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Essential Surfaces in Graph Link Exteriors

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Abstract. An irreducible graph manifold M contains an essential torus if it is not a special Seifert manifold. Whether M contains a closed essential surface of negative Euler characteristic or not depends on the difference of Seifert fibrations from the two sides of a torus system which splits M into Seifert manifolds. However, it is not easy to characterize geometrically the class of irreducible graph manifolds which contain such surfaces. This article studies this problem in the case of graph link exteriors.

1 Preliminaries

Let V be a solid torus in S^3 with preferred framing. A loop l on ∂V is said to be of type (p, q), if it wraps around V in the longitudinal direction p times and in the meridional direction q times. Note that (p, q) and $(-p, -q)$ denote the same type. A link L on ∂V consisting of n parallel copies of l is said to be of type (np, nq). In particular, L is a *torus link* of type (np, nq) if V is unknotted. The exterior of a torus link of type (np, nq) is called a torus link space of type (np, nq).

We say V is a *fibered solid torus* of type (p, q) if V is Seifert fibered so that each fiber on ∂V is of type (p, q). A manifold obtained from V by removing an open regular neighborhood of n regular fibers in the interior is called a *cable space* of type (np, nq). A manifold homeomorphic to a cable space of type $(n, 0)$ is called an n-fold *composing space*.

A Seifert fibration of S^3 is said to be of type (p, q) if a regular fiber is a torus knot of type (p, q). A *singular fibration* of S^3 is a trivial fibration of a trivial knot complement which extends to no Seifert fibration of S^3. The trivial knot is called a *singular circle*. A link L in S^3 is called a *Seifert link* if the exterior is a Seifert manifold. It is shown by Burde and Murasugi [1] that any Seifert link is either a union of fibers of a Seifert fibration of S^3 or a union of fibers of a singular fibration and its singular circle.

A link L in S^3 is called a *graph link* if the exterior E is a graph manifold, *i.e.,* E is split by a system of disjoint, embedded tori into pieces that are Seifert manifolds. Suppose that L is non-splittable. Then the splitting of E is realized by the JSJ decomposition [4, 5]. Each piece P is bounded by a system $T_1 \cup \cdots \cup T_n$ of tori such that T_i is an essential torus in E or a component of ∂E. Denote by V_i a solid torus in S^3 bounded by T_i. Each T_i is called an *outer torus* of P if $P \subset V_i$, and an *inner torus* of P otherwise. An inner torus T_i is said to be *regular* if the Seifert fibration of T_i extends to a trivial fibration of V_i, and *exceptional* otherwise. We classify the pieces P into a torus link space of type (np, nq), where $|p| > 1$ and $|q| > 1$, with two exceptional fibers, a cable space of type $((n-1)p, (n-1)q)$, where $|p| > 1$, with an exceptional fiber,

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or an $(n-1)$-fold composing space with no exceptional fiber (see [2]). If P is a composing space, some T_i can be regarded as an outer torus and P has the following three possibilities:

Type I: the Seifert fibration of P does not extend to that of V_i,
Type II: the Seifert fibration of P extends to a non-trivial fibration of V_i, or
Type III: the Seifert fibration of P extends to a trivial fibration of V_i.

We assume that any composing space of Type I is bounded by an exceptional inner torus and several outer tori, and that any composing space of Type II or III is bounded by an outer torus and several inner tori. Note that the type of a composing space depends on the choice of the outer tori. For example, we can regard a composing space of type I with an unknotted exceptional inner torus T_i as a composing space of Type III with the outer torus T_i. Figure 1 illustrates the types of the Seifert link exteriors.

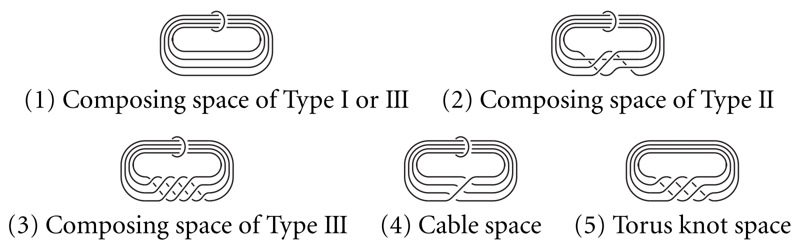

Figure 1: Exterior of Seifert links.

Let F be a closed essential surface of negative Euler characteristic in E. Isotope F so that afterwards F intersects any P in a system of essential surfaces. It follows from [3, VI.34] that each component of $F \cap P$ has the following two possibilities:

(1) a fiber in a fibration of P as a surface bundle over S^1, which we call a *surface fiber*, or
(2) an annulus saturated in some Seifert fibration of P.

Let L be a link in S^3 and K a component of L. Take a regular neighborhood V of K. Let C be a link on ∂V of type (np, nq) where $\gcd(p, q) = 1$ and $n \geq 1$. We say the link $(L - K) \cup C$ is obtained from L by taking an (np, nq)-*cable* C of K if $|np| > 1$, *i.e.,* C is not parallel to K in V. The link $L \cup C$ is said to be obtained from L by taking an (np, nq)-*special cable* $K \cup C$ of K if $|p| \neq 1$, *i.e.,* $K \cup C$ is not a cable of K.

2 Essential Surfaces in Seifert Manifold Pieces

Let P be an n-fold composing space of Type II bounded by tori $T_1, \dots, T_{n+1}$ where $n \geq 2$. Without loss of generality, each T_i bounds a solid torus V_i in S^3 so that T_1 is an exceptional inner torus, $T_2, \dots, T_n$ are regular inner tori, and T_{n+1} is an outer torus. By extending the Seifert fibration of P, we assume that V_i is a fibered solid torus of type (p, q) where $|p| > 1$ for $i = 1$ or $n + 1$, and $(1, pq)$ otherwise.

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Table 1: Boundary types of an essential surface in a piece.

		n-fold composing space				cable space	torus link space
type		I	II		III	(np, nq)	(np, nq)
inner tori	number of tori	1	1 (exceptional)	$n-1$ (regular)	n	n	n
	annulus	$(0, k)$	(kp, kq)	(k, kpq)	(k, kq)	(k, kpq)	(k, kpq)
	surface fiber	$\left(\lambda, \sum_{i=1}^{n} \mu_i\right)$	$\left(\lambda_1, \frac{\sigma\mu + q\lambda_1}{p}\right)$	$(\lambda_2, \mu + pq\lambda_2)$, $\dots$, $(\lambda_n, \mu + pq\lambda_n)$	$(\lambda_1, \mu + q\lambda_1)$, $\dots$, $(\lambda_n, \mu + q\lambda_n)$	$(\lambda_1, \mu + pq\lambda_1)$, $\dots$, $(\lambda_n, \mu + pq\lambda_n)$	$(\lambda_1, \mu + pq\lambda_1)$, $\dots$, $(\lambda_n, \mu + pq\lambda_n)$
outer tori	number of tori	n	1		1	1	0
	annulus	$(0, 2-k)$	$((2-k)p, (2-k)q)$		$(1, (2-k)q)$	$((2-k)p, (2-k)q)$	—
	surface fiber	(λ, μ_1), $\dots$, (λ, μ_n)	$\left(\overline{\lambda}, \frac{\sigma\mu + q\overline{\lambda}}{p}\right)$		$(\overline{\lambda}, \mu + q\overline{\lambda})$	$\left(\overline{\lambda}, \frac{\sigma\mu + q\overline{\lambda}}{p}\right)$	—
remark		$0 \le k \le 2$, $\lambda \ne 0$	$\gcd(p, q) = 1$, $\lvert p\rvert > 1$, $0 \le k \le 2$, $\mu \ne 0$, $\sigma = \frac{p}{\lvert p\rvert}$, $\overline{\lambda} = \lambda_1 + \lvert p\rvert \sum_{i=2}^{n} \lambda_i$		$0 \le k \le 2$, $\mu \ne 0$, $\overline{\lambda} = \sum_{i=1}^{n} \lambda_i$	$\gcd(p, q) = 1$, $\lvert p\rvert > 1$, $0 \le k \le 2$, $\mu \ne 0$, $\sigma = \frac{p}{\lvert p\rvert}$, $\overline{\lambda} = \lvert p\rvert \sum_{i=1}^{n} \lambda_i$	$\gcd(p, q) = 1$, $\lvert p\rvert > 1$, $\lvert q\rvert > 1$, $0 \le k \le 2$, $\mu = -pq \sum_{i=1}^{n} \lambda_i \ne 0$

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Let $\widetilde{V}_{n+1}$ be a $|p|$-cover of V_{n+1}. Denote by $\widetilde{P}$ the induced $|p|$-cover of P and by $\widetilde{V}_i$ the induced $|p|$-covers of V_i for $1 \leq i \leq n$. We consider each $\widetilde{V}_i$ endowed with the induced fibration. With respect to the lift of the preferred framing of V_{n+1}, $\widetilde{V}_i$ is a fibered solid torus of type $(1, \sigma q)$, where $\sigma = p/|p|$, for $i = 1$ or $n + 1$, and a union of $|p|$ copies of $\widetilde{V}_1$ otherwise.

Suppose that a surface fiber F in P intersects T_i in loops of type (λ_i, μ_i). Then $p\mu_i - q\lambda_i \neq 0$ for $i = 1$ or $n+1$, and $\mu_i - pq\lambda_i \neq 0$ otherwise. The induced $|p|$-cover $\widetilde{F}$ of F intersects each component of $\widetilde{T}_i = \partial\widetilde{V}_i$ in a link of type $(\lambda_i, |p|\mu_i)$ for $i = 1$ or $n+1$, and $(\lambda_i, \mu_i - (pq - \sigma q)\lambda_i)$ otherwise. By twisting $\widetilde{V}_n$ in the meridional direction $-\sigma q$ times, any component of $\widetilde{V}_i$ is a fibered solid torus of type $(1, 0)$, and $\widetilde{F}$ intersects $\widetilde{T}_i$ in a link of type $(\lambda_i, \sigma(p\mu_i - q\lambda_i))$ for $i = 1$ or $n+1$, and $(\lambda_i, \mu_i - pq\lambda_i)$ otherwise. Note that $H_1(\widetilde{P})$ is a free abelian group generated by meridians of the components of $\widetilde{T}_1 \cup \cdots \cup \widetilde{T}_n$ and a longitude of $\widetilde{T}_1$. A meridian of $\widetilde{T}_{n+1}$ is homologous to the sum of these meridians. Longitudes of the components of $\widetilde{T}_1 \cup \cdots \cup \widetilde{T}_{n+1}$ are mutually homologous. Since $\widetilde{F} \cap \widetilde{T}_{n+1}$ is homologous to $\widetilde{F} \cap (\widetilde{T}_1 \cup \cdots \cup \widetilde{T}_n)$, we obtain, by replacing (λ_i, μ_i) with $(-\lambda_i, -\mu_i)$ if necessary,

$$\sigma(p\mu_1 - q\lambda_1) = \mu_2 - pq\lambda_2 = \cdots = \mu_n - pq\lambda_n = \sigma(p\mu_{n+1} - q\lambda_{n+1}) \neq 0 \text{ and}$$
$$\lambda_{n+1} = \lambda_1 + |p|\lambda_2 + \cdots + |p|\lambda_n.$$

Set $\mu = \sigma(p\mu_1 - q\lambda_1)$ and $\overline{\lambda} = \lambda_{n+1}$. Then $\mu_1 = (\sigma\mu + q\lambda_1)/p$, $\mu_{n+1} = (\sigma\mu + q\overline{\lambda})/p$ and $\mu_i = \mu + pq\lambda_i$ for $2 \leq i \leq n$.

We can apply a similar argument to the case where P is a composing space of Type III or a cable space. The argument for a cable space is applicable to the case of a torus link space P by splitting P into a composing space of Type III and a torus knot space. Furthermore, the argument for a composing space of Type III is applicable to the case of a composing space of Type I by exchanging the role of inner and outer tori. These arguments are summarized in Table 1.

3 Essential Surfaces in Graph Link Exteriors

In this section, we focus on several classes of graph links to derive from Table 1 criteria for determining whether a given graph link has a closed essential surface of negative Euler characteristic in the exterior or not.

Theorem 3.1 *Let L be a non-splittable graph link whose exterior E contains a closed essential surface F of negative Euler characteristic. Suppose that E is split by a JSJ decomposition into pieces of Seifert manifolds. Then we have the following:*

(1) *If E consists of two pieces, L is an $(n, 0)$-cable of a non-trivial torus knot.*

(2) *If E consists of three pieces, L has the following possibilities:*

   (2-1) *L is obtained from a link stated in (1) by taking a (p, q)-cable or a (p, q)-special cable of some component where $q \neq 0$.*

   (2-2) *L is an $(n, 0)$-cable of an (r, s)-cable of a non-trivial torus knot of type (p, q), where $\gcd(r, s) = 1$ and $s \neq pqr$.*

(2-3) *L is obtained from a non-Hopf torus link $L_1 \cup L_2$ of type $(2p, 2q)$ by taking an (r_i, s_i)-cable or an (r_i, s_i)-special cable of each L_i where $s_i \neq pqr_i$ and $p^2q^2r_1r_2 = s_1s_2$.*

(2-4) *L is obtained from a non-Hopf link consisting of fibers and the singular circle of a singular fibration of S^3 by taking either a $(0, n)$-special cable of an (r, s)-cable of the singular circle where $\gcd(r, s) = 1$, or an $(n, 0)$-cable of an (r, s)-cable of a fiber where $\gcd(r, s) = 1$ and $s \neq 0$.*

(2-5) *L is obtained from a non-trivial non-Hopf link consisting of fibers of a Seifert fibration of S^3 of type (p, q) by taking either an $(n, npqr^2)$-cable of an (r, s)-cable of a regular fiber where $\gcd(r, s) = 1$ and $s \neq pqr$, or a (t, u)-cable or a (t, u)-special cable of an (r, s)-cable of an exceptional fiber of order $|p|$ where $\gcd(r, s) = 1$, $ps \neq qr$ and $pu = qr^2t$.*

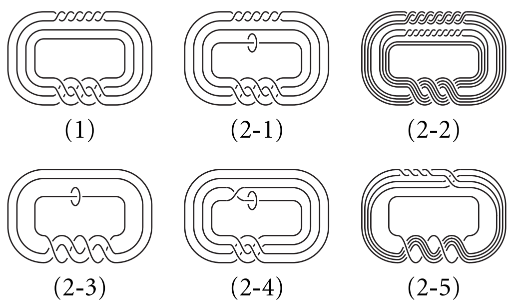

Figure 2: Examples of links stated in Theorem 3.1.

Proof Isotope F so that afterwards F consists of essential surfaces in the Seifert manifold pieces. Note that any component surface fiber appears in a piece which is disjoint from ∂E, and that no component annulus is connected to another.

(1) Suppose that E consists of pieces P_1 and P_2. Without loss of generality, F intersects P_1 in surface fibers. Then P_1 is a torus knot space bounded by the essential torus splitting E into P_1 and P_2. Table 1 implies that F intersects ∂P_1 in preferred longitudes. Since $\partial E \subset \partial P_2$, F intersects P_2 in annuli. Therefore Table 1 implies that P_2 is an n-fold composing space of Type III. It immediately follows that E is the exterior of an $(n, 0)$-cable of a non-trivial torus knot.

(2) Suppose that E consists of pieces P_1, P_2, and P_3. If F is disjoint from some P_i, (1) implies that L is classified into (2-1). Assume that F intersects every P_i. Without loss of generality, F intersects P_1 in surface fibers. Then P_1 is bounded by one or two essential tori. Therefore P_1 is a torus link space or a cable space.

Assume that P_1 is a torus knot space of type (p, q). Then F intersects ∂P_1 in preferred longitudes of type $(n, 0)$. Without loss of generality, P_1 is connected to P_2 over the torus $T_1 = \partial P_1$ and P_2 is connected to P_3 over a torus T_2. Since Table 1 implies that P_2 is a composing space of Type III, P_3 is contained in a solid torus bounded by T_2 and therefore intersects ∂E. Thus F intersects P_2 in surface fibers

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and P_3 in annuli. Since P_2 is disjoint from ∂E, P_2 is a cable space of type (r, s), where $\gcd(r, s) = 1$ and $s \neq pqr$. Table 1 implies that a surface fiber in P_2 joins a link on T_1 of type $(|r|\lambda, (\mu + rs\lambda)/|r|)$ and a link on T_2 of type $(\lambda, \mu + rs\lambda)$. Then $(|r|\lambda, (\mu + rs\lambda)/|r|) = (n, 0)$ implies $(\lambda, \mu + rs\lambda) = (n/|r|, 0)$. Thus F intersects T_2 in preferred longitudes. Table 1 implies that P_3 is a composing space of Type III with preferred longitudinal fibers on ∂P_3. Hence L is classified into (2-2).

Assume that P_1 is a torus link space of type $(2p, 2q)$ bounded by two tori T_1 and T_2. Table 1 implies that each component of $F \cap P_1$ joins a link on T_1 of type $(\lambda_1, -pq\lambda_2)$ and a link on T_2 of type $(\lambda_2, -pq\lambda_1)$, where $\lambda_1 + \lambda_2 \neq 0$. Without loss of generality, P_1 is connected to P_2 and P_3 over T_1 and T_2 respectively. Each of P_2 and P_3 is a cable space or a composing space that F intersects in annuli. Suppose that a fiber of P_{i+1} on T_i is of type (r_i, s_i). Then $-pqr_1\lambda_2 = s_1\lambda_1$ and $-pqr_2\lambda_1 = s_2\lambda_2$. One easily obtain $s_i \neq pqr_i$ and $p^2q^2r_1r_2 = s_1s_2$. Hence L is classified into (2-3).

Assume that P_1 is a cable space of type (r, s), where $\gcd(r, s) = 1$. Denote by T_1 and T_2 the outer torus and the inner torus of P_1 respectively. Table 1 implies that a surface fiber in P_1 joins a link on T_1 of type $(|r|\lambda, (\mu + rs\lambda)/|r|)$ and a link on T_2 of type $(\lambda, \mu + rs\lambda)$. Suppose that P_1 is connected to P_2 and P_3 over T_1 and T_2 respectively. If F intersects P_2 in surface fibers, P_2 is a torus knot space, in which case L is classified into (2-2) by the argument presented above. We may therefore assume that F intersects P_2 in annuli. Since P_3 is contained in the solid torus bounded by T_2, P_3 intersects ∂E. Thus F intersects P_3 in annuli.

Assume that the Seifert fibration of P_2 extends to a singular fibration of S^3. We consider P_2 to be a composing space of Type I by switching the inner and outer tori of P_2 if necessary. If T_1 is the inner torus of P_2, Table 1 implies that F intersects T_1 in meridians of type $(0, n)$. Then $(|r|\lambda, (\mu + rs\lambda)/|r|) = (0, n)$ implies $(\lambda, \mu + rs\lambda) = (0, n|r|)$. Therefore F intersects T_2 in meridians and hence P_3 is a composing space of Type I. If T_1 is an outer torus of P_2, $s \neq 0$ and F intersects T_1, as the outer torus of P_1, in preferred longitudes of type $(n, 0)$. Then $(|r|\lambda, (\mu + rs\lambda)/|r|) = (n, 0)$ implies $(\lambda, \mu + rs\lambda) = (n/|r|, 0)$. Therefore F intersects T_2, which is the outer torus of P_3, in preferred longitudes and hence P_3 is a composing space of Type III. In both cases, L is classified into (2-4).

Assume that the Seifert fibration of P_2 extends to a Seifert fibration of S^3 of type (p, q). In this case, P_2 is a torus link space, cable space, or a composing space of Type II or III. We may assume that F intersects T_1, as the outer torus of P_1, in loops of type (n, npq) or (np, nq), corresponding to a regular or exceptional fiber of the Seifert fibration of S^3. Suppose that F intersects T_1 in loops of type (n, npq), where $s \neq pqr$. Then $(|r|\lambda, (\mu+rs\lambda)/|r|) = (n, npq)$ implies $(\lambda, \mu+rs\lambda) = (n/|r|, npq|r|)$. Therefore F intersects T_2, which is the outer torus of P_3, in longitudes each of which is of type $(1, pqr^2)$. Hence P_3 is a composing space of Type III. Suppose that F intersects T_1 in loops of type (np, nq), where $|p| \geq 2$ and $ps \neq qr$. Then $(|r|\lambda, (\mu + rs\lambda)/|r|) = (np, nq)$ implies $(\lambda, \mu + rs\lambda) = (np/|r|, nq|r|)$. Therefore F intersects T_2 in loops of type (t, u), where $pu = qr^2t$, and hence P_3 is a cable space, or a composing space of Type II or III. In both cases, L is classified into (2-5). ∎

Theorem 3.2 *Let L be a non-splittable graph link and E the exterior of L. Suppose that no composing space is obtained by the JSJ decomposition of E. For any closed essential*

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Table 2: Surface fiber types in cable spaces.

$\chi(F_0)$	$(k, n, \lvert p\rvert)$
-1	$(2, 1, 2)$
-2	$(3, 1, 3), (4, 1, 2)$
-3	$(6, 1, 2), (4, 1, 4), (2, 2, 2)$
-4	$(8, 1, 2), (6, 1, 3), (5, 1, 5)$

surface F in E, the Euler characteristic $\chi(F)$ of F satisfies $\chi(F) \geq 0$ or $\chi(F) \leq -6$. Moreover, $(p, q) \neq (4, 2r)$ for any cable space type (p, q) and for any odd r implies $\chi(F) \geq 0$ or $\chi(F) \leq -10$, and $(p, q) \neq (2, r)$ for any cable space type (p, q) and for any odd r implies $\chi(F) \geq 0$ or $\chi(F) \neq -8$.

Proof Assume that $-8 \leq \chi(F) \leq -2$. Suppose that F is split by a JSJ decomposition of E into essential surfaces in Seifert manifold pieces. Then $\chi(F)$ is the sum of the Euler characteristics $\chi(F_0)$ of the component surfaces F_0. Since F separates E, it separates any piece which F intersects. Therefore, there is no piece which F intersects in an odd number of surface fibers. Hence $-4 \leq \chi(F_0) \leq -1$ for any component surface fiber F_0.

Let P be a cable space of type (np, nq), where $\gcd(p, q) = 1$ and $|p| \geq 2$. The orbit manifold O of P is a disk with n holes and an exceptional point C of order $|p|$. We may consider a surface fiber F_0 in P to be a k-fold branched cover of O branched over C, where k is a multiple of $|p|$. This implies $\chi(F_0) = k(1 - n|p|)/|p|$. We say F_0 is of type $(k, n, |p|)$. For example, $\chi(F_0) = -1$ implies $k/|p| = 1$ and $n|p| = 2$ because $k/|p| > 0$ and $(1 - n|p|) < 0$ are integers, and therefore $(k, n, |p|) = (2, 1, 2)$. The possible types of F_0 are listed in Table 2.

Let P be a torus link space of type (np, nq), where $\gcd(p, q) = 1$, $|p| \geq 2$ and $|q| \geq 2$. The orbit manifold O of P is a disk with $n - 1$ holes and two exceptional points C_1 and C_2 of order $|p|$ and $|q|$. Then a surface fiber F_0 in P is a k-fold branched cover of O branched over C_1 and C_2, where k is a multiple of $|pq|$. Therefore $\chi(F_0) = k(|p|+|q|-n|pq|)/|pq|$. We say F_0 is of type $(k, n, |p|, |q|)$. For example, $\chi(F_0) = -1$ implies $k/|pq| = 1$ and $(|p|-1)(|q|-1)+(n-1)|pq| = 2$. Since $(|p|-1)(|q|-1) = 1$ implies $|p| = |q| = 2$ which contradicts $\gcd(p, q) = 1$, we have $(|p| - 1)(|q| - 1) = 2$ and therefore $(k, n, |p|, |q|) = (6, 1, 2, 3)$ or $(6, 1, 3, 2)$. The possible types of F_0 are listed in Table 3.

Assume that F contains a surface fiber F_1 of type $(2, 2, 2)$ in a cable space P_1. Clearly, P_1 is of type $(4, 2r)$ where r is odd. Table 2 implies $\chi(F_1) = -3$. Then F intersects P_1 in two surface fibers and therefore $\chi(F) = -6$ or -8. Assume that F contains a surface fiber F_2 of type $(6, 1, 2, 3)$ or $(6, 1, 3, 2)$ in a torus knot space P_2. Since F_2 is bounded by a preferred longitude on ∂P_2, Table 1 implies that F_2 is connected to no essential annulus in a cable space over ∂P_2. Therefore, ∂P_2 is the outer torus of P_1, and two surface fibers in P_2 is connected separately to two surface fibers in P_1. Hence F_1 intersects ∂P_2 in a preferred longitude. This is impossible by Table 1. Consequently $\chi(F) = -8$ implies that F contains a surface fiber of type

Table 3: Surface fiber types in torus link spaces.

$\chi(F_0)$	$(k, n, \|p\|, \|q\|)$
-1	$(6, 1, 2, 3), (6, 1, 3, 2)$
-2	$(12, 1, 2, 3), (12, 1, 3, 2)$
-3	$(18, 1, 2, 3), (18, 1, 3, 2), (10, 1, 2, 5), (10, 1, 5, 2)$
-4	$(24, 1, 2, 3), (24, 1, 3, 2)$

$(2, 1, 2)$, *i.e.*, F intersects a cable space of of type $(2, r)$ where r is odd.

Assume that F contains no surface fiber of type $(2, 2, 2)$. Then any component surface fiber of F appears in a torus knot space or a cable space bounded by two tori. If F contains a surface fiber F_1 in a torus knot space P_1, F_1 intersects $T_1 = \partial P_1$ in a longitude. Table 1 implies that F_1 is connected over T_1 to a surface fiber F_2 in a cable space P_2 and that F_2 intersects the inner torus T_2 of P_2 in a longitude by the argument presented for the proof of Theorem 3.1. Repeating this argument three times, we obtain $\chi(F) < -8$ and a contradiction occurs. If F contains no surface fiber in a torus knot space, F contains an essential annulus F_1 in a piece P_1 bounded by (possibly not preferred) longitudes in the inner torus T_1 of P_1. Applying the argument presented in the previous case, a contradiction occurs again. ■

Now we show an example in the case $\chi(F) = -6$. Let L be an iterated torus link obtained from a $(4, 22)$-cable of a torus knot of type $(2, 3)$ by taking a $(2, 45)$-cable of a component and a $(3, 64)$-cable of the another. The exterior of L is split by a JSJ decomposition into a torus knot space P_1 of type $(2, 3)$, a cable space P_2 of type $(4, 22)$, a cable space P_3 of type $(2, 45)$ and a cable space P_4 of type $(3, 64)$. Denote by T_1 the outer torus of P_2, and by T_2 and T_3 the inner tori of P_2. Suppose that P_2 is connected to P_3 and P_4 over T_2 and T_3 respectively. Clearly P_2 admits a surface bundle structure over $S^1 = \{e^{2\pi\theta i} | \theta \in \mathbb{R}\}$ with a surface fiber intersecting each T_i in a meridian. Denote by F_θ the surface fiber of level $e^{2\pi\theta i}$. Then $\chi(F_\theta) = -3$. Take disjoint essential annuli A_1, A_2, A_3, and A_4 in P_2 so that any F_θ intersects each A_i in essential arc, A_1 joins T_1 and T_2, and each of A_2, A_3, and A_4 joins T_2 and T_3. By a cutting and pasting technique illustrated in Figure 3, we obtain a surface fiber F'_θ of Euler characteristic -3 which joins links on T_1, T_2, and T_3 of types $(2, 12), (4, 90)$, and $(-3, -64)$ respectively. By glueing essential annuli in P_1, P_3, and P_4 to $F'_0 \cup F'_{1/2}$ along their boundary loops, we obtain a closed essential surface of Euler characteristic -6.

4 Graph Knot Case

In this section, we focus on the graph knot case. Note that any non-trivial graph knot exterior is split by a JSJ decomposition into a torus knot space, a cable space bounded by two tori, or a composing space of Type I.

Theorem 4.1 *Let E be a graph knot exterior which contains a closed essential surface F of negative Euler characteristic. Then F is split by a JSJ decomposition of E into essential annuli in composing spaces and surface fibers in cable spaces.*

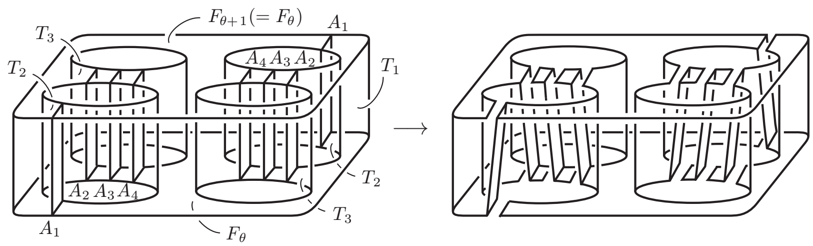

Figure 3: Cutting and pasting of surface fibers.

Proof We may consider F split into essential surfaces in Seifert manifold pieces.

Assume that F intersects an inner torus T_1 of a piece P_1 in non-meridional non-longitudinal loops. Table 1 implies that P_1 is a cable space or a composing space which F intersects in surface fibers, otherwise F intersects P_1 in meridians or longitudes. A surface fiber in a cable space of type (p,q) joins a link on the inner torus of type $(\lambda, \mu + pq\lambda)$ and a link on the outer torus of type $(|p|\lambda, (\mu + pq\lambda)/|p|)$. If λ and $\mu + pq\lambda$ are non-zero integers such that $\mu + pq\lambda$ is not a multiple of λ, $|p|\lambda$ and $(\mu + pq\lambda)/|p|$ are non-zero integers such that $(\mu + pq\lambda)/|p|$ is not a multiple of $|p|\lambda$. Furthermore, a surface fiber in an n-fold composing space joins a link on the inner torus of type $(\lambda, \sum_{i=1}^{n} \mu_i)$ and links on the outer tori of type (λ, μ_i) for $1 \leq i \leq n$. If λ and $\sum_{i=1}^{n} \mu_i$ are non-zero integers such that $\sum_{i=1}^{n} \mu_i$ is not a multiple of λ, some μ_i is a non-zero integer which is not a multiple of λ. In both cases, we can find an outer torus T_2 of P_1 which F intersects in non-meridional non-longitudinal loops. Suppose that P_1 is connected over T_2 to a piece P_2. Then P_2 is a cable space or a composing space which F intersects in surface fibers as before. Repeating this argument, we obtain an infinite sequence of the pieces, which contradicts the compactness of E.

Assume that F intersects an outer torus T_1 of a piece P_1 in longitudes. Table 1 implies that P_1 is a cable space or a composing space which F intersects in surface fibers, otherwise F cannot intersect T_1 in longitudes. Denote by T_2 the inner torus of P_1. If P_1 is a cable space, the argument presented in the proof of Theorem 3.1 implies that F intersects T_2 in longitudes. If P_1 is an n-fold composing space, the first half of this proof implies that F intersects any component of ∂P_1 in meridians or in longitudes, and therefore Table 1 implies that F intersects T_2 in longitudes. In both cases, P_1 is connected over T_2 to a cable space or a composing space P_2 which F intersects in surface fibers. Repeating this argument, we obtain an infinite sequence of the pieces and a contradiction occurs again.

Consequently, F intersects the splitting tori in meridians. Hence the theorem follows from Table 1. ■

Let K be a connected sum of a trefoil knot and a cable of a granny knot. The exterior E of K is split into a cable space and two composing spaces. Figure 4 illustrates an essential surface F in E of Euler characteristic -2. One easily checks that F consists of three essential annuli in the composing spaces and two surface fibers, which are twice-punctured disks, in the cable space.

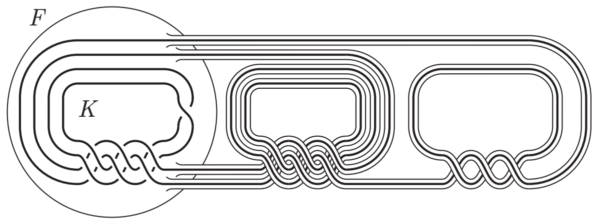

Figure 4: A genus two essential surface in a graph knot exterior.

Corollary 4.2 *Any iterated torus knot exterior contains no closed essential surface of negative Euler characteristic.*

Proof No composing space is obtained by the JSJ decomposition of the iterated torus knot exterior. Hence this is immediate from Theorem 4.1. ∎

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