7

Classical and Fractional Inequalities of Rellich Type

7.1 The Classical Inequalities

Rellich's classical inequality in [153] asserts that for all $u \in C_0^\infty(\mathbb{R}^n \setminus \{0\})$ and $n \in \mathbb{N} \setminus \{2\}$,

$$\int_{\mathbb{R}^n} |\Delta u(x)|^2 \, dx \geq \frac{n^2(n-4)^2}{16} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^4} \, dx, \tag{7.1.1}$$

with sharp constant $n^2(n-4)^2/16$. The inequality also holds for $n = 2$ (with constant 1), but only for those functions $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$ which, in terms of polar co-ordinates (r, θ), satisfy

$$\int_0^\infty u(r,\theta) \cos\theta \, d\theta = \int_0^\infty u(r,\theta) \sin\theta \, d\theta = 0. \tag{7.1.2}$$

In [73], Rellich-type inequalities involving magnetic Laplacians with magnetic potentials of Aharonov–Bohm type are studied, which are valid for all $n \in \mathbb{N}$ in some circumstances. What is of particular significance for (7.1.1) is that they clarify the situation for the case $n = 2$ and also the trivial case $n = 4$. The study was motivated by the Laptev–Weidl inequality (5.1.9) in which a magnetic Hardy inequality is shown to be valid in the case $n = 2$ (when there is no non-trivial Hardy inequality) if the magnetic potential is of Aharonov–Bohm type

$$\mathbf{A}(x) = \Psi \left(-\frac{x_2}{|x|^2}, \frac{x_1}{|x|^2} \right), x = (x_1, x_2) \tag{7.1.3}$$

with non-integer flux Ψ; the magnetic field curl $\mathbf{A} = 0$ in $\mathbb{R}^2 \setminus \{0\}$. The following two theorems are proved in [73]: in them

$$\Delta_{\mathbf{A}} = (\nabla - \mathbf{A})^2 \text{ is the magnetic Laplacian.}$$

Theorem 7.1 *For all* $u \in C_0^\infty(\mathbb{R}^2 \setminus \{0\})$,

$$\int_{\mathbb{R}^2} |\Delta_{\mathbb{A}} u(x)|^2 \frac{dx}{|x|^s} \, dx \geq C(2,s) \int_{\mathbb{R}^2} |u(x)|^2 \frac{dx}{|x|^{s+4}} \, dx, \tag{7.1.4}$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

where

$$C(2,s) = \min_{m\in\mathbb{Z}} \left\{ (m+\Psi)^2 - \frac{(s+2)^2}{4} \right\}^2. \tag{7.1.5}$$

If $\Psi \notin \mathbb{Z}$ *(*$\Psi \in (0,1)$ *without loss of generality), we have*

$$\begin{aligned} C(2,0) &= \min\{(\Psi^2-1)^2, \Psi^2(\Psi-2)^2\} \\ &= \begin{cases} (\Psi^2-1)^2 & \text{if} \quad \Psi \in [\frac{1}{2},1), \\ \Psi^2(\Psi-2)^2 & \text{if} \quad \Psi \in [0,\frac{1}{2}). \end{cases} \end{aligned} \tag{7.1.6}$$

Remark 7.2

If $\Psi \in \mathbb{Z}$, then $C(2,0) = 0$. However, if (7.1.2) is satisfied, then the minimum in (7.1.5) is over $m \in \mathbb{Z}\setminus\{-1,1\}$ and this recovers the result $C(2,0) = 1$.

Theorem 7.3 *Let* $u \in C_0^\infty(\mathbb{R}^4 \setminus \mathcal{L}_4)$, *where* $\mathcal{L}_4 := \{x = (r,\theta_1,\theta_2,\theta_3;\ \theta_1,\theta_2 \in (0,\pi),\ \theta_3 \in (0,2\pi)\colon r\sin\theta_1\sin\theta_2 = 0\}$. *Then* curl $A = 0$ *on* $\mathbb{R}^4\setminus\mathcal{L}_4$ *and*

$$\int_{\mathbb{R}^4} |\Delta_{\mathbf{A}} u(x)|^2 \frac{dx}{|x|^s} \ge C(4,s) \int_{\mathbb{R}^4} |u(x)|^2 \frac{dx}{|x|^{s+4}}, \tag{7.1.7}$$

where

$$C(4,s) := \inf_{m\in\mathbb{Z}'} \left\{ \left[(m-\Psi)^2 - 1 - \frac{s(s+4)}{4} \right]^2 \right\}, \tag{7.1.8}$$

and $\mathbb{Z}' := \{m \in \mathbb{Z}\colon (m-\Psi)^2 \ge 1\}$. *In particular, when* $s = 0$ *and* $\Psi \in (0,1)$*,*

$$C(4,0) = \min\{[(1-\Psi)^2 - 1]^2, [(-2-\Psi)^2 - 1]^2\} > 0.$$

When $\Psi = 0$, (7.1.7) is satisfied on $C_0^\infty(\mathbb{R}^4\setminus\{0\})$. The inequality is trivial if $C(4,0) = 0$, but there is a restricted class of functions which is such that the infimum is attained for $m = \pm 2$, and so $C(4,0) = 9$; this is an analogue for $n = 4$ of the result for $n = 2$ in Remark 7.2. We refer to [15], Corollary 6.4.10, for further details.

Similar results to Theorems 7.1 and 7.3 are given in the case $n = 3$ in [15], and for $n > 4$ in [166].

Next, we consider Rellich inequalities on a domain $\Omega \subset \mathbb{R}^n,\ n \ge 2$. Let Ω be a proper, non-empty open subset of $\mathbb{R}^n (n \ge 2)$ and $\delta(x) := \operatorname{dist}(x, \partial\Omega)$, the distance from $x \in \Omega$ to the boundary of Ω. The following Rellich inequality in $L_2(\Omega)$ is established in [15], Corollary 6.2.7,

$$\int_\Omega |\Delta u(x)|^2\, dx \ge \frac{9}{16} \int_\Omega \frac{|u(x)|^2}{\delta(x)^4}\, dx, \quad u \in C_0^\infty(\Omega), \tag{7.1.9}$$

under the assumption that δ is superharmonic, i.e., $\Delta\,\delta \le 0$ in the distributional sense; this requirement is met if Ω is convex or if Ω is weakly mean convex with

$\Sigma(\Omega) = \Omega \setminus G(\Omega)$ a null set; see Section 5.3. The proof in [15] is based on the abstract Hardy-type inequality

$$\int_\Omega |\Delta V(x)||u(x)|^2 dx \leq 4 \int_\Omega \frac{|\nabla V(x)|^2}{|\Delta V(x)|} |\nabla u(x)|^2 dx, \quad u \in C_0^\infty(\Omega),$$

which was proved for ΔV of one sign in [123], Lemma 2 (see also [49]). For $s \neq 0$, choose $V(x) = -[(s+1)/s]\delta(x)^{-s}$ and for $s = 0$ let $V(x) = \ln \delta(x)$. Then $|\nabla V(x)|^2 = (s+1)^2 \delta(x)^{-2(s+1)}$, and when $\Delta\delta(x) \leq 0$,

$$-\Delta V(x) = (s+1)^2 \delta(x)^{-(s+2)} + (s+1)\delta(x)^{-(s+1)}(-\Delta\delta(x)) \geq (s+1)^2 \delta(x)^{-(s+2)}.$$

It follows that for $n \geq 2$,

$$(s+1)^2 \int_\Omega \frac{|u(x)|^2}{\delta(x)^{s+2}} dx \leq 4 \int_\Omega \frac{|\nabla u(x)|^2}{\delta(x)^s} ds, \quad u \in C_0^\infty(\Omega). \tag{7.1.10}$$

We show that (7.1.9) is a consequence of (7.1.10). With the notation $u_j = \partial_j u,\ u_{jk} = \partial_j \partial_k u$ and $u_{jkl} = \partial_j \partial_k \partial_l u$, we have

$$\begin{aligned}
\int_\Omega |\Delta u(x)|^2 dx &= \Sigma_{j,k=1}^n \int_\Omega u_{jj} \overline{u_{kk}}\, dx \\
&= \Sigma_{j=k} \int_\Omega |ujj|^2 dx - \Sigma_{j\neq k} \int_\Omega u_j \overline{u_{jkk}}\, dx \\
&= \Sigma_{j=k} \int_\Omega |ujj|^2 dx + \Sigma_{j\neq k} \int_\Omega u_{jk} \overline{u_{jk}}\, dx \\
&= \Sigma_{j=1}^n \int_\Omega |\nabla(u_j)|^2\, dx.
\end{aligned} \tag{7.1.11}$$

Hence from (7.1.10),

$$\begin{aligned}
\int_\Omega |\Delta u(x)|^2\, dx &= \Sigma_{j=1}^n \frac{1}{4} \int_\Omega \frac{|\nabla u(x)|^2}{\delta(x)^2}\, dx \\
&\geq \frac{9}{16} \int_\Omega \frac{|u(x)|^2}{\delta(x)^4}\, dx,
\end{aligned}$$

thus (7.1.9),

In the L_p setting, results have been obtained by Davies and Hinz in [49]. Their main tool is an abstract Rellich-type inequality reminiscent of the abstract Hardy-type inequality in [123], but now depending on the existence of a positive function V which is such that $\Delta V < 0$ and $\Delta(V^a) \leq 0$ for some $a > 1$: this is that if $p \in (1, \infty)$ and $u \in C_0^\infty(\Omega)$,

$$\int_\Omega |\Delta V(x)||u(x)|^p\, dx \leq \left(\frac{p^2}{(p-1)a+1} \right)^p \int_\Omega \frac{V^p(x)}{|\Delta V(x)|^{p-1}} |\Delta u(x)|^p\, dx.$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

When $\Omega = \mathbb{R}^n \setminus \{0\}$ and $n > s > 2$, the choice $V(x) = |x|^{-(s-2)}$, $a = \frac{n-2}{s-2}$ gives

$$\int_{\mathbb{R}^n} \frac{|\Delta u(x)|^p}{|x|^{s-2p}}\, dx \geq \left(p^{-2}(n-s)[(p-1)n+s-2p]\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^s}\, dx,$$

whence the Rellich inequality

$$\int_{\mathbb{R}^n} |\Delta u(x)|^p dx \geq \left(\frac{n(p-1)(n-2p)}{p^2}\right)^p \int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{2p}}\, dx; \tag{7.1.12}$$

the constant is shown to be sharp in [15], Corollary 6.3.5.

In [63], the mean distance function M_p defined in (6.2) by

$$\frac{1}{M_p(x)^p} := \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+p}{2}\right)}{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{S}^{n-1}} \frac{1}{\delta_\nu^p(x)}\, d\omega(\nu) \tag{7.1.13}$$

is used to obtain Rellich inequalities of the form

$$\int_\Omega |\Delta u(x)|^p\, dx \geq C \int_\Omega \frac{|u(x)|^p}{M_{2p}(x)^{2p}}\, dx, \;\; u \in C_0^\infty(\Omega),$$

for general domains Ω. The following is proved in [63]:

Theorem 7.4 *Let Ω be a non-empty, proper, open subset of $\mathbb{R}^n$, let $p \in (1, \infty)$ and suppose that $u \in C_0^2(\Omega)$. Then*

$$\int_\Omega \frac{|u(x)|^p}{M_{2p}(x)^{2p}}\, dx \leq K(p,n) \int_\Omega |\Delta u(x)|^p\, dx, \tag{7.1.14}$$

where

$$K(p,n) = c_p B(n,2p) n^d \cot^{2p}\left(\frac{\pi}{2p^*}\right). \tag{7.1.15}$$

Here

$$d = 2 \text{ if } 1 < p < 2, \;\; d = 2p/p' \text{ if } 2 < p < \infty,$$

$$p^* = \max\left\{p, p'\right\}, \; c_p = \left(\frac{p}{2p-1}\right)^p \left(\frac{p}{p-1}\right)^p,$$

and

$$B(n,2p) = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+2p}{2}\right)}{\Gamma\left(\frac{2p+1}{2}\right)\Gamma\left(\frac{n}{2}\right)}.$$

If $p = 2$, then

$$\int_\Omega \frac{|u(x)|^2}{M_4(x)^4}\, dx \leq \frac{16}{9} \int_\Omega |\Delta u(x)|^2\, dx. \tag{7.1.16}$$

For Ω convex, $M_{2p}(x) \leq \delta(x) := \inf\{|y-x| \colon y \in \mathbb{R}^n \setminus \Omega\}$.

We refer to [63] for a proof but some comments might be helpful. The cotangent factor in (7.1.15) appears in the inequality

$$\|D_j D_k u\|_p \leq \cot^2\left(\frac{\pi}{2p*}\right)\|\Delta u\|_p,\ u \in C_0^\infty(\mathbb{R}^n)$$

for $j, k = 1, 2, ..., n$, which follows from the identity

$$D_j D_k u = -R_j R_k \Delta u$$

involving the Riesz transform R_j in $L_p(\mathbb{R}^n)$ and the remarkable result

$$\|R_j \colon L_p(\mathbb{R}^n) \to L_p(\mathbb{R}^n)\| = \cot^2\left(\frac{\pi}{2p*}\right)$$

proved in [105] and [16]. A brief background discussion of Riesz transforms may be found in [65], Section 1.4.

7.2 Fractional Rellich Inequalities in $\mathbb{R}^n$

Frank and Seiringer establish the following Hardy inequality in [82], Theorem 1.1. Let $0 < s < 1$ and suppose that $u \in C_0^\infty(\mathbb{R}^n)$ if $1 \leq p < n/s$, while $u \in C_0^\infty(\mathbb{R}^n \backslash \{0\})$ if $n/s < p < \infty$. Then

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^p}{|x-y|^{n+ps}}\,dx\,dy \geq \mathcal{C}_{n,s,p}\int_{\mathbb{R}^n}\frac{|u(x)|^p}{|x|^{ps}}\,dx, \tag{7.2.1}$$

where

$$\mathcal{C}_{n,s,p} := 2\int_0^1 r^{ps-1}\left|1 - r^{(n-ps)/p}\right|^p \Phi_{n,s,p}(r)\,dr, \tag{7.2.2}$$

$$\Phi_{n,s,p}(r) = \left|\mathbb{S}^{n-2}\right|\int_{-1}^1 \frac{\left(1-t^2\right)^{(n-3)/2}}{\left(1-2rt+r^2\right)^{(n+ps)/2}}\,dt \text{ if } n \geq 2, \tag{7.2.3}$$

and

$$\Phi_{1,s,p}(r) = (1-r)^{-1-ps} + (1+r)^{-1-ps} \text{ if } n = 1. \tag{7.2.4}$$

In the case $p = 2$, (7.2.1) follows from Proposition 3.28, and the identity

$$\left\|(-\Delta)^{s/2}u\right\|_{2,\mathbb{R}^n}^2 = C(n,s)\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\,dx\,dy, \tag{7.2.5}$$

in which

$$C(s,n) = 2^{2s}\pi^{-n/2}\Gamma\left(\frac{n}{2}+s\right) / |\Gamma(-s)| \tag{7.2.6}$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

and thus

$$\lim_{s\to 1-} C(s,n)/(1-s) = 2n\pi^{-n/2}\Gamma(n/2) = 4n/\omega_{n-1}.$$

For $s \in (0,1),\ p=2,\ n>2s$ and $u \in C_0^\infty(\mathbb{R}^n)$ (7.2.1) is then a consequence of (7.2.5) and the Herbst inequality [96]

$$\int_{\mathbb{R}^n} \left|(-\Delta)^{s/2} u(x)\right|^2 dx \geq C_{s,n} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^{2s}}\, dx, \tag{7.2.7}$$

which has the sharp constant

$$C_{s,n} = 2^{2s} \frac{\Gamma^2\left(\frac{n+2s}{4}\right)}{\Gamma^2\left(\frac{n-2s}{4}\right)}. \tag{7.2.8}$$

Therefore, for $s \in (0,1)$, $n > 2s$ and $u \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|u(x)-u(y)|^2}{|x-y|^{n+2s}}\, dx\, dy \geq \mathcal{C}_{n,s,2} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^{2s}}\, dx, \tag{7.2.9}$$

with sharp constant

$$\mathcal{C}_{n,s,2} = 2C_{s,n}/C(s,n) = 2\pi^{n/2} \frac{\Gamma^2\left(\frac{n+2s}{4}\right)|\Gamma(-s)|}{\Gamma^2\left(\frac{n-2s}{4}\right)\Gamma\left(\frac{n+2s}{4}\right)}. \tag{7.2.10}$$

Another consequence of (7.2.5) is

Corollary 7.5 *Let $\sigma \in (0,1)$ and $n > 2\sigma$. Then for all $C_0^\infty(\mathbb{R}^n)$,*

$$\sum_{i=1}^{n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D_i u(x) - D_i u(y)|^2}{|x-y|^{n+2\sigma}}\, dx\, dy \geq 2C(\sigma,n)^{-1} C_{\sigma+1,n} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^{2+2\sigma}}\, dx, \tag{7.2.11}$$

where the constant is sharp.

Proof From (7.2.5) with $s = 1+\sigma$ and $\sigma \in (0,1)$, we have

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \frac{|D_i u(x) - D_i u(y)|^2}{|x-y|^{n+2\sigma}}\, dx\, dy \\
&= 2C(\sigma,n)^{-1} \sum_{i=1}^{n} \int_{\mathbb{R}^n} |\xi|^{2\sigma} \left|\left(F\left(D_i u\right)\right)(\xi)\right|^2 d\xi \\
&= 2C(\sigma,n)^{-1} \sum_{i=1}^{n} \int_{\mathbb{R}^n} \left|F^{-1}\left(|\xi|^{\sigma+1} F(u)(\xi)\right)\right|^2 d\xi \\
&= 2C(\sigma,n)^{-1} \left\|(-\Delta)^{(\sigma+1)/2} u\right\|_{2,\mathbb{R}^n}^2 \\
&\geq 2C(\sigma,n)^{-1} C_{\sigma+1,n} \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^{2+2\sigma}}\, dx.
\end{aligned} \tag{7.2.12}$$

□

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

Note that we also have from (7.2.1)

$$\begin{aligned}
&\sum_{i=1}^{n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|D_i u(x) - D_i u(y)|^2}{|x-y|^{n+2\sigma}} \, dx \, dy \\
&\quad \geq \mathcal{C}_{n,\sigma,2} \int_{\mathbb{R}^n} \sum_{i=1}^{n} \frac{|D_i u(x)|^2}{|x|^{2\sigma}} \, dx \\
&\quad = \mathcal{C}_{n,\sigma,2} \int_{\mathbb{R}^n} \frac{|\nabla u(x)|^2}{|x|^{2\sigma}} \, dx \\
&\quad \geq \mathcal{C}_{n,\sigma,2} \left| \frac{2+2\sigma-n}{2} \right|^2 \int_{\mathbb{R}^n} \frac{|u(x)|^2}{|x|^{2+2\sigma}} \, dx,
\end{aligned}$$

the final step being the weighted Hardy inequality (7.1.10). Since the constant in (7.2.12) is sharp, it follows that

$$\mathcal{C}_{n,\sigma,2} \left| \frac{2+2\sigma-n}{2} \right|^2 \leq 2C(\sigma,n)^{-1} C_{\sigma+1,n};$$

hence by (7.2.10),

$$C_{\sigma,n} \left| \frac{2+2\sigma-n}{2} \right|^2 \leq C_{\sigma+1,n}. \tag{7.2.13}$$

For $n > 2$ the inequality (7.2.13) is strict since, on allowing $\sigma \to 1-$, the left-hand side tends to $((n-2)(n-4))^2/16$ while the right-hand side tends to $(n(n-4))^2/16$, the optimal constant in the Rellich inequality (7.1.1). If $\sigma \to 0+$, (7.2.11) becomes the Hardy inequality and the constants on both sides of (7.2.13) tend to the optimal Hardy constant $(n-2)^2/4$.

Remark 7.6

The inequality (7.2.5) is the special case $p = 2$ of Herbst's inequality in [96] which is, for $1 < p < \infty$, $s > 0$, $n > ps$ and $u \in C_0^\infty(\mathbb{R}^n)$, that

$$\int_{\mathbb{R}^n} \frac{|u(x)|^p}{|x|^{ps}} \, dx \leq K_{n,p,s}^p \left\| (-\Delta)^{s/2} u \right\|_{p,\mathbb{R}^n}^p, \tag{7.2.14}$$

with best possible constant

$$K_{n,p,s} = 2^{-s} \frac{\Gamma\left(\frac{n(p-1)}{2p}\right) \Gamma\left(\frac{n-ps}{2p}\right)}{\Gamma\left(\frac{n}{2p}\right) \Gamma\left(\frac{n(p-1)+ps}{2p}\right)}. \tag{7.2.15}$$

This is also established in [156]; moreover, Samko determines a sharp constant for the Hardy–Stein–Weiss inequality for fractional Riesz operators in $L_p(\mathbb{R}^n, \rho)$ with a power weight $\rho(x) = |x|^\beta$ and as a corollary finds the sharp

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

constant for a similar weighted inequality for fractional powers of the Laplace–Beltrami operator on the unit sphere. A proof of (7.2.14) in the case $p = 2$ was given in [172]; moreover, Yafaev shows that if $n < 2(1 + \sigma)$, $1 + \sigma - n/2 \notin \mathbb{Z}$ and $k := \left[1 + \sigma - n/2\right]$, then

$$\int_{\mathbb{R}^n} |x|^{-2-2\sigma} \left| u(x) - \sum_{|\alpha| \le k} (\alpha!)^{-1} \left(D^\alpha u\right)(0) x^\alpha \right|^2 dx$$
$$\le K_{n,\sigma}^2 \left\| (-\Delta)^{(1+\sigma)/2} u \right\|_{2,\mathbb{R}^n}^2 ,$$

where

$$K_{n,\sigma} = 2^{-1-\sigma} \max \left\{ \frac{\Gamma\left(\frac{n-2-2\sigma}{4}\right)}{\Gamma\left(\frac{n+2+2\sigma}{4}\right)}, \frac{\Gamma\left(\frac{n-2\sigma}{4}\right)}{\Gamma\left(\frac{n+4+2\sigma}{4}\right)} \right\}.$$

Thus in particular, (7.2.14), with $p = 2$, $s = 1 + \sigma$ and constant $K_{n,\sigma}^2$, holds for all $u \in C_0^\infty\left(\mathbb{R}^n \backslash \{0\}\right)$ if $n < 2(1 + \sigma)$ and $1 + \sigma - n/2 \notin \mathbb{Z}$.

7.3 Fractional Rellich Inequalities in General Domains

The mean distance defined in (6.2.1), namely

$$\frac{1}{M_{s,p}(x)^{ps}} := \frac{\pi^{1/2}\Gamma\left(\frac{n+ps}{2}\right)}{\Gamma\left(\frac{1+ps}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{\mathbb{S}^{n-1}} \frac{1}{\delta_\nu^{ps}(x)}\, d\omega(\nu), \tag{7.3.1}$$

will again feature prominently in this and following sections, with the range of the parameter s specific to the problem being considered. It remains true that if Ω is convex with non-empty boundary, then $M_{s,p}(x) \le \delta(x) := \inf\{|y - x| : y \notin \Omega\}$.

The main theorem on fractional Rellich inequalities comes from [66] and sets the scene for much of what follows in this chapter. The method of proof is that in [130] which was what was used to prove Theorem 6.8 and (6.2.20). We shall use the following notation, some of which is reminiscent of that in Section 6.2. Suppose that $1 < p < \infty$, $1/p < \sigma < 1$ and define

$$e_p(n) := \begin{cases} 1 & \text{if } p \le 2, \\ n^{-(p-2)/2} & \text{if } p > 2, \end{cases} \tag{7.3.2}$$

and

$$\mathcal{D}_{n,p,\alpha} := \frac{\pi^{(n-1)/2}\Gamma\left(\frac{1+p+\alpha}{2}\right)}{\Gamma\left(\frac{n+p+\alpha}{2}\right)} \mathcal{D}_{1,p,\alpha}, \tag{7.3.3}$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

where

$$\mathcal{D}_{1,p,\alpha} = 2\int_0^1 \frac{\left|1 - r^{(\alpha-1)/p}\right|^p}{(1-r)^{1+\alpha}}\, dr. \tag{7.3.4}$$

In the special case in which $p = 2$, we have from the appendix of [22] that

$$\mathcal{D}_{1,2,\alpha} = \frac{2}{\alpha}\left\{\frac{2^{-\alpha}}{\sqrt{\pi}}\Gamma\left(\frac{1+\alpha}{2}\right)\Gamma\left(\frac{2-\alpha}{2}\right) - 1\right\}. \tag{7.3.5}$$

Theorem 7.7 *Let Ω be an open subset of $\mathbb{R}^n$ with non-empty boundary, let $p \in (1,\infty)$ and suppose that $\sigma \in (1/p, 1)$. Then for all $f \in C_0^\infty(\Omega)$,*

$$\sum_{i=1}^n \int_\Omega\int_\Omega \frac{|D_i f(x) - D_i f(y)|^p}{|x-y|^{n+p\sigma}}\, dx\, dy \\ \geq e_p(n)\mathcal{D}_{n,p,p\sigma}\,(\sigma + 1 - 1/p)^p \int_\Omega \frac{|f(x)|^p}{M_{1+\sigma,p}(x)^{p+p\sigma}}\, dx. \tag{7.3.6}$$

The following improvement is possible in the case $p = 2$ with $\sigma \in (1/2, 1)$:

$$\sum_{i=1}^n \int_\Omega\int_\Omega \frac{|D_i f(x) - D_i f(y)|^2}{|x-y|^{n+2\sigma}}\, dx\, dy \\ \geq 2\kappa_{n,2\sigma}\,(\sigma + 1/2)^2 \int_\Omega \frac{|f(x)|^2}{M_{1+\sigma,2}(x)^{2+2\sigma}}\, dx, \tag{7.3.7}$$

where

$$2\kappa_{n,2\sigma} := 2\frac{\pi^{(n-1)/2}\Gamma\left(\frac{3+2\sigma}{2}\right)}{\sigma\Gamma\left(\frac{n+2+2\sigma}{2}\right)}\left\{\frac{2^{-2\sigma}}{\sqrt{\pi}}\Gamma\left(\frac{1+2\sigma}{2}\right)\Gamma\left(\frac{2-2\sigma}{2}\right) - \frac{1}{2}\right\} \\ > \mathcal{D}_{n,2,2\sigma}. \tag{7.3.8}$$

The proof of this theorem will be given later, after the establishment of various one-dimensional inequalities based on ones in [130]. Only the case $k = 1$ of the first lemma is needed in this section, but the general case will be required when we consider higher-order inequalities later.

Lemma 7.8 *Let $-\infty < a < b < \infty$, $p + s - 1 > 0$, $1 < p < \infty$, $k \in \mathbb{N}$ and for $t \in (a,b)$, set $\delta_{(a,b)}(t) := \min\{t-a, b-t\}$. Then, for all $f \in C_0^\infty(a,b)$,*

$$\int_a^b |f^k(x)|^p \left\{\frac{1}{(x-a)} + \frac{1}{(b-x)}\right\}^s \\ \geq \Pi_{j=1}^k \left(\frac{jp+s-1}{p}\right)^p \int_a^b \frac{|f(x)|^p}{\delta_{(a,b)}(x)^{kp+s}}\, dx. \tag{7.3.9}$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

Proof Let $c = (a+b)/2,\ j \in \{1, 2, \cdots, k\}$ and $q = p/(p-1)$. On integration by parts and the use of Hölder's inequality, we have (cf. [66], Lemma 4.7)

$$\begin{aligned}
I_a &:= \int_a^c \frac{|f(x)|^p}{|x-a|^{jp+s}}\,dx \\
&= \frac{p}{jp+s-1}\int_a^c |f(x)|^{p-2}\Re[\bar{f}f']\left[(x-a)^{-(jp+s)+1} - (c-a)^{-(jp+s)+1}\right] \\
&\leq \frac{p}{jp+s-1}\int_a^c \frac{|f|^{p-1}}{(x-a)^{(jp+s)/q}}(x-a)^{(jp+s)/q}|f'|[(x-a)^{-(jp+s)+1} \\
&\quad - (c-a)^{-(jp+s)+1}]\,dx \\
&\leq \frac{p}{jp+s-1}\int_a^c \left(\frac{|f|^p}{(x-a)^{jp+s}}\right)^{1/q} |f'|(x-a)^{-(jp+s)/p+1} \\
&\quad \times \left[1 - \left(\frac{x-a}{c-x}\right)^{jp+s-1}\right]dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
I_a \leq &= \left(\frac{p}{jp+s-1}\right)^p \int_a^c \frac{|f'|^p}{(x-a)^{(j-1)p+s}}\left[1 - \left(\frac{x-a}{c-a}\right)^{jp+s-1}\right]^p dx \\
&\leq \left(\frac{p}{jp+s-1}\right)^p \int_a^c \frac{|f'|^p}{(x-a)^{(j-1)p+s}}\,dx. \qquad (7.3.10)
\end{aligned}$$

We also have for $j = 1$,

$$I_a \leq \left(\frac{p}{p+s-1}\right)^p \int_a^c \frac{|f'|^p}{(x-a)^s}\,dx\left[1 - \left(\frac{x-a}{c-x}\right)^{p+s-1}\right]^p dx.$$

Let

$$h_a(x) := \left[1 - \left(\frac{x-a}{c-a}\right)^{p+s-1}\right]^p - \left[1 + \left(\frac{x-a}{b-x}\right)\right]^s.$$

Then, $h_a(a) = 0, h_a(c) = -2^s$ and $h_a'(x) \leq 0$ on (a, c), and so $h_a(x) < 0$ on (a, c). Hence when $j = 1$,

$$\begin{aligned}
I_a &\leq \int_a^c \frac{|f'(x)|^p}{(x-a)^s}\left\{h_a(x) + \left[1 + \left(\frac{x-a}{b-x}\right)\right]^s\right\}dx \\
&= \int_a^c |f'(x)|^p \left\{\frac{1}{(x-a)} + \frac{1}{(b-x)}\right\}^s dx. \qquad (7.3.11)
\end{aligned}$$

Similar inequalities to (7.3.10) and (7.3.11) hold for the integral

$$I_b := \int_c^b \frac{|f(x)|^p}{|x-b|^{jp+s}}\,dx$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

and hence for

$$I := \int_a^b \frac{|f(x)|^p}{\delta_{(a,b)}(x)^{jp+s}}\,dx.$$

It follows that for $j \in 2, \ldots k$,

$$\int_a^b \frac{\left|f^{(k-j)}(x)\right|^p}{\delta_{(a,b)}(x)^{jp+s}}\,dx \le \left(\frac{p}{jp+s-1}\right)^p \int_a^b \frac{|f^{(k-j+1)}(x)|^p}{\delta_{(a,b)}(x)^{(j-1)p+s}}\,dx, \tag{7.3.12}$$

and for $j = 1$,

$$\int_a^b \frac{\left|f^{(k-1)}(x)\right|^p}{\delta_{(a,b)}(x)^{p+s}}\,dx \le \left(\frac{p}{p+s-1}\right)^p \int_a^b |f^k(x)|^p \left\{\frac{1}{(x-a)} + \frac{1}{(b-x)}\right\}^s dx. \tag{7.3.13}$$

Therefore

$$\begin{aligned}\int_a^b |f^k(x)|^p \left\{\frac{1}{(x-a)} + \frac{1}{(b-x)}\right\}^s dx \\ \ge \left(\frac{p+s-1}{p}\right)^p \left(\frac{2p+s-1}{p}\right)^p \cdots \left(\frac{kp+s-1}{p}\right)^p \int_a^b \frac{|f(x)|^p}{\delta_{(a,b)}(x)^{kp+s}}\,dx\end{aligned}$$

and the lemma is proved. □

The following lemma is a key result in the proof of Theorem 7.7, and is a consequence of Lemma 7.8, and Theorems 2.1 and 2.6 in [130].

Lemma 7.9 *Let* $-\infty < a < b < \infty$ *and* $1 < \alpha < 2$. *Then for* $f \in C_0^\infty(a,b)$,

$$\int_{(a,b)\times(a,b)} \frac{|f'(x) - f'(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy \ge 2\left(\frac{\alpha+1}{2}\right)^2 \kappa_{1,\alpha} \int_a^b \frac{|f(x)|^2}{\delta_{(a,b)}(x)^{\alpha+2}}\,dx. \tag{7.3.14}$$

For $1 < \alpha < p < \infty$,

$$\begin{aligned}\int_{(a,b)\times(a,b)} \frac{|f'(x) - f'(y)|^p}{|x-y|^{1+\alpha}}\,dx\,dy \\ \ge \left(\frac{\alpha+p-1}{p}\right)^p \mathcal{D}_{1,p,\alpha} \int_a^b \frac{|f(x)|^p}{|\delta_{(a,b)}(x)|^{\alpha+p}}\,dx.\end{aligned} \tag{7.3.15}$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

Proof From Theorem 2.1 in [130],

$$\begin{aligned}
&\int_{(a,b)\times(a,b)} \frac{|f'(x)-f'(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy \\
&\quad\geq 2\kappa_{1,\alpha}\int_a^b |f'(x)|^2\left[\frac{1}{x-a}+\frac{1}{b-x}\right]^\alpha dx \\
&\quad= 2\kappa_{1,\alpha}\left(\int_a^c+\int_c^b\right)|f'(x)|^2\left[\frac{1}{x-a}+\frac{1}{b-x}\right]^\alpha dx \\
&\quad\geq 2\kappa_{1,\alpha}\int_a^b \frac{|f'(x)|^2}{\delta_{(a,b)}(x)^\alpha}\,dx, \qquad (7.3.16)
\end{aligned}$$

and (7.3.14) follows from the case $k=1$ of Lemma 7.8. The inequality (7.3.15) follows from Theorem 2.6 in [130] and Lemma 7.8. □

The same argument as in the proof of (6.2.6) in Lemma 6.2.2, gives

Corollary 7.10 *Let J be an open subset of* $\mathbb{R}$ *and*

$$\delta_J(t) := \min\{|s| \colon t+s\notin J\}. \qquad (7.3.17)$$

For $1<\alpha<2$ *and* $f\in C_0^\infty(J)$,

$$\int_{J\times J}\frac{|f'(x)-f'(y)|^2}{|x-y|^{1+\alpha}}\,dx\,dy \geq 2\left(\frac{\alpha+1}{2}\right)^2\kappa_{1,\alpha}\int_J \frac{|f(x)|^2}{\delta_J(x)^{\alpha+2}}\,dx. \qquad (7.3.18)$$

If $1<\alpha<p<\infty$,

$$\int_{J\times J}\frac{\left|f'(x)-f'(y)\right|^p}{|x-y|^{1+\alpha}}\,dx\,dy \geq \left(\frac{\alpha+p-1}{p}\right)^p\mathcal{D}_{1,p,\alpha}\int_J \frac{|f(t)|^p}{\delta_J(t)^{p+\alpha}}\,dt. \qquad (7.3.19)$$

Corollary 7.11 *For each x in the domain* $\Omega\subset\mathbb{R}^n$ *and* $\nu\in\mathbb{S}^{n-1}$, *define*

$$J(x,\nu) := \{t\colon x+t\nu\in\Omega\}, \qquad (7.3.20)$$

$$\delta_{J(\mathbf{x},\nu)} := \min\{|t|\colon t\notin J(\mathbf{x},\nu)\}. \qquad (7.3.21)$$

Let $1<\alpha<p<\infty$ *and set* $D=(D_1,D_2,\dots D_n)$, $D_i=\partial/\partial x_i$. *Then for* $x\in\Omega,\ f\in C_0^\infty(J(x,\nu))$ *and* $\nu\in\mathbb{S}^{n-1}$,

$$\begin{aligned}
&\int_{J(x,\nu)\times J(x,\nu)}\frac{|(Df\cdot\nu)(x+s\nu)-(Df\cdot\nu)(x+t\nu)|^p}{|s-t|^{1+\alpha}}\,ds\,dt \\
&\quad\geq E(\alpha,p)\int_{J(x,\nu)}|(Df\cdot\nu)(x+t\nu)|^p\frac{1}{\delta_{J(x,\nu)}(t)^\alpha}\,dt \\
&\quad\geq E(\alpha,p)\left(\frac{\alpha+p-1}{p}\right)^p\int_{J(x,\nu)}|f(x+t\nu)|^p\frac{1}{\delta_{J(x,\nu)}(t)^{\alpha+p}}\,dt, \qquad (7.3.22)
\end{aligned}$$

where $E(\alpha,p)=\mathcal{D}_{1,p,\alpha}$ *for* $1<p<\infty$ *and* $2\kappa_{1,\alpha}$ *when* $p=2$.

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

Proof Since Ω is an open connected set, then each $J(x,\nu)$ is an open set in $\mathbb{R}$. As a function of $t, f(x+t\nu) \in C_0^\infty(J(x,\nu))$ and by the chain rule,

$$\frac{d}{dt} f(x+t\nu) = (\nu \cdot Df)\,(x+t\nu).$$

Thus, (7.3.22) follows from Corollary 7.10 applied to $f(x+t\nu)$. □

Lastly we need a lower bound for

$$e_p(n) := \left(\sum_{i=1}^n |\nu_i|^{p'}\right)^{-p/p'}, \quad \nu = (\nu_i) \in \mathbb{S}^{n-1}.$$

If $1 < p \leq 2$,

$$\sum_{i=1}^n |\nu_i|^{p'} \leq \sum_{i=1}^n |\nu_i|^2 = 1,$$

and if $p > 2$,

$$\sum_{i=1}^n |\nu_i|^{p'} \leq \left(\sum_{i=1}^n |\nu_i|^2\right)^{p'/2} \left(\sum_{i=1}^n 1\right)^{1-p'/2} = n^{1-p'/2}.$$

Thus

$$\left(\sum_{i=1}^n |\nu_i|^{p'}\right)^{-p/p'} \geq \begin{cases} 1, & \text{if } 1 < p \leq 2, \\ n^{-(p-2)/2}, & \text{if } p > 2. \end{cases} \tag{7.3.23}$$

Hence, for $\nu = (\nu_i) \in \mathbb{S}^{n-1}$,

$$\begin{aligned} &|(\nu \cdot Df)\,(x+s\nu) - (\nu \cdot Df)\,(x+t\nu)|^p \\ &\quad = \left|\Sigma_{i=1}^n \nu_i D_i f(x+s\nu) - \Sigma_{i=1}^n \nu_i D_i f(x+t\nu)\right|^p \\ &\quad \leq e_p(n)^{-1} \Sigma_{i=1}^n |D_i f(x+s\nu) - D_i f(x+t\nu)|^p, \end{aligned}$$

and we have as in Lemma 6.10,

Lemma 7.12 *Let* $1/p < \sigma < 1,\ 1 < p < \infty$ *and* $f \in C_0^\infty(\Omega)$. *Then*

$$\begin{aligned} &\sum_{i=1} \int_\Omega \int_\Omega \frac{|D_i f(x) - D_i f(y)|^p}{|x-y|^{n+p\sigma}}\, dx\, dy \\ &\quad \geq \frac{\omega_{n-1}}{2} \int_{S^{n-1}} e_p(n)\, d\omega(\nu) \int_{x:\ x\cdot\nu=0} d\mathcal{L}_\nu(x) \int_{x+s\nu\in\Omega} ds \\ &\qquad \times \int_{x+t\nu\in\Omega} \left\{\frac{|(\nu\cdot Df)\,(x+s\nu) - (\nu\cdot Df)\,(x+t\nu)|^p}{|s-t|^{1+p\sigma}}\right\} dt, \end{aligned} \tag{7.3.24}$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

where $\mathcal{L}_\nu(x)$ denotes the $(n-1)$-dimensional Lebesgue measure on the plane $x \cdot \nu = 0$.

We now have all we need for the proof of Theorem 7.7.

Proof of Theorem 7.7 From Lemma 2.4 in [130],

$$\int_\Omega \int_\Omega \frac{|D_i f(x) - D_i f(y)|^2}{|x-y|^{n+2\sigma}}\,dx\,dy$$
$$= \frac{\omega_{n-1}}{2} \int_{S^{n-1}} d\omega(\nu) \int_{x:\ x\cdot\nu=0} d\mathcal{L}_\nu(x) \int_{x+s\nu\in\Omega} ds$$
$$\times \int_{x+t\nu\in\Omega} \left\{ \frac{|D_i f\,(x+s\nu) - D_i f\,(x+t\nu)|^2}{|s-t|^{1+2\sigma}} \right\} dt.$$

Thus, on applying (7.3.24),

$$\sum_{i=1}^n \int_\Omega \int_\Omega \frac{|D_i f(x) - D_i f(y)|^2}{|x-y|^{n+2\sigma}}\,dx\,dy$$
$$\geq \frac{\omega_{n-1}}{2} \int_{S^{n-1}} d\omega(\nu) \int_{x:\ x\cdot\nu\ =0} d\mathcal{L}_\nu(x) \int_{x+s\nu\in\Omega} ds$$
$$\times \int_{x+t\nu\in\Omega} \left\{ \frac{|(\nu\cdot Df)\,(x+s\nu) - (\nu\cdot Df)\,(x+t\nu)|^2}{|s-t|^{1+2\sigma}} \right\} dt.$$

From Corollary 7.11, we therefore have

$$\sum_{i=1}^n \int_\Omega \int_\Omega \frac{|D_i f(x) - D_i f(y)|^2}{|x-y|^{n+2\sigma}}\,dx\,dy$$
$$\geq \omega_{n-1}\ \kappa_{1,2\sigma} \int_{S^{n-1}} d\omega(\nu) \int_{x:\ x\cdot\nu\ =0} d\mathcal{L}_\nu(x)$$
$$\times \int_{x+s\nu\in\Omega} |(\nu\cdot Df)(x+s\nu)|^2\, \frac{1}{\delta_\nu(x+s\nu)^{2\sigma}}\,ds$$
$$\geq \left(\frac{2\sigma+1}{2}\right)^2 \omega_{n-1}\ \kappa_{1,2\sigma} \int_{S^{n-1}} d\omega(\nu) \int_{x:\ x\cdot\nu=0} d\mathcal{L}_\nu(x)$$
$$\times \int_{x+s\nu\in\Omega} |f(x+s\nu)|^2 \frac{1}{\delta_\nu^{2+2\sigma}(x+s\nu)}\,ds$$
$$\geq 2\left(\frac{2\sigma+1}{2}\right)^2 \kappa_{n,2\sigma} \int_\Omega \frac{|f(x)|^2}{M_{1+\sigma,2}(x)^{2+2\sigma}}\,dx.$$

The proof for $p=2$ is complete. For general p the proof is similar. □

7.4 Higher-Order Fractional Hardy–Rellich Inequalities

Some more notation and preliminary remarks are required before stating the main theorem.

For $\nu = (\nu_i) \in \mathbb{S}^{n-1}$, $\alpha = (\alpha_1, ..., \alpha_n) \in \mathbb{N}_0^n$ and $k \in \mathbb{N}$, use of the multinomial theorem shows that

$$\begin{aligned}(\nu \cdot D)^k &= (\nu_1 D_1 + \cdots + \nu_n D_n)^k \qquad (7.4.1)\\ &= \sum_{|\alpha|=k} \frac{k!}{\alpha_1!...\alpha_n!} (\nu_1 D_1)^{\alpha_1} ... (\nu_n D_n)^{\alpha_n}\\ &:= \sum_{|\alpha|=k} \frac{k!}{\alpha!} (\nu_\alpha D^\alpha), \quad \nu_\alpha = \nu_1^{\alpha_1} ... \nu_n^{\alpha_n}. \qquad (7.4.2)\end{aligned}$$

For $p \in (1, \infty)$, $k \in \mathbb{N}$ and $\nu \in \mathbb{S}^{n-1}$, we shall need

$$S_{k,p'}(\nu) := \left(\sum_{|\alpha|=k} \left(\frac{k!}{\alpha!}\right)^{p'} |\nu_\alpha|^{p'} \right)^{1/p'}, S_{k,p'} := \max_{\nu \in \mathbb{S}^{n-1}} S_{k,p'}(\nu), \qquad (7.4.3)$$

where $|\nu_\alpha|^2 := \nu_1^{2\alpha_1} + \cdots + \nu_n^{2\alpha_n}$.

Remark 7.13

1. It follows from (7.3.23) that

$$S_{1,p'}^p \le \begin{cases} 1, & \text{if } 1 < p \le 2, \\ n^{(p-2)/2(p-1)} & \text{if } 2 < p < \infty. \end{cases} \qquad (7.4.4)$$

2. Estimation of $S_{k,p'}$ when $k > 1$ requires more effort. For example, suppose that $k = 2$. Then there are two possibilities:
   (a) two components of α, say α_i and α_j, are 1 and the others are zero;
   (b) one component of α, say α_j, is 2 and the others are zero.

In case (a), $\alpha! = 1$ and $\nu_1^{2\alpha_1} + \cdots + \nu_n^{2\alpha_n} = \nu_i^2 + \nu_j^2 + n - 2$, so that $n - 2 \le |\nu_\alpha|^2 \le n - 1$ and

$$4(n-2) \le \left(\frac{2}{\alpha!}\right)^2 |\nu_\alpha|^2 \le 4(n-1).$$

In case (b), $\alpha! = 2$ and $n - 1 \le |\nu_\alpha|^2 \le \nu_j^2 + n - 1 \le n$, showing that

$$n - 1 \le \frac{2}{\alpha!} |\nu_\alpha|^2 \le n.$$

In the sum for $S_{2,2}^2$ there are $n(n-1)/2$ terms of type (a) and n of type (b). Thus

$$2n(n-1)(2n-3) \le S_{2,2}^2(\nu) \le 2n(n-1)^2 + n^2,$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

and so

$$n(n-1)(2n-3) \le S_{2,2}^2(\nu) \le n(2n^2-3n+2). \tag{7.4.5}$$

The mean distance function for the higher-order inequalities is, for $1 < p < \infty$ and $1/p < \sigma < 1$,

$$\frac{1}{M_{k+\sigma,p}(x)^{p\sigma+kp}} = \frac{\sqrt{\pi}\,\Gamma\left(\frac{n+p\sigma+kp}{2}\right)}{\Gamma\left(\frac{1+p\sigma+kp}{2}\right)\Gamma\left(\frac{n}{2}\right)} \int_{S^{n-1}} \frac{1}{\delta_{\nu,\Omega}^{p\sigma+kp}(x)} d\omega(\nu), \tag{7.4.6}$$

and the following constants are analogous to those in Section 7.3:

$$\mathcal{D}_{k,n,p,p\sigma} := \frac{2\pi^{(n-1)/2}\Gamma\left(\frac{1+pk+p\sigma}{2}\right)}{\Gamma\left(\frac{n+pk+p\sigma}{2}\right)} \mathcal{D}_{1,p,p\sigma}, \tag{7.4.7}$$

$$\kappa_{k,n,2\sigma} = \frac{2\pi^{(n-1)/2}\Gamma\left(\frac{1+2k+2\sigma}{2}\right)}{\Gamma\left(\frac{n+2k+2\sigma}{2}\right)} \kappa_{1,2\sigma}, \tag{7.4.8}$$

where $\mathcal{D}_{1,p,p\sigma}$ and $\kappa_{1,2\sigma}$ are given in (7.3.4) and (7.3.5). If Ω is convex with non-empty boundary, $0 < \sigma < 1$ and $1/\sigma < p < \infty$, then for all values of k, we have

$$M_{k+\sigma,p}(x) \le \delta(x) := \inf\{|y-x| : y \notin \Omega\}. \tag{7.4.9}$$

Note that in the case $k = 1$, our notation for (7.4.7) and (7.4.8) was $\mathcal{D}_{n,p,p\sigma}$ and $\kappa_{n,2\sigma}$. The constant

$$G(m\sigma, k, p) = \begin{cases} \Pi_{j=1}^{k} \left(\frac{jp+m\sigma-1}{p}\right)^p, & k \in \mathbb{N}, \\ 1, & k = 1 \end{cases} \tag{7.4.10}$$

appears in our main theorem.

Theorem 7.14 *Let Ω be a domain in $\mathbb{R}^n$ with non-empty boundary, $1 < p < \infty$ and $1/p < \sigma < 1$. Then, for all $f \in C_0^\infty(\Omega)$,*

$$\begin{aligned} S_{k,p'}^p \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|(D^\alpha f(x) - D^\alpha f(y))|^p}{|x-y|^{n+p\sigma}} dx\,dy \\ = S_{k,p'}^p \sum_{j_1,\cdots,j_k=1}^{n} \int_\Omega \int_\Omega \frac{\left|\left(D_{j_1}\cdots D_{j_k} f\right)(x) - \left(D_{j_1}\cdots D_{j_k} f\right)(y)\right|^p}{|x-y|^{n+p\sigma}} dx\,dy \\ \ge E_{k,n,p,p\sigma} G(p\sigma, k, p) \int_\Omega \frac{|f(x)|^p}{M_{k+\sigma,p}(x)^{p\sigma+kp}} dx, \end{aligned} \tag{7.4.11}$$

where $E_{k,n,p,p\sigma} = \mathcal{D}_{k,n,p,p\sigma}$; when $p = 2$ the inequality holds with $E_{k,n,2,2\sigma} = 2\kappa_{k,n,2\sigma}$.

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

Proof Corollaries 7.10 and 7.11 have the following analogues for any $k \in \mathbb{N}_0$:

$$\int_{J\times J} \frac{|f^{(k)}(x) - f^{(k)}(y)|^p}{|x-y|^{1+p\sigma}}\, dx\, dy \geq E(p\sigma, p)G(p\sigma, k, p) \int_J \frac{|f(x)|^p}{|\delta_J(x)|^{kp+p\sigma}}\, dx,$$

where $E(p\sigma, p) = \mathcal{D}_{1,p,p\sigma}$, $E(2\sigma, 2) = \kappa_{1,2\sigma}$, and

$$\int_{J(x,\nu)\times J(x,\nu)} \frac{|(\nu\cdot D)^k f(x+s\nu) - (\nu\cdot D)^k f(x+t\nu)|^p}{|s-t|^{1+p\sigma}}\, ds\, dt \geq E(p\sigma, p)G(p\sigma, k, p) \int_{J(x,\nu)} \frac{|f(x+t\nu)|^p}{\delta_{J(x,\nu)}(t)^{kp+p\sigma}}\, dt.$$

To proceed with the proof, we need the following inequality to obtain an analogue of Lemma 7.3.6, and thus of Lemma 2.4 in [130]. From (7.4.1) and (7.4.2),

$$(\nu\cdot D)^k = \sum_{|\alpha|=k} \frac{k!}{\alpha!} \nu_\alpha D^\alpha$$

for $\nu_\alpha = \nu_1^{\alpha_1} \cdots \nu_n^{\alpha_n}$, and

$$\begin{aligned}
&\left|\left((\nu\cdot D)^k f\right)(x+s\nu) - \left((\nu\cdot D)^k f\right)(x+t\nu)\right|^p \\
&= \left|\sum_{|\alpha|=k} \frac{k!}{\alpha!} \{(\nu_\alpha\cdot D^\alpha) f(x+s\nu) - (\nu_\alpha\cdot D^\alpha) f(x+t\nu)\}\right|^p \\
&\leq \left(\sum_{|\alpha|=k} \left(\frac{k!}{\alpha!}\right)^{p'} |\nu_\alpha|^{p'}\right)^{p/p'} \left(\sum_{|\alpha|=k} |(D^\alpha f)(x+s\nu) - (D^\alpha f)(x+t\nu)|^p\right) \\
&\leq S_{k,p'}^p \left(\sum_{|\alpha|=k} |(D^\alpha f)(x+s\nu) - (D^\alpha f)(x+t\nu)|^p\right),
\end{aligned}$$

where $S_{k,p'}$ is defined in (7.4.3). Then, for $1/p < \sigma < 1,\ 1 < p < \infty$ and $f \in C_0^\infty(\Omega)$,

$$\begin{aligned}
&S_{k,p'}^p \int_\Omega \int_\Omega \frac{\sum_{|\alpha|=k} |(D^\alpha f)(x) - (D^\alpha f)(y)|^p}{|x-y|^{n+p\sigma}}\, dx\, dy \\
&\quad \geq \frac{\omega_{n-1}}{2} \int_{S^{n-1}} d\omega(\nu) \int_{x:\ x\cdot\nu=0} d\mathcal{L}_\nu(x) \int_{x+s\nu\in\Omega} ds \\
&\quad\quad \times \int_{x+t\nu\in\Omega} \left\{ \frac{\left|(\nu\cdot D)^k f(x+s\nu) - (\nu\cdot D)^k f(x+t\nu)\right|^p}{|s-t|^{1+p\sigma}} \right\} dt,
\end{aligned}$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

where $\mathcal{L}_\nu(x)$ denotes the $(n-1)$-dimensional Lebesgue measure on the plane $x \cdot \nu = 0$. It follows that

$$\begin{aligned}
S^p_{k,p'} \int_\Omega \int_\Omega & \frac{\sum_{|\alpha|=k} |(D^\alpha f)(x) - (D^\alpha f)(y)|^p}{|x-y|^{n+p\sigma}}\, dx\, dy \\
&\geq E(p\sigma, p) G(p\sigma, k, p) \int_{S^{n-1}} d\omega(\nu) \int_{x:\, x\cdot\nu=0} d\mathcal{L}_\nu(x) \\
&\quad \times \int_{x+s\nu\in\Omega} \frac{|f(x+s\nu)|^p}{\delta_\nu(x+s\nu)^{p\sigma+kp}}\, ds \\
&\geq E(p\sigma, p) G(p\sigma, k, p) \int_{S^{n-1}} d\omega(\nu) \int_\Omega \frac{|f(x)|^p}{\delta_{\nu,\Omega}(x)^{p\sigma+kp}}\, dx \\
&\geq E_{k,n,p,p\sigma} G(p\sigma, k, p) \int_\Omega \frac{|f(x)|^p}{M_{k+\sigma,p}(x)^{p\sigma+kp}}\, dx.
\end{aligned}$$

This completes the proof. □

7.5 Higher-Order Inequality with a Remainder

An analogue of Proposition 6.14 is now readily established for higher-order Hardy–Rellich inequalities. First we note that Corollary 7.3.5 has the extension

Corollary 7.15 *Let Ω be a bounded domain in $\mathbb{R}^n$, and for $x \in \Omega$ and $\nu \in \mathbb{S}^{n-1}$, define*

$$\begin{aligned}
J(x,\nu) &:= \{t : x + t\nu \in \Omega\}, \\
\delta_{J(x,\nu)} &:= \min\{|t| : t \notin J(x,\nu)\}.
\end{aligned}$$

Then for $1/2 < \sigma < 1, f \in C_0^\infty(\Omega)$ and $k \in \mathbb{N}_0$,

$$\begin{aligned}
\int_{J(x,\nu)\times J(x,\nu)} & \frac{|(\nu\cdot D)^k f(x+r\nu) - (\nu\cdot D)^k f(x+t\nu)|^2}{|r-t|^{1+2\sigma}}\, dr\, dt \\
&\geq 2\kappa_{1,2\sigma} G(2\sigma, k, 2) \int_{J(x,\nu)} \frac{|f(x+t\nu)|^2}{\delta_{J(x,\nu)}(t)^{2k+2\sigma}}\, dt \\
&\quad + 2\frac{4-2^{3-2\sigma}}{2\sigma \operatorname{diam}(J(x,\nu))} G(2\sigma-1, k, 2) \int_{J(x,\nu)} \frac{|f(x)|^2}{|\delta_J(x)|^{2k+2\sigma-1}}\, dx, \quad (7.5.1)
\end{aligned}$$

using (7.4.8).

We may now follow a similar argument to that in the proof of Theorem 7.4.2, and in Dyda's Theorem 6.13, to derive

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

Theorem 7.16 *Let Ω be a bounded domain in $\mathbb{R}^n$ with non-empty boundary and $k \in \mathbb{N}_0$, $1/2 < \sigma < 1$. Then, for all $f \in C_0^\infty(\Omega)$,*

$$S_{k,2}^2 \sum_{|\alpha|=k} \int_\Omega \int_\Omega \frac{|(D^\alpha f(x) - D^\alpha f(y))|^2}{|x-y|^{n+2\sigma}}\, dx\, dy$$
$$\geq 2\kappa_{k,n,2\sigma} G(2\sigma, k, 2) \int_\Omega \frac{|f(x)|^p}{M_{k+\sigma,2}(x)^{2\sigma+2k}}\, dx$$
$$+ 2\frac{4-2^{3-2\sigma}}{2\sigma \text{ diam } (\Omega)} \frac{\kappa_{k,n,2\sigma-1}}{\kappa_{1,2\sigma-1}} G(2\sigma-1, k, 2) \int_\Omega \frac{|f(x)|^2}{M_{k+\sigma-1/2,2}(x)|^{2k+2\sigma-1}}\, dx, \tag{7.5.2}$$

where, by (7.4.8),

$$\frac{\kappa_{k,n,2\sigma-1}}{\kappa_{1,2\sigma-1}} = \frac{2\pi^{(n-1)/2}\Gamma\left(\frac{2k+2\sigma}{2}\right)}{\Gamma\left(\frac{n+2k+2\sigma-1}{2}\right)}$$

and $M_{k+\alpha,2}$ is defined in (6.2.1). If Ω is convex, $M_{k+\alpha,2}(x) \leq \delta(x) := \inf\{|y - x| : y \notin \Omega\}$.

The constant multiplying the first integral on the right-hand side of (7.5.2) cannot be replaced by a larger one in the case $k = 0$, but this is not proved for $k \geq 1$.

7.6 Higher-Order Classical Inequalities

It is proved by Bourgain, Brezis and Mironescu in [23] that if Ω is a connected open subset of $\mathbb{R}^n$ and $1 < p < \infty$, then for all $f \in C_0^\infty(\Omega)$,

$$\lim_{\sigma \to 1-} (1-\sigma) \int_\Omega \int_\Omega \frac{|f(x) - f(y)|^p}{|x-y|^{n+p\sigma}}\, dx\, dy = K(n,p) \int_\Omega |\nabla f(x)|^p\, dx$$

for some positive constant $K(n,p)$ depending only on n and p; see Corollary 3.20 and Remark 3.21. If $p = 2$, the following precise information is established in [78], Lemma 3.1:

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{|f(x) - f(y)|^2}{|x-y|^{n+2\sigma}}\, dx\, dy = 2C(n,\sigma)^{-1} \int_{\mathbb{R}^n} \left|(-\Delta)^{\sigma/2} f(x)\right|^2\, dx \tag{7.6.1}$$

for $0 < \sigma < 1$ and

$$\frac{1}{2} C(n,\sigma) = 2^{2\sigma-1}\pi^{-n/2} \frac{\Gamma\left(\frac{n}{2}+\sigma\right)}{|\Gamma(-\sigma)|}. \tag{7.6.2}$$

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press

In (6.6.1), $(-\Delta)^{\sigma/2}f(x) := \left[F^{-1}\left(|\xi|^{\sigma}\hat{f}(\xi)\right)\right](x)$, where $\hat{f} = F(f)$, and it follows that

$$\begin{aligned}
&\sum_{|\alpha|=k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|(D^{\alpha}f(x)-D^{\alpha}f(y))|^2}{|x-y|^{n+2\sigma}}\,dx\,dy \\
&\quad = 2C(n,\sigma)^{-1}\sum_{|\alpha|=k}\int_{\mathbb{R}^n}\left|(-\Delta)^{\sigma/2}D^{\alpha}f(x)\right|^2 dx \\
&\quad = 2C(n,\sigma)^{-1}\sum_{|\alpha|=k}\int_{\mathbb{R}^n}\left|\left(|\xi|^2\right)^{\sigma/2}(i\xi)^{\alpha}\hat{f}(\xi)\right|^2 d\xi \\
&\quad = 2C(n,\sigma)^{-1}\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{\sigma+k}{2}}f(x)\right|^2 dx.
\end{aligned}$$

Hence, for $f \in C_0^{\infty}(\Omega)$,

$$\begin{aligned}
&\int_{\mathbb{R}^n}\left|(-\Delta)^{\frac{\sigma+k}{2}}f(x)\right|^2 dx \\
&\quad = \frac{1}{2}C(n,\sigma)\sum_{|\alpha|=k}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n}\frac{|(D^{\alpha}f(x)-D^{\alpha}f(y))|^2}{|x-y|^{n+2\sigma}}\,dx\,dy \\
&\quad \geq \frac{1}{2}C(n,\sigma)\sum_{|\alpha|=k}\int_{\Omega}\int_{\Omega}\frac{|(D^{\alpha}f(x)-D^{\alpha}f(y))|^2}{|x-y|^{n+2\sigma}}\,dx\,dy. \qquad (7.6.3)
\end{aligned}$$

In (7.4.11), the constant multiple of the integral on the right-hand side is

$$2G(2\sigma,k,2)\kappa_{n,2\sigma}$$

in which, as $\sigma \to 1-$, $G(2,k,2) = \left(\frac{(2k+1)!}{k!2^{2k+1}}\right)^2 = \left(\frac{2}{\sqrt{\pi}}\Gamma\left(k+\frac{3}{2}\right)\right)^2$ and $\kappa_{n,2\sigma}$ is asymptotic to

$$\begin{aligned}
&\frac{\pi^{(n-1)/2}\Gamma\left(\frac{3+2k}{2}\right)}{\Gamma\left(\frac{n+2k+2}{2}\right)}\frac{2^{-2}}{\sqrt{\pi}}\Gamma(3/2)(1-\sigma)^{-1} \\
&\quad = \frac{1}{8}\pi^{(n-1)/2}\frac{\Gamma\left(\frac{3+2k}{2}\right)}{\Gamma\left(\frac{n+2k+2}{2}\right)}(1-\sigma)^{-1}. \qquad (7.6.4)
\end{aligned}$$

Also, as $\sigma \to 1-$, $\frac{1}{2}C(n,\sigma)$ in (7.6.3) satisfies

$$\frac{1}{2}C(n,\sigma) \sim 2\pi^{-n/2}\Gamma(n/2+1)(1-\sigma), \qquad (7.6.5)$$

and for $f \in C_0^{\infty}(\Omega)$,

$$I := \lim_{\sigma\to 1-}\int_{\mathbb{R}^n}\left|(-\Delta)^{(\sigma+k)/2}f(x)\right|^2 dx = \int_{\mathbb{R}^n}\left|(-\Delta)^{(1+k)/2}f(x)\right|^2 dx.$$

This follows by dominated convergence, on noting that

$$I = \lim_{\sigma \to 1-} \int_{\mathbb{R}^n} \left| (|\xi|^2)^{(\sigma+k)/2} \hat{f}(\xi) \right|^2 d\xi,$$

and, for $0 \le \sigma \le 1$,

$$\begin{aligned} \left| (|\xi|^2)^{(\sigma+k)/2} \hat{f}(\xi) \right|^2 &\le \left| \left[(|\xi|^2)^{(1+k)/2} + 1 \right] \hat{f}(\xi) \right|^2 \\ &= \left| F\left(\left[(-\Delta)^{(1+k)/2} + 1 \right] f \right)(\xi) \right|^2 \in L_1(\mathbb{R}^n). \end{aligned}$$

Hence from (7.4.11) and (7.6.3), the inequality we get in the limit as $\sigma \to 1-$ is

$$\int_{\mathbb{R}^n} \left| \Delta^{\frac{1+k}{2}} f(x) \right|^2 dx \ge K(n,k) \lim_{\sigma \to 1-} \int_\Omega \frac{|f(x)|^2}{M_{k+\sigma,2}(x)^{2k+2\sigma}} dx, \tag{7.6.6}$$

where

$$K(n,k) = 2S_{k,2}^{-2} \frac{\left[\Gamma\left(\frac{3+2k}{2} \right) \right]^3 \Gamma\left(\frac{n}{2} + 1 \right)}{\pi^{3/2} \Gamma\left(\frac{n+2k+2}{2} \right)}. \tag{7.6.7}$$

As $\sigma \to 1-$, $M_{k+\sigma,2}^{-2k-2\sigma}$ is bounded by $M_{k+1,2}^{-2k-2}$ on the support of f. Since $|f|^2 M_{k+1,2}^{-2k-2} \in L_1(\Omega)$ is proved in [143] (see (7.6.10) below) it follows by dominated convergence from (7.6.6) that

$$\int_{\mathbb{R}^n} \left| \Delta^{\frac{1+k}{2}} f(x) \right|^2 dx \ge K(n,k) \int_\Omega \frac{|f(x)|^2}{M_{k+1,2}(x)^{2k+2}} dx. \tag{7.6.8}$$

In [143], Owen establishes the following Hardy–Rellich inequality for polyharmonic operators with a sharp constant:

$$\int_\Omega \bar{f}(x) \left[(-\Delta)^m f \right](x)\, dx \ge \frac{\Gamma\left(\frac{n}{2} + m \right) \Gamma\left(m + \frac{1}{2} \right)}{\Gamma\left(\frac{n}{2} \right) \Gamma\left(\frac{1}{2} \right)} \int_\Omega \frac{|f(x)|^2}{a_m^{2m}(x)} dx$$

for all $f \in C_0^\infty(\Omega)$, $m \in \mathbb{N}$, and where

$$\frac{1}{a_m^{2m}(x)} = \int_{\mathbb{S}^{n-1}} \frac{1}{\delta_\nu(x)^{2m}} d\omega(\nu).$$

Owen expresses his result in the quadratic form sense

$$((-\Delta)^m f, f) \ge \frac{\Gamma\left(\frac{n}{2} + m \right) \Gamma\left(m + \frac{1}{2} \right)}{\Gamma\left(\frac{n}{2} \right) \Gamma\left(\frac{1}{2} \right)} (Af, f),$$

where $(\cdot,\cdot)$ is the $L_2(\Omega)$ inner-product, $(-\Delta)^m$ is the polyharmonic operator of order $2m$ and A is the operator of multiplication by $1/a_m^{2m}(x)$. On $C_0^\infty(\Omega)$,

$(-\Delta)^m$ is the restriction of $F^{-1}(|\cdot|^{2m})$, where F is the Fourier transform. In our notation, with $m = k+1$,

$$\frac{1}{a_m^{2m}(x)} = \frac{\Gamma\left(k+\frac{3}{2}\right)\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi}\,\Gamma\left(\frac{n}{2}+k+1\right)}\frac{1}{M_{k,1,2,\Omega}(x)^{2+2k}}.$$

Owen's inequality therefore implies

$$\int_\Omega \bar{f}(x)\left[(-\Delta)^{k+1}f\right](x)\,dx \geq K_0(k)\int_\Omega \frac{|f(x)|^2}{M_{k+1,2}(x)^{2+2k}}\,dx, \tag{7.6.9}$$

where

$$K_0(k) = \frac{\left[\Gamma\left(\frac{3+2k}{2}\right)\right]^2}{\pi}. \tag{7.6.10}$$

Hence, in particular, $|f|^2M_{k+1,2}^{-2-2k} \in L_1(\mathbb{R}^n)$, as noted earlier.

The inequality (7.6.6) can also be expressed in the form sense, namely

$$\int_\Omega \bar{f}(x)\left[(-\Delta)^{k+1}f\right](x)\,dx \geq K(n,k)\int_\Omega \frac{|f(x)|^2}{M_{k+1,2}(x)^{2+2k}}\,dx,$$

and we have from (7.6.7) and (7.6.10),

$$\frac{K_0(k)}{K(n,k)} = S_{k,2}^2\frac{\sqrt{\pi}\,\Gamma\left(\frac{n+2k+2}{2}\right)}{2\Gamma\left(\frac{3+2k}{2}\right)\Gamma(\frac{n}{2}+1)}. \tag{7.6.11}$$

When $k = 0$ we have $K(n,0) = K_0(0) = 1/4$, which confirms that the constant in the Hardy case of Theorem 7.4.11 with $p = 2$ is sharp, as already proved in [130]. However we cannot claim this for $k \geq 1$; for instance, when $k = 1$, the value $K_0(1) = 9/16$ is sharp, but $K(n,1) = (9/16)(3/(n+2)) < K_0(1)$ for $n > 1$.

When $p \neq 2$ it seems harder to use Fourier transform techniques. However, there is an analogue of Corollary 1.4.8 of [63] that can be established by induction, namely that if $p \in (1,\infty)$ and $m \in \mathbb{N}$, then for all $\alpha \in \mathbb{N}_0^n$ with $|\alpha| = 2m$,

$$\left\|D^\alpha f|L_p\left(\mathbb{R}^n\right)\right\| \leq c_p^m\left\|\Delta^m f|L_p\left(\mathbb{R}^n\right)\right\| \tag{7.6.12}$$

for all smooth f with compact support, where

$$c_p = \cot^2\left(\frac{\pi}{2p^*}\right), p^* = \max(p,p'). \tag{7.6.13}$$

This enables higher-order counterparts of Theorem 7.4 to be proved.

https://doi.org/10.1017/9781009254625.009 Published online by Cambridge University Press