; v')$ does not Kim-divide over $\emptyset$.

Secondly, suppose $R_\varphi(u, v)$ holds witnessed by v' and $S_\varphi(u, v)$ holds witnessed by v''. Then we have $v \equiv^L v' \equiv^L v''$, $u \mathop{\smile\!\!\!\!\!\!|}^K v'$, $u \mathop{\smile\!\!\!\!\!\!|}^K v''$, $\varphi(x; u) \wedge \varphi(x; v')$ does not Kim-divide over $\emptyset$, and $\varphi(x; u) \wedge \varphi(x; v'')$ Kim-divides over $\emptyset$. Choose c' realizing $\varphi(x; u) \wedge \varphi(x; v')$ with $c' \mathop{\smile\!\!\!\!\!\!|}^K u, v'$, and pick c'' so that $c'v' \equiv^L c''v''$. Then, in particular, we have $c' \equiv^L c''$, $c' \mathop{\smile\!\!\!\!\!\!|}^K u$, $c'' \mathop{\smile\!\!\!\!\!\!|}^K v''$, and $u \mathop{\smile\!\!\!\!\!\!|}^K v''$ so, by the independence theorem, there is c with $c \models \mathrm{Lstp}(c'/u) \cup \mathrm{Lstp}(c/v'')$ and $c \mathop{\smile\!\!\!\!\!\!|}^K u, v''$. Since c realizes $\varphi(x; u) \wedge \varphi(x; v'')$, we obtain a contradiction. Together with the first part, this establishes the claim. ⊣

By the claim and compactness, there is a formula $\sigma(x, y) = \sigma_\varphi(x, y)$ such that $p(x) \cup p(y) \vdash R_\varphi(x, y)$ if and only if $\sigma(x, y)$. Moreover, it is clear from the definitions that $R_\varphi(x, y) \wedge r(y, y')$ implies $R_\varphi(x, y')$. Again by compactness, there is a formula $\delta(x) \in p(x)$ such that

(*) $\delta(x) \wedge \delta(y) \wedge \sigma(x, y) \wedge \bigwedge r(x, z) \models \sigma(x, z)$.

It follows from $(*)$ and the symmetry of r that, for any $z \models \delta$, we have $\models \sigma(z, x) \leftrightarrow \sigma(z, y)$, for all $x, y \models \delta$. Therefore, we obtain a definable equivalence relation $E_\varphi(x, y)$ as follows:

$$E_\varphi(x, y) \equiv \left[\neg\delta(x) \wedge \neg\delta(y)\right] \vee \left[\delta(x) \wedge \delta(y) \wedge (\forall z) \left(\delta(z) \to (\sigma(z, x) \leftrightarrow \sigma(z, y)\right)\right].$$

To conclude, we establish the following:

Claim 2. *The partial type $p(x) \cup p(y)$ implies $r(x, y)$ holds if and only if $\bigwedge_{\varphi \in r} E_\varphi(x, y)$ holds.*

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Proof of claim. First, we will show $r(x, y)$ entails $E_\varphi(x, y)$ for all $\varphi \in r$. Clearly $r(x, y)$ implies $\delta(x) \leftrightarrow \delta(y)$. Moreover, as noted above, if $r(x, y)$ and $\delta(x) \wedge \delta(y)$ both hold, then for any $z \models \delta$ we have $\sigma(z, x) \leftrightarrow \sigma(z, y)$ by (*) above and the symmetry of $r(x, y)$. Hence $E_\varphi(x, y)$ holds.

Second, we will show that if $(a, b) \models p(x) \cup p(y)$ does not realize $r(x, y)$, then $\neg \bigwedge_{\varphi \in r} E_\varphi(a, b)$. Choose $\psi(x, y) \in r(x, y)$ such that $\neg\psi(a, b)$. Because r is an equivalence relation, there is some $\varphi(x, y) \in r(x, y)$ such that

$$(\exists x, x') \left(\varphi(x, y) \wedge \varphi(x, x') \wedge \varphi(x', z)\right) \vdash \psi(y, z).$$

Then if $E_\varphi(a, b)$ holds, then, because $R_\varphi(a, a)$ holds, we have $\sigma_\varphi(a, a)$ and $\sigma_\varphi(a, b)$ and therefore $R_\varphi(a, b)$. By the definition of $R_\varphi(a, b)$, this implies there is some $c \equiv^L b$ such that $\varphi(x, a) \wedge \varphi(x, c)$ is consistent, and therefore $\models \psi(a, b)$, a contradiction. This concludes the proof of the claim, and hence the theorem. ⊣

§5. Rank. In this section, we introduce a family of ranks, suitable for the study of NSOP_1 theories. This makes critical use of witnessing over arbitrary sets and provides a clear context in which working over arbitrary sets greatly simplifies the situation. Our definition is close to the definition of D-rank familiar from simple theories, but we are required to add a new parameter in the rank, which keeps track of the type of the parameters that appear in instances of Kim-dividing.

Definition 5.1. Suppose $q(y) \in S(B)$, $\Delta(x; y)$ is a finite set of $L(B)$-formulas, and $k < \omega$. Then for any set of formulas $\pi(x)$ over $\mathbb{M}$, we define $D_1(\pi, \Delta, k, q) \geq 0$ if π is consistent, and $D_1(\pi, \Delta, k, q) \geq n + 1$ if there is a sequence $I = \langle c_i : i < \omega\rangle$ such that the following conditions hold:

(1) The sequence I is an $\mathop{\smile\hspace{-0.9em}\mid}^K$-Morley sequence over B with $c_i \models q$.
(2) The sequence I is indiscernible over $\mathrm{dom}(\pi)B$ (and hence $\mathrm{dom}(\pi) \mathop{\smile\hspace{-0.9em}\mid}^K_B c_i$ for all i).
(3) We have $\{\varphi(x; c_i) : i < \omega\}$ is k-inconsistent for some formula $\varphi(x; y) \in \Delta$.
(4) We have $D_1(\pi \cup \{\varphi(x; c_i)\}, \Delta, k, q) \geq n$ for all $i < \omega$.

We define $D_1(\pi, \Delta, k, q) = n$ if n is least such that $D_1(\pi, \Delta, k, q) \geq n$ and also $D_1(\pi, \Delta, k, q) \not\geq n + 1$. We say $D_1(\pi, \Delta, k, q) = \infty$ if $D_1(\pi, \Delta, k, q) \geq n$ for all $n < \omega$.

Lemma 5.2. *Suppose $q(y) \in S(B)$, $\Delta(x; y)$ is a finite set of $L(B)$-formulas, and $k < \omega$. Then we have the following*:

(1) *For all $\sigma \in \mathrm{Aut}(\mathbb{M}/B)$,*

$$D_1(\sigma(\pi), \Delta, k, q) = D_1(\pi, \Delta, k, q).$$

(2) *If π and π' are partial types over $\mathbb{M}$ such that $\pi(x) \vdash \pi'(x)$, then*

$$D_1(\pi, \Delta, k, q) \leq D_1(\pi', \Delta, k, q).$$

(3) *If $n \geq m$, then $D_1(\pi, \Delta, k, q) \geq n$ implies $D_1(\pi, \Delta, k, q) \geq m$.*
(4) *If $\psi_0(x), \ldots, \psi_{m-1}(x)$ are formulas over $\mathbb{M}$, then*

$$D_1\left(\pi \cup \left\{\bigvee_{j<m} \psi_i(x)\right\}, \Delta, k, q\right) = \max_{j<m} D_1(\pi \cup \{\psi_j(x)\}, \Delta, k, q).$$

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PROOF. (1) is clear.

(2) By induction on n, we will show

$$D_1(\pi, \Delta, k, q) \geq n \implies D_1(\pi', \Delta, k, q) \geq n.$$

For $n = 0$ there is nothing to show. Suppose the statement holds for n, and assume $D_1(\pi, \Delta, k, q) \geq n + 1$. Then there is a $\mathrm{dom}(\pi)B$-indiscernible sequence $I = \langle c_i : i < \omega \rangle$ which is additionally an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence over B in q and $\varphi(x; y) \in \Delta$ such that $\{\varphi(x; c_i) : i < \omega\}$ is k-inconsistent and $D_1(\pi \cup \{\varphi(x; c_i)\}, \Delta, k, q) \geq n$. Let $I' = \langle c_i' : i < \omega \rangle$ be a $B\mathrm{dom}(\pi)\mathrm{dom}(\pi')$-indiscernible sequence locally based on I. Clearly we have that I' is an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence in q over B and $D_1(\pi \cup \{\varphi(x; c_i')\}, \Delta, k, q) \geq n$ for all $i < \omega$, by (1). As $\pi \cup \{\varphi(x; c_i')\} \vdash \pi' \cup \{\varphi(x; c_i')\}$, we have $D_1(\pi' \cup \{\varphi(x; c_i')\}, \Delta, k, q) \geq n$ by induction, which implies $D_1(\pi', \Delta, k, q) \geq n + 1$.

(3) We will prove by induction on $l < \omega$ that $D_1(\pi, \delta, k, q) \geq n + l$ implies $D_1(\pi, \delta, k, q) \geq n$. For $l = 0$, this is trivial. Assume it has been shown for l and that $D_1(\pi, \Delta, k, q) \geq n + l + 1$. Then there is a sequence $I = \langle c_i : i < \omega \rangle$ satisfying the conditions of Definition 5.1 such that $D_1(\pi \cup \{\varphi(x; c_i)\}, \Delta, k, q) \geq n + l$ for all i. By (2), this entails, in particular, that $D_1(\pi, \Delta, k, q) \geq n + l$ and hence $D_1(\pi, \Delta, k, q) \geq n$ by induction.

(4) By (2), we have

$$\max_{j<m} D_1(\pi \cup \{\psi_j(x)\}, \Delta, k, q) \leq D_1\left(\pi \cup \left\{\bigvee_{j<m} \psi_j(x)\right\}, \Delta, k, q\right).$$

Hence, it suffices to show, for $n < \omega$, that

$$D_1\left(\pi \cup \left\{\bigvee_{j<m} \psi_j(x)\right\}, \Delta, k, q\right) \geq n \implies \max_{j<m} D_1(\pi \cup \{\psi_j(x)\}, \Delta, k, q) \geq n.$$

Let C be the (finite) set of parameters appearing in the formulas $\psi_0, \ldots, \psi_{n-1}$. For $n = 0$, this is clear. If $D_1\left(\pi \cup \left\{\bigvee_{j<m} \psi_j(x)\right\}, \Delta, k, q\right) \geq n + 1$, then, as in (2), there is an $\mathop{\smile\!\!\!\!\!\mid}^K$-Morley sequence $I = \langle c_i : i < \omega \rangle$ over B in q which is $\mathrm{dom}(\pi)BC$-indiscernible and $\varphi(x; y) \in \Delta$ such that $\{\varphi(x; c_i) : i < \omega\}$ is k-inconsistent with

$$D_1\left(\pi \cup \left\{\bigvee_{j<m} \psi_j(x)\right\} \cup \{\varphi(x; c_i)\}, \Delta, k, q\right) \geq n,$$

which implies $\max_{j<m} D_1(\pi \cup \{\psi_j(x), \varphi(x; c_i)\}, \Delta, k, q) \geq n$, by the induction hypothesis for n, for each i. By the pigeonhole principle, we may assume this maximum witnessed by the same j for all $i < \omega$. This shows $\max_{j<m} D_1(\pi \cup \{\psi_j(x)\}, \Delta, k, q) \geq n + 1$. ⊣

REMARK 5.3. Lemma 5.2(2) implies, in particular, that if $\pi(x)$ and $\pi'(x)$ are equivalent then the ranks (with respect to a choice of Δ, k, and q) will be the same, even if they have different domains.

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LEMMA 5.4. *Suppose T is NSOP$_1$ with existence. Suppose $q \in S(B)$, $\Delta(x; y)$ is a finite set of $L(B)$-formulas, and $k < \omega$. Then for all $n < \omega$, we have $D_1(\pi, \Delta, k, q) \geq n$ if and only if there are $(c_\eta)_{\eta \in \omega^{\leq n} \setminus \{\emptyset\}}$ and $\varphi_i(x; y) \in \Delta$ for $i < n$ satisfying the following*:

(a) *For all $\eta \in \omega^n$,*

$$\pi(x) \cup \{\varphi_i(x; c_{\eta|(i+1)}) : i < n\}$$

is consistent.

(b) *For all $\eta \in \omega^{<n}$, $\{\varphi_{l(\eta)}(x; c_{\eta \frown \langle i \rangle}) : i < \omega\}$ is k-inconsistent.*

(c) *For all $\eta \in \omega^{<n}$, $\langle c_{\eta \frown \langle i \rangle} : i < \omega \rangle$ is an $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^K$-Morley sequence over B in q.*

(d) *The tree $(c_\eta)_{\eta \in \omega^{\leq n} \setminus \{\emptyset\}}$ is s-indiscernible over $B\mathrm{dom}(\pi)$.*

PROOF. For the case of $n = 0$, (a) is satisfied if and only if π is consistent and (b)–(d) hold trivially, which gives the desired equivalence.

Now assume, for a given n, that $D_1(\pi, \Delta, k, q) \geq n$ if and only if there are $(c_\eta)_{\eta \in \omega^{\leq n} \setminus \{\emptyset\}}$ and $\varphi_i(x; y) \in \Delta$ for $i < n$ such that (a)–(d) hold. First, suppose $D_1(\pi, \Delta, k, q) \geq n + 1$. Then we can find a $\mathrm{dom}(\pi)B$-indiscernible sequence $I = \langle c_i : i < \omega \rangle$ which is also $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^K$-Morley over B in q and $\varphi(x; y) \in \Delta$ such that $\{\varphi(x; c_i) : i < \omega\}$ is k-inconsistent and $D_1(\pi \cup \{\varphi(x; c_i)\}, \Delta, k, q) \geq n$ for all $i < \omega$. By induction, for each $i < \omega$, there is a tree $(c_{i,\eta})_{\eta \in \omega^{\leq n} \setminus \{\emptyset\}}$ and sequence of formulas $(\varphi_{i,j}(x; y))_{1 \leq j \leq n}$ from Δ satisfying (a)–(d), with π replaced by $\pi \cup \{\varphi(x; c_i)\}$. As Δ is a finite set of formulas, we may, by the pigeonhole principle, assume that there are $\varphi_j \in \Delta$ such that $\varphi_{i,j}(x; y) = \varphi_j(x; y)$ for all $i < \omega$ and $1 \leq j \leq n$. Now define a tree $(c'_\eta)_{\eta \in \omega^{\leq n+1} \setminus \{\emptyset\}}$ such that $c'_{\langle i \rangle} = c_i$ and $c'_{\langle i \rangle \frown \eta} = c_{i,\eta}$ for all $i < \omega$ and $\eta \in \omega^{\leq n} \setminus \{\emptyset\}$. Define $\varphi_0(x; y) = \varphi(x; y)$. Let $(c_\eta)_{\eta \in \omega^{\leq n+1} \setminus \{\emptyset\}}$ be a tree that is s-indiscernible tree over $\mathrm{dom}(\pi)B$ and locally based on $(c'_\eta)_{\eta \in \omega^{\leq n+1} \setminus \{\emptyset\}}$ over B. Note that for all $\eta \in \omega^{<n+1}$, the sequence $\langle c_{\eta \frown \langle i \rangle} : i < \omega \rangle$ has the same type over B as a Morley sequence in q and hence is a Morley sequence in q. It is clear that $(c_\eta)_{\eta \in \omega^{\leq n+1} \setminus \{\emptyset\}}$ and $(\varphi_j(x; y))_{j < n+1}$ satisfy the requirements.

Conversely, given $(c_\eta)_{\eta \in \omega^{\leq n+1} \setminus \{\emptyset\}}$ and $(\varphi_j(x; y))_{j < n+1}$ satisfying (a)–(d), we observe by induction that the tree $(c_{\langle i \rangle \frown \eta})_{\eta \in \omega^{\leq n} \setminus \{\emptyset\}}$ witnesses

$$D_1(\pi \cup \{\varphi_0(x; c_{\langle i \rangle})\}, \Delta, k, q) \geq n,$$

for all $i < \omega$. As the sequence $\langle c_{\langle i \rangle} : i < \omega \rangle$ is $\mathop{\smile\hspace{-0.9em}\mid\hspace{0.5em}}^K$-Morley over B and $\mathrm{dom}(\pi)B$-indiscernible, and $\{\varphi_0(x; c_{\langle i \rangle}) : i < \omega\}$ is k-inconsistent, it follows that

$$D_1(\pi, \Delta, k, q) \geq n + 1,$$

completing the proof. ⊣

COROLLARY 5.5. *If T is NSOP$_1$ with existence, then given $q(y) \in S(B)$, a finite set of $L(B)$-formulas $\Delta(x; y)$, and $k < \omega$, there is some $n < \omega$ such that $D_1(x = x, \Delta, k, q) = n$.*

PROOF. Suppose towards contradiction that there are q, Δ, and $k < \omega$ such that $D(x = x, \Delta, k, q) > n$ for all $n < \omega$. By Lemma 5.4, Lemma 2.21, compactness, and the pigeonhole principle (by the finiteness of Δ), we can find a tree $(c_\eta)_{\eta \in \omega^{<\omega} \setminus \{\emptyset\}}$ and $\varphi(x; y) \in \Delta$ satisfying the following:

(a) For all $\eta \in \omega^\omega$,

$$\{\varphi(x; c_{\eta|(i+1)}) : i < \omega\}$$

is consistent.

(b) For all $\eta \in \omega^{<\omega}$, $\{\varphi(x; c_{\eta\frown\langle i \rangle}) : i < \omega\}$ is k-inconsistent.

(c) For all $\eta \in \omega^{<\omega}$, $\langle c_{\eta\frown\langle i \rangle} : i < \omega \rangle$ is an $\mathop{\smile\!\!\!\!\!\!\!\;|}^K$-Morley sequence over B in q.

(d) The tree $(c_\eta)_{\eta \in \omega^{<\omega} \setminus \{\emptyset\}}$ is s-indiscernible over B.

Note that, by s-indiscernibility, we have $\langle c_{0^n\frown\langle i \rangle} : i < \omega \rangle$ is an $\mathop{\smile\!\!\!\!\!\!\!\;|}^K$-Morley sequence over B which is $Bc_{0\frown 0^{<n}}$-indiscernible, for all $n < \omega$. By witnessing, it follows that $c_{0\frown 0^{<n}} \mathop{\smile\!\!\!\!\!\!\!\;|}^K_B c_{0^{n+1}}$ for all n. Let $\langle d_i : i < \omega \rangle$ be a B-indiscernible sequence locally based on $\langle c_{0^n} : 1 \leq n < \omega \rangle$ over B. As each $c_{0^n} \models q$, for $1 \leq n < \omega$, we have that $d_{<n} \mathop{\smile\!\!\!\!\!\!\!\;|}^K_B d_n$ for all n as well, and therefore $\langle d_i : i < \omega \rangle$ is an $\mathop{\smile\!\!\!\!\!\!\!\;|}^K$-Morley sequence over B in q'. Moreover, since $\{\varphi(x; c_{0^n}) : 1 \leq n < \omega\}$ is consistent, we know $\{\varphi(x; d_n) : n < \omega\}$ is consistent. However, we know $\varphi(x; c_{0^n})$ Kim-divides over B and hence $\varphi(x; d_n)$ Kim-divides over B. This contradicts witnessing (Theorem 2.5). ⊣

Remark 5.6. Corollary 5.5 has a converse: if T has SOP$_1$, then for some B, there is a $q(y) \in S(B)$, a finite set of $L(B)$-formulas $\Delta(x; y)$, and $k < \omega$ such that $D_1(x = x, \Delta, k, q) = \infty$. One way to see this is to note that by [5, Corollary 3.7] Kim's lemma for non-forking Morley sequences fails over some set B in any theory with SOP$_1$ (this doesn't use existence). Concretely, this means there are Morley sequences $I = \langle a_i : i < \omega \rangle$ and $J = \langle b_i : i < \omega \rangle$ over B with $a_0 = b_0$ and a formula $\varphi(x; y)$ such that $\{\varphi(x; a_i) : i < \omega\}$ is k-inconsistent and $\{\varphi(x; b_i) : i < \omega\}$ is consistent. Then, by [5, Lemma 3.4], this implies there is a tree $(c_\eta)_{\eta \in \omega^{<\omega}}$ satisfying the following properties:

(1) For all $\eta \in \omega^{<\omega}$, $(c_{\eta\frown\langle i \rangle})_{i<\omega} \equiv_B I$.
(2) For all $\eta \in \omega^{<\omega}$, $(c_\eta, c_{\eta|l(\eta)-1}, \ldots, c_\emptyset) \equiv_B (b_0, b_1, \ldots, b_{l(\eta)})$.
(3) $(c_\eta)_{\eta \in \omega^{<\omega}}$ is s-indiscernible over B.

Then, by Lemma 5.4, for $q = \mathrm{tp}(a_0/B)$, we have $D_1(x = x, \Delta, k, q) \geq n$ for all n.

Lemma 5.7. *Suppose T is NSOP$_1$ with existence. Suppose $q(y) \in S(B)$, $\Delta(x, y)$ is a finite set of $L(B)$-formulas, and $k < \omega$.*

(1) *For any partial type p, there is a finite $r \subseteq p$ such that*

$$D_1(p, \Delta, k, q) = D_1(r, \Delta, k, q).$$

(2) *For any small set $A \subseteq \mathbb{M}$ and any partial type p over A, there is $p' \in S(A)$ extending p such that*

$$D_1(p, \Delta, k, q) = D_1(p, \Delta, k, q).$$

Proof. (1) By Lemma 5.2(1), if r is a subtype of p then $D_1(r, \Delta, k, q) \geq D_1(p, \Delta, k, q)$ so it suffices to find a finite $r \subseteq p$ with $D_1(r, \Delta, k, q) \leq D_1(p, \Delta, k, q)$. Suppose $D_1(p, \Delta, k, q) = n$. Consider, for each sequence $\overline{s} = (\varphi_i(x; y))_{i<n+1}$ of

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formulas from Δ, the set of formulas $\Gamma_{\overline{s}}(x, (z_\eta)_{\eta \in \omega^{\leq n+1} \setminus \{\emptyset\}})$ over $\mathrm{dom}(p)B$ expressing the following:

(1) For all $\eta \in \omega^{n+1}$,

$$p(x) \cup \{\varphi_i(x; z_{\eta|(i+1)}) : i < n+1\}$$

is consistent.

(2) For all $\eta \in \omega^{<n+1}$, $\{\varphi_{l(\eta)}(x; z_{\eta \frown \langle i \rangle}) : i < \omega\}$ is k-inconsistent.

(3) For all $\eta \in \omega^{<n+1}$, $\langle z_{\eta \frown \langle i \rangle} : i < \omega \rangle$ is an $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}^K$-Morley sequence over B in q (possible by Lemma 2.21(2)).

(4) The tree $(z_\eta)_{\eta \in \omega^{\leq n+1} \setminus \{\emptyset\}}$ is s-indiscernible over $B\mathrm{dom}(p)$.

By Lemma 5.4 and the fact that $D_1(p, \Delta, k, q) = n < n+1$, we know that $\Gamma_{\overline{s}}$ is inconsistent. By compactness, there is some finite $r_{\overline{s}}(x) \subseteq p(x)$ such that, replacing $p(x)$ with $r_{\overline{s}}(x)$ in (1), the formulas remain inconsistent. Let r be the union of $r_{\overline{s}}$ as $\overline{s}$ ranges over all length $n+1$ sequences of formulas of Δ. As Δ is finite, this is a finite set, so r is a finite extension of each $r_{\overline{s}}$ and a subtype of p. Then by Lemma 5.4 again, $D_1(r, \Delta, k, q) \leq n = D_1(p, \Delta, k, q)$.

(2) Let $\Gamma(x)$ be defined as follows:

$$\Gamma(x) = \{\neg\psi(x) \in L(A) : D_1(p(x) \cup \{\psi(x)\}, \Delta, k, q) < D_1(p, \Delta, k, q)\}.$$

If $p(x) \cup \Gamma(x)$ is inconsistent, then by compactness, there are $\psi_0, \dots, \psi_{n-1} \in \Gamma$ such that $p(x) \vdash \bigvee_{j<n} \psi_j$. By Lemma 5.2(1) and (2), this gives

$$\begin{aligned} D_1(p, \Delta, k, q) &= D_1\left(p \cup \left\{\bigvee_{j<n} \psi_j(x)\right\}, \Delta, k, q\right) \\ &= \max_j D_1(p \cup \{\psi_j(x)\}, \Delta, k, q) \\ &< D_1(p, \Delta, k, q), \end{aligned}$$

a contradiction. Therefore, we can choose a complete type $p' \in S(A)$ extending $p(x) \cup \Gamma(x)$. By (1), if $D_1(p', \Delta, k, q) < D_1(p, \varphi, k, q)$, then there is a formula $\psi(x) \in p'$ such that $D_1(p \cup \{\psi(x)\}, \Delta, k, q) < D_1(p, \Delta, k, q)$ but this is impossible by the definition of Γ. Therefore p' is as desired. ⊣

Theorem 5.8. *Assume T is NSOP$_1$ with existence.*

(1) *Suppose π is a partial type over B and $\pi \subseteq \pi'$. If π' Kim-divides over B, witnessed by the formula $\varphi(x; c_0)$, and $I = \langle c_i : i < \omega \rangle$ is an $\mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}^K$-Morley sequence over B in q such that $\{\varphi(x; c_i) : i < \omega\}$ is k-inconsistent, then we have*

$$D_1(\pi', \varphi, k, q) < D_1(\pi, \varphi, k, q).$$

(2) *If T is simple, $a \mathop{\smile\hskip-0.9em\raise0.6ex\hbox{$\mid$}}_{BD} C$, then for any $q(y) \in S(B)$, any finite set of $L(B)$-formulas $\Delta(x, y)$, and any $k < \omega$, we have*

$$D_1(p', \Delta, k, q) = D_1(p, \Delta, k, q),$$

where $p' = \mathrm{tp}(a/BDC)$ and $p = \mathrm{tp}(a/BD)$.

PROOF. Suppose we are given $\pi \subseteq \pi'$, φ, k, I, and q as in the statement. We claim that $D_1(\pi \cup \{\varphi(x; c_0)\}, \varphi, k, q) < D_1(\pi, \varphi, k, q)$. If not, then, again by Lemma 5.2(1), we have

$$n := D_1(\pi \cup \{\varphi(x; c_0)\}, \varphi, k, q) = D_1(\pi, \varphi, k, q),$$

and therefore, by B-indiscernibility, $D_1(\pi \cup \{\varphi(x; c_i)\}, \varphi, k, q) = n$ for all i. This implies, by definition of the rank, that $D_1(\pi, \varphi, k, q) \geq n + 1$, a contradiction. Therefore, since $\varphi(x; c_0) \in \pi'$, we have

$$D_1(\pi', \varphi, k, q) \leq D_1(\pi \cup \{\varphi(x; c_0)\}, \varphi, k, q) < D_1(\pi, \varphi, k, q),$$

which proves (1).

Now we prove (2). As we are working in a simple theory, Kim-dividing and forking coincide by Kim's lemma [10]. Fix an arbitrary $q \in S(B)$, finite set of $L(B)$-formulas $\Delta(x, y)$ and $k < \omega$. By Lemma 5.2(2), we have

$$D_1(p', \Delta, k, q) \leq D_1(p, \Delta, k, q).$$

Hence, it suffices to show, by induction on $n < \omega$,

$$D_1(p, \Delta, k, q) \geq n \implies D_1(p', \Delta, k, q) \geq n.$$

For $n = 0$, this is clear, so assume it holds for n and suppose $D_1(p, \Delta, k, q) \geq n + 1$. Then there is a Morley sequence $I = \langle c_i : i < \omega \rangle$ over B in q which is BD-indiscernible such that, for some $\varphi(x, y) \in \Delta$, $\{\varphi(x; c_i) : i < \omega\}$ is k-inconsistent and $D_1(p \cup \{\varphi(x; c_i)\}, \Delta, k, q) \geq n$ for all $i < \omega$. Let $p_* = p_*(x; c_0)$ be a completion of $p \cup \{\varphi(x; c_0)\}$ over BDc_0 with

$$D_1(p_*, \Delta, k, q) = D_1(p \cup \{\varphi(x; c_0)\}, \Delta, k, q) \geq n,$$

which is possible by Lemma 5.7(2). Without loss of generality, $a \models p_*$.

By extension and an automorphism over aBD, we may assume $C \mathop{\smile\!\!\!\!\!\!|}_{BD} ac_0$, and hence there is $I' = \langle c'_i : i < \omega \rangle \equiv_{BDc_0} I$ such that I' is BCD-indiscernible. Note that I' is still Morley in q. Moreover, by base monotonicity and symmetry, this gives $a \mathop{\smile\!\!\!\!\!\!|}_{BDc'_0} C$. Let $p'_* = \text{tp}(a/BCDc'_0) \supseteq p'$. By the inductive hypothesis, the fact that $D_1(p_*, \Delta, k, q) \geq n$ gives

$$D_1(p'_*, \Delta, k, q) \geq n.$$

By Lemma 5.2(1) and (2), we obtain

$$D_1(p' \cup \{\varphi(x; c'_i)\}, \Delta, k, q) \geq n$$

for all $i < \omega$, which allows us to conclude $D_1(p', \Delta, k, q) \geq n + 1$, completing the proof. ⊣

REMARK 5.9. By witnessing (Theorem 3.8), we know that if π is a partial type over B and π' Kim-divides over B, then this will be witnessed by some formula $\varphi(x; c_0)$ implied by π' and an $\mathop{\smile\!\!\!\!\!\!|}^K$-Morley sequence over B. Therefore, Theorem 5.8(1) implies that, if for all $\varphi(x; y) \in L(B)$, $q(y) \in S(B)$, and $k < \omega$,

$$D_1(\pi', \varphi, k, q) = D_1(\pi, \varphi, k, q),$$

then π' does not Kim-divide over B.

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QUESTION 5.10. *Does Theorem 5.8(2) hold for $\mathop{\smile\!\!\!\!\!\!\;\mid}^K$ in all NSOP$_1$ theories satisfying existence? Evidently, the proof above makes use of base monotonicity, which is known to fail in all non-simple NSOP$_1$ theories.*

§6. The Kim–Pillay theorem over arbitrary sets. The Kim–Pillay-style criterion for NSOP$_1$ of [4] proceeds, essentially, by first showing that any relation that satisfies axioms (1)–(5) over models in the axioms below must be weaker than coheir independence, in the sense that if $M \models T$ and $\text{tp}(a/Mb)$ is finitely satisfiable in M then $a \mathop{\smile\!\!\!\!\!\!\;\mid}_M b$. Consequently, this proof does not adapt to arbitrary sets. Instead, we relate any relation satisfying the axioms to Kim-independence directly, using a tree-induction, to prove the following theorem.

THEOREM 6.1. *Assume T satisfies existence. The theory T is NSOP$_1$ if and only if there is an Aut$(\mathbb{M})$-invariant ternary relation $\mathop{\smile\!\!\!\!\!\!\;\mid}$ on small subsets of the monster $\mathbb{M} \models T$ which satisfies the following properties, for an arbitrary set of parameters A and arbitrary tuples from $\mathbb{M}$.*

(1) *Strong finite character: if $a \not\mathop{\smile\!\!\!\!\!\!\;\mid}_A b$, then there is a formula $\varphi(x,b,m) \in tp(a/bA)$ such that for any $a' \models \varphi(x,b,m)$, $a' \not\mathop{\smile\!\!\!\!\!\!\;\mid}_A b$.*
(2) *Existence and Extension: $a \mathop{\smile\!\!\!\!\!\!\;\mid}_A A$ always holds and, if $a \mathop{\smile\!\!\!\!\!\!\;\mid}_A b$, then, for any c, there is $a' \equiv_{Ab} a$ such that $a' \mathop{\smile\!\!\!\!\!\!\;\mid}_A bc$.*
(3) *Monotonicity: $aa' \mathop{\smile\!\!\!\!\!\!\;\mid}_A bb' \implies a \mathop{\smile\!\!\!\!\!\!\;\mid}_A b$.*
(4) *Symmetry: $a \mathop{\smile\!\!\!\!\!\!\;\mid}_A b \iff b \mathop{\smile\!\!\!\!\!\!\;\mid}_A a$.*
(5) *The independence theorem: $a \mathop{\smile\!\!\!\!\!\!\;\mid}_A b$, $a' \mathop{\smile\!\!\!\!\!\!\;\mid}_A c$, $b \mathop{\smile\!\!\!\!\!\!\;\mid}_A c$ and $a \equiv^L_A a'$ implies there is a'' with $a'' \equiv_{Ab} a$, $a'' \equiv_{Ac} a'$ and $a'' \mathop{\smile\!\!\!\!\!\!\;\mid}_A bc$.*

Moreover, any such relation $\mathop{\smile\!\!\!\!\!\!\;\mid}$ satisfying (1)–(5) strengthens $\mathop{\smile\!\!\!\!\!\!\;\mid}^K$, i.e., $a \mathop{\smile\!\!\!\!\!\!\;\mid}_A b$ implies $a \mathop{\smile\!\!\!\!\!\!\;\mid}^K_A b$. If, additionally, $\mathop{\smile\!\!\!\!\!\!\;\mid}$ satisfies the following:

(6) *Transitivity: if $a \mathop{\smile\!\!\!\!\!\!\;\mid}_A b$ and $a \mathop{\smile\!\!\!\!\!\!\;\mid}_{Ab} c$ then $a \mathop{\smile\!\!\!\!\!\!\;\mid}_A bc$.*
(7) *Local character: if $\kappa \geq |T|^+$ is a regular cardinal, $\langle A_i : i < \kappa \rangle$ is an increasing continuous sequence of sets of size $< \kappa$, $A_\kappa = \bigcup_{i<\kappa} A_i$ and $|A_\kappa| = \kappa$, then for any finite d, there is some $\alpha < \kappa$ such that $d \mathop{\smile\!\!\!\!\!\!\;\mid}_{A_\alpha} A_\kappa$,*

then $\mathop{\smile\!\!\!\!\!\!\;\mid} = \mathop{\smile\!\!\!\!\!\!\;\mid}^K$.

PROOF. We know that if T is NSOP$_1$ with existence then $\mathop{\smile\!\!\!\!\!\!\;\mid}^K$ satisfies (1)–(5) by Fact 2.7, so we will show the other direction.

First, assume $\mathop{\smile\!\!\!\!\!\!\;\mid}$ satisfies axioms (1)–(5), and we will show that $\mathop{\smile\!\!\!\!\!\!\;\mid}$ strengthens $\mathop{\smile\!\!\!\!\!\!\;\mid}^K$. By [4, Theorem 6.1], the existence of such a relation over models entails that the theory is NSOP$_1$. Towards contradiction, suppose there is some set of parameters A and tuples a, b such that $a \mathop{\smile\!\!\!\!\!\!\;\mid}_A b$ but $a \not\mathop{\smile\!\!\!\!\!\!\;\mid}^K_A b$, witnessed by the formula $\varphi(x;b) \in \text{tp}(a/Ab)$. By induction on the ordinals α, we will construct trees $(b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ satisfying the following conditions for all α:

(a) For all $\eta \in \mathcal{T}_\alpha$, $b^\alpha_\eta \models \text{tp}(b/A)$.
(b) $(b^\alpha_\eta)_{\eta \in \mathcal{T}_\alpha}$ is s-indiscernible and weakly spread out over A.

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(c) For α successor, $b^\alpha_\emptyset \mathop{\smile\!\!\!\!\!\!|}_A b^\alpha_{\rhd\emptyset}$.
(d) If $\beta < \alpha$, then $b^\alpha_{\iota_{\beta\alpha}(\eta)} = b^\beta_\eta$ for all $\eta \in \mathcal{T}_\beta$.

To begin, we may take $b^0_\emptyset = b$ and at limits we take unions. Given $(b^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$, let $I = \langle (b^\alpha_{i,\eta})_{\eta\in\mathcal{T}_\alpha} : i < \omega\rangle$ be a mutually s-indiscernible Morley sequence over A with $(b^\alpha_{0,\eta})_{\eta\in\mathcal{T}_\alpha} = (b^\alpha_\eta)_{\eta\in\mathcal{T}_\alpha}$, which exists by Lemma 2.15. We may apply extension to find $b_* \equiv_A b$ such that $b_* \mathop{\smile\!\!\!\!\!\!|}_A I$. Define a tree $(c_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$ by setting $c_\emptyset = b_*$ and $c_{\langle i\rangle\frown\eta} = b^\alpha_{i,\eta}$ for all $i < \omega$ and $\eta \in \mathcal{T}_\alpha$. Finally, we may define $(b^{\alpha+1}_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$ to be an s-indiscernible tree over A locally based on $(c_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$. After moving by an automorphism, we can assume $b^{\alpha+1}_{\iota_{\alpha\alpha+1}(\eta)} = b^\alpha_\eta$ for all $\eta \in \mathcal{T}_\alpha$. This completes the construction. By strong finite character and invariance, our construction ensures (b) and (c) will be satisfied for $(b^{\alpha+1}_\eta)_{\eta\in\mathcal{T}_{\alpha+1}}$.

Applying Erdős–Rado, we obtain a weak Morley tree $(b_\eta)_{\eta\in\mathcal{T}_\omega}$ over A such that $b_{\zeta_\alpha} \mathop{\smile\!\!\!\!\!\!|}_A b_{\rhd\zeta_\alpha}$ for all $\alpha < \omega$. In particular, $(b_{\zeta_\alpha})_{\alpha<\omega}$ is an $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence over A. Define $v_\alpha = \zeta_{\alpha+1} \frown \langle 1\rangle$ for all $\alpha < \omega$. Because the tree is weakly spread out over A, we have for all $\alpha < \omega$, $b_{\unrhd\zeta_{\alpha+1}\frown\langle 1\rangle} \mathop{\smile\!\!\!\!\!\!|}^K_A b_{\unrhd\zeta_{\alpha+1}\frown 0}$ and hence $b_{v_\alpha} \mathop{\smile\!\!\!\!\!\!|}^K_A b_{v_{<\alpha}}$ since $(b_{v_\beta})_{\beta<\alpha}$ was enumerated in $b_{\unrhd\zeta_{\alpha+1}\frown 0}$. Since the tree is a weak Morley tree, we have that both $(b_{\zeta_\alpha})_{\alpha<\omega}$ and $(b_{v_\alpha})_{\alpha<\omega}$ are A-indiscernible.

As $b \equiv_A b_{\zeta_0}$, there is a_0 such that $a_0 b_{\zeta_0} \equiv_A ab$. As $(b_{\zeta_\alpha})_{\alpha<\omega}$ is A-indiscernible, for each $\alpha > 0$, there is $\sigma_\alpha \in \text{Autf}(\mathbb{M}/A)$ such that $\sigma_\alpha(b_{\zeta_0}) = b_{\zeta_\alpha}$. Setting $a_\alpha = \sigma_\alpha(a_0)$, we have $a_\alpha b_{\zeta_\alpha} \equiv^L_A a_0 b_{\zeta_0}$. By the independence theorem for $\mathop{\smile\!\!\!\!\!\!|}$, we may find some a_* such that $a_* b_{\zeta_\alpha} \equiv_A ab$ for all $\alpha < \omega$. In particular, this implies $\{\varphi(x; b_{\zeta_\alpha}) : \alpha < \omega\}$ is consistent. However, since $\varphi(x;b)$ Kim-divides over A, $\{\varphi(x; b_{v_\alpha}) : \alpha < \omega\}$ is k-inconsistent for some k. The s-indiscernibility of the tree implies that $b_{\zeta_\alpha} \equiv_{Ab_{\zeta_{>\alpha}} b_{v_{>\alpha}}} b_{v_\alpha}$ for all α, so by compactness, we have shown φ has SOP$_1$. This contradiction shows that $\mathop{\smile\!\!\!\!\!\!|}$ strengthens $\mathop{\smile\!\!\!\!\!\!|}^K$.

Second, assume additionally that $\mathop{\smile\!\!\!\!\!\!|}$ satisfies (6) and (7). The proof of Theorem 3.8 shows that (6) and (7) imply *witnessing*: if $I = (b_i)_{i<\omega}$ is an A-indiscernible sequence with $b_0 = b$ satisfying $b_i \mathop{\smile\!\!\!\!\!\!|}_A b_{<i}$, then whenever $a \not\mathop{\smile\!\!\!\!\!\!|}_A b$, there is $\varphi(x;c,b) \in \text{tp}(a/Ab)$ such that $\{\varphi(x;c,b_i) : i < \omega\}$ is inconsistent.

Suppose that $a \mathop{\smile\!\!\!\!\!\!|}^K_A b$ and, by extension and Erdős–Rado, find an $\mathop{\smile\!\!\!\!\!\!|}$-Morley sequence $I = \langle b_i : i < \omega\rangle$ over A with $b_0 = b$. By the remarks above, we know that I is, in particular, an $\mathop{\smile\!\!\!\!\!\!|}^K$-Morley sequence over A, so there is $a' \equiv_{Ab} a$ such that I is Aa-indiscernible (using $a \mathop{\smile\!\!\!\!\!\!|}^K_A b$). By witnessing, this entails $a \mathop{\smile\!\!\!\!\!\!|}_A b$. In other words, $\mathop{\smile\!\!\!\!\!\!|}$ and $\mathop{\smile\!\!\!\!\!\!|}^K$ coincide. ⊣

REMARK 6.2. It is clear from the proof of Theorem 6.1 that, in order to get the same conclusion, we can replace (6) and (7) with witnessing: if $I = (b_i)_{i<\omega}$ is an A-indiscernible sequence with $b_0 = b$ satisfying $b_i \mathop{\smile\!\!\!\!\!\!|}_A b_{<i}$, then whenever $a \not\mathop{\smile\!\!\!\!\!\!|}_A b$, there is $\varphi(x;c,b) \in \text{tp}(a/Ab)$ such that $\{\varphi(x;c,b_i) : i < \omega\}$ is inconsistent.

To do so gives a result that more closely resembles [8, Theorem 6.11], while the formulation of Theorem 6.1 is closer to the original Kim–Pillay theorem for simple theories [15, Theorem 4.2] (see also [12, Theorem 3.3.1]).

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