ON THE MEAN CONVERGENCE OF MARKOV OPERATORS

by ROBERT E. ATALLA

(Received 21st July 1973)

1. Introduction

Throughout the paper, T will be a Markov operator on $C(X)$ (X compact T_2), i.e. a continuous positive operator such that $Te = e$ (e the unit function). P will be the set of Borel probability measures on X, which we shall often think of as linear functionals on $C(X)$, and $P_T = \{m \in P\colon\ T'm = m\}$, where T' is the adjoint of T. Let

$$F = \text{closure} \cup \{\text{carrier } (m)\colon\ m \in P_T\}.$$

Our main result, Theorem 3.1, is the following. Suppose S is a Markov projection on $C(X)$ such that $T_n f \to 0$ iff $Sf = 0$, where $T_n = (1/n)(T + \ldots + T^n)$. Then T and S induce operators T_0 and S_0 on $C(F)$ such that $(T_0)_n f \to S_0 f$ for all $f \in C(F)$.

Our results may be motivated as follows. If T is a continuous operator on a topological vector space and S is a projection, it is an easy exercise to prove that for $T_n x \to Sx$ (all x) to occur, it is necessary and sufficient that

(i) $T_n x \to 0$ iff $Sx = 0$, and

(ii) $ST = TS = S$.

Now (i) easily implies $ST = S$. (We must show that if (i) holds, then $0 = S(T - \mathrm{I})$, or $Tx - x \in$ kernel (S) for all x. But $T_n(Tx - x) \to 0$, so

$$S(Tx - x) = 0,$$

by condition (i).) Thus (ii) may be replaced by (ii)′ $TS = S$. What our main result shows is that in the special case of Markov operators on $C(X)$, (ii) may be discarded altogether, and positive convergence results still obtained.

Section 4 will show how condition (i) arises naturally in the theory of regular matrices.

We shall assume throughout the paper that S and T are Markov operators such that

$$\text{(I)} \quad T_n f \to 0 \text{ iff } Sf = 0 \quad (f \in C(X)).$$

2. Preliminaries

This section is devoted to lemmas needed for the main result in Section 3. The following notation will be used. If $x \in X$, let t_x be the element of P representing the functional $f \to Tf(x)(f \in C(X))$. Thus, $Tf(x) = \int f dt_x$. Likewise, $Sf(x) = \int f ds_x$. Recall that $P_T = \{m \in P\colon\ T'm = m\}$.

We shall assume throughout this section that S and T are operators satisfying condition (I) of the Introduction.

Remark 2.1. We proved in the Introduction that condition (I) implies that $ST = S$, so that for each $x \in X$ and $f \in C(X)$ we have $\int f ds_x = \int Tf ds_x$. Thus, $S'\delta_x = s_x \in P_T$ (where δ_x is the point mass at x), and hence $S'P \subset P_T$.

Our first result shows that if $S^2 = S$, then $S'P = P_T$. The condition $S^2 = S$ is the hypothesis for all results after the first, although it may be of interest to note that the results are valid under the weaker assumption that $S'P = P_T$.

Lemma 2.2. *(a) $S^2 = S$ iff $P_T = P_S$.*

(b) Hence if $S^2 = S$, then $S'P = P_T$.

Proof. (*a*) First we note that in any case, $P_S \subset P_T$. To show this, recall that $ST = S$ (Remark 2.1). If $m \in P_S$, then $T'm = T'S'm = S'm = m$, i.e. $m \in P_T$.

Assume $S^2 = S$. If $f \in C(X)$, then $Sf - f \in$ kernel (S), so by condition (I) we have $T_n(Sf-f) \to 0$. If $m \in P_T$, then $m(Sf-f) = m(T_n(Sf-f)) \to 0$, so $m(Sf) = m(f)$, i.e. $m \in P_S$. Thus, $P_T = P_S$.

Assume $P_T = P_S$. Let $x \in X$. Since $ST = S$, we have $\int Tf ds_x = \int f ds_x$ for all $f \in C(X)$, and hence $s_x \in P_T = P_S$, whence $S's_x = s_x$. This implies $S^2 = S$.

(*b*) Assume $S^2 = S$. We know that $S'P \subset P_T$. If $m \in P_T$, then by (*a*), $S'm = m$, so $m \in S'P$. Thus $S'P = P_T$.

Lemma 2.3. *Assume $S^2 = S$. Then (a) all extreme points of P_T have the form s_x for some $x \in X$, and (b)*

$$F = \mathrm{cl} \cup \{\mathrm{car}\,(s_x) : s_x \in (P_T)^e\},$$

where $(P_T)^e$ is the set of extreme points of P_T.

Proof. (*a*) If $m \in (P_T)^e$, then since $S'P = P_T$ (by Lemma 2.2 (*b*)), the set $(S')^{-1}(m) = \{\rho \in P: S'\rho = m\}$ is a non-empty face of P, and hence contains an extreme point (by, e.g., Theorem 9 on page 9 of (3)). Since all extreme points of P have the form $\delta_x (x \in X)$, there exists $x \in X$ with $m = S'\delta_x = s_x$.

(*b*) Let F_0 be the set in question. Clearly, $F_0 \subset F$. Suppose $y \in F \backslash F_0$. Then there exists a non-negative $f \in C(X)$ with $f = 0$ on F_0 and $f(y) > 0$. Then $m(f) > 0$ for some $m \in P_T$, while (*a*) implies $\rho(f) = 0$ for each $\rho \in (P_T)^e$. But this is impossible, by the Krein-Milman theorem. Hence $F_0 = F$.

Lemma 2.4. *Assume $S^2 = S$. Let $F_T(F) = \{f \in C(X) : Tf \mid F = f \mid F\}$. Then $S(C(X)) \subset F_T(F)$.*

Proof. Since $S^2 = S$, it follows that $s_x \in P_S$ for each $x \in X$. Furthermore, Lemma 2.2 implies $P_T = P_S$, and Lemma 2.3 (*a*) implies that each extreme point of P_S has the form s_x for some $x \in X$. Let s_x be a fixed extreme point of P_S. By Theorem 1.11 of (4), car (s_x) is contained in an " S-ergodic set ", i.e. a set of constancy for any $g \in C(X)$ such that $Sg = g$. Since $S^2 = S$, it follows that Sf is constant (say t) on car (s_x) for any $f \in C(X)$. If $y \in$ car (s_x), then since $s_x \in P_T$ we have car $(t_y) \subset$ car (s_x) (**4**, Theorem 1.3), and hence

$$T(Sf)(y) = t = Sf(y).$$

Lemma 2.3 (*b*) implies $T(Sf) \mid F = Sf \mid F$. Hence $Sf \in F_T(F)$ for each $f \in C(X)$.

Remark 2.5. The preceding proof suggests the following: let S be a Markov projection on $C(X)$. If $Sf = 0$, then $S(fSg) = 0$ for all $g \in C(X)$.

Proof. Let s_x be an extreme point of $P_S = S'P$. Then for each $g \in C(X)$, Sg is constant (say t) on car (s_x), and hence

$$S(fSg)(x) = \int f(w)Sg(w)ds_x(w) = t \int f(w)ds_x(w) = 0.$$

By Krein-Milman, $S(fSg)(y) = \int f(w)Sg(w)ds_y(w) = 0$ for all $y \in X$, since each $s_y \in P_S$.

Lemma 2.6. *Assume* $S^2 = S$. *If* $f = 0$ *on* F, *then* $Tf = 0$ *on* F, *and* $Sf = 0$ *everywhere.*

Proof. To show $Tf = 0$ on F we note that since $s_x \in P_T$ for each x, car (s_x) is *T-self-supporting* (**4**, Theorem 1.3), and hence, by (**4**, Theorem 1.1), if $g = 0$ on car (s_x), then $Tg = 0$ on car (s_x). Thus, since $f = 0$ on F we have $Tf = 0$ on each car (s_x), and hence $Tf = 0$ on F, by Lemma 2.3 (*b*).

To show $Sf(x) = 0$ for all x, it is enough to show car $(s_x) \subset F$ for each $x \in X$. By Remark 2.1, $s_x \in P_T$ for each $x \in X$, and hence car $(s_x) \subset F$ by the definition of F.

3. Mean convergence theorem

In this section we assume condition (I), and also that $S^2 = S$. By Lemma 2.6, $f \mid F = 0$ implies $Tf \mid F = 0$ and $Sf \mid F = 0$. We define T_0 on $C(F)$ by $T_0 f(x) = T\bar{f}(x)$ for each $f \in C(F)$ *and* $x \in F$, where $\bar{f} \in C(X)$ is any extension of f. Likewise, $S_0 f = S\bar{f}$.

Since $f \mid F = 0$ implies $Tf \mid F = Sf \mid F = 0$, the set F is both T-self-supporting and S-self-supporting, i.e. $x \in F$ implies car $(t_x) \subset F$ and car $(s_x) \subset F$ (**4**, Theorem 1.1). Hence for $f \in C(F)$ and $x \in F$ we have

$$T_0 f(x) = T\bar{f}(x) = \int \bar{f} dt_x = \int f dt_x, \quad \text{and} \quad S_0 f = \int f ds_x.$$

Theorem 3.1. *Assume* (I) $Sf = 0$ *iff* $T_n f \to 0$, *and* (II) $S^2 = S$. *Then* $(T_0)_n f \to S_0 f$ *for each* $f \in C(F)$.

Proof. We first prove the following:

(i) $T_0 S_0 = S_0$,

(ii) $S_0^2 = S_0$, and

(iii) $S_0 f = 0$ implies $(T_0)_n f \to 0$.

https://doi.org/10.1017/S0013091500010324 Published online by Cambridge University Press

For (i), if $f \in C(F)$, then by Lemma 2.4, $S\bar{f} \in F_T(F)$, and hence for each $y \in F$, $T(S\bar{f})(y) = S\bar{f}(y)$, or $T_0(S_0 f)(y) = S_0 f(y)$.

For (ii), if $f \in C(F)$, $S_0{}^2 f = S^2 \bar{f} \mid F = S\bar{f} \mid F = S_0 f$.

For (iii), let $x \in X$. Then since $S\bar{f} = 0$ on car (s_x) $(\subset F)$, we have

$$S\bar{f}(x) = S^2\bar{f}(x) = \int S\bar{f} ds_x = 0.$$

By condition (I), $(T_0)_n f = T_n \bar{f} \mid F \to 0$.

We now show that if $f \in C(F)$, then $(T_0)_n f \to S_0 f$. Since $S_0{}^2 = S_0$, we have $S_0 f - f \in$ kernel (S_0), and hence (iii) implies $(T_0)_n(S_0 f - f) \to 0$. Using (i), we have $(T_0)_n f - S_0 f = (T_0)_n(f - S_0 f) \to 0$.

Theorem 3.2. *Suppose the conditions of Theorem* 3.1 *are satisfied, and that T also satisfies the condition*

$$\text{car } (t_x) \subset F \textit{ for all } x \in X.$$

Then $T_n f \to TSf$ for all $f \in C(X)$ (and hence TS is a projection).

Proof. By Theorem 3.1 we have $T_n f \mid F \to Sf \mid F$ for each $f \in C(X)$. Let $R_n = (1/n)(I + T + \ldots + T^{n-1})$, so $T_n = TR_n$. Clearly, $R_n f \mid F \to Sf \mid F$ for each $f \in C(X)$. If $x \in X$,

$$T_n f(x) = T(R_n f)(x) = \int_F R_n f dt_x$$
$$\to \int_F Sf dt_x = T(Sf)(x),$$

by dominated convergence. By Theorem 1.1 of (**2**), $T_n f \to T(Sf)$ uniformly.

4. Final remarks

We conclude with a remark on how condition (I) arises naturally in matrix summability. Let $T = (t_{mn})$ be a regular matrix, considered as a linear operator on $C^*(N)$, and m_T the set of T-invariant means on $C^*(N)$, as defined in (**1**). $f \in C^*(N)$ is *T-almost convergent* if $\rho_1(f) = \rho_2(f)$ for all $\rho_1, \rho_2 \in m_T$. Let V_T be the space of T-almost convergent functions. By a well-known result of G. G. Lorentz, if T is the shift matrix ($Tf(n) = f(n+1)$), then V_T is not equal to the bounded convergence field C_S of any regular matrix S. It is natural to ask under what circumstances the equation $V_T = C_S$ is possible. (This question has been asked by J. P. Duran. To the author's knowledge it remains open.)

As in (**1**), let T_1 and S_1 be the operators induced on $C(\beta N \backslash N)$ by the matrices T and S. Then it can be shown that the condition $V_T = C_S$ is equivalent to (I) $(T_1)_n f \to 0$ iff $S_1 f = 0$ ($f \in C(\beta N \backslash N)$). (Necessity is easy, and sufficiency follows from an easy generalisation of the Mazur-Orlicz consistency theorem.)

REFERENCES

(1) R. ATALLA, On the inclusion of a bounded convergence field in the space of almost convergent sequences, *Glasgow Math. J.* **13** (1972), 82-90.

(2) B. JAMISON, Ergodic decompositions induced by certain Markov operators, *Trans. Amer. Math. Soc.* **117** (1965), 451-468.

(3) G. LEIBOWITZ, *Lectures on complex function algebras* (Scott, Foresman and Company, Glenview, Illinois, 1970).

(4) R. SINE, Geometric theory of a single Markov operator, *Pacific J. Math*· **27** (1968), 155-166.

OHIO UNIVERSITY
ATHENS, OHIO 45701