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DISCRETE WIRTINGER AND ISOPERIMETRIC TYPE INEQUALITIES

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In this paper we give some new types of inequalities which can be regarded as discrete forms of the Wirtinger inequality and several new isoperimetric-type inequalities.

0. INTRODUCTION

The Wirtinger inequality states that if $f(x)$ is a continuous periodic function of period 2π whose derivative $f'(x)$ is also continuous, and $\int_0^{2\pi} f(x)dx = 0$, then

$$\int_0^{2\pi} [f'(x)]^2 dx \geqslant \int_0^{2\pi} [f(x)]^2 dx,$$

with equality if and only if $f(x) = a\cos x + b\sin x$. It is also well-known that the plane isoperimetric inequality is an easy consequence of the Wirtinger inequality (see [4]), that is,

$$L^2 \geqslant 4\pi A,$$

with equality if and only if C is a circle, where L is the length of a simple closed curve C and A is the area it bounds.

In this paper we shall prove some useful new types of inequalities which can be regarded as discrete forms of the Wirtinger inequality. We are able to obtain some new isoperimetric-type inequalities from the discrete Wirtinger type inequalities including a new and easy proof of the plane isoperimetric-type inequality

$$L_n^2 \geqslant 4d_n A_n, \tag{1}$$

where L_n is the length of an n-sided plane polygonal curve c_n, A_n is the area enclosed by c_n and $d_n = n \cdot \tan(\pi/n)$ (see [1, 4]). Moreover, these discrete Wirtinger-type inequalities are very interesting themselves and may be useful in analysis and other branches of mathematics and mathematical physics.

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1. Discrete Wirtinger-type inequalities

In this section we will state and prove some discrete Wirtinger inequalities.

Theorem 1. *Let $f(\theta)$ be a C^2-function, $0 \leqslant \theta \leqslant l$, such that $f(\theta) > 0$ and $f'(\theta) > 0$ over $(0, l)$. Suppose $0 < \theta_i < l$, for $i = 1, \ldots, n$; $\sum_{i=1}^{n} \theta_i = ml$, for some positive constant m with $m < n$, and*

$$[f'(\theta)]^2 - f(\theta) \cdot f''(\theta) = \mu,$$

for $0 < \theta < l$, where μ is a positive constant and $\mu > [f'(\theta)]^2$. Then

$$\left(\sum_{j=1}^{n} f(\theta_j)\right)^2 \geqslant c_n \sum_{j=1}^{n} f(\theta_j) \cdot f'(\theta_j),$$

where $c_n = n \cdot f(ml/n)/f'(ml/n)$. Equality holds if and only if $\theta_1 = \ldots = \theta_n = ml/n$.

Proof: Set $L_n = \sum_{j=1}^{n} f(\theta_j)$ and

$$F(\theta_1, \cdots, \theta_n) = L_n^2 - c_n \cdot \sum_{j=1}^{n} f(\theta_j) \cdot f'(\theta_j).$$

It suffices to show that $F(\theta_1, \cdots, \theta_n) \geqslant 0$, or 0 is the minimum of F. By the method of Lagrange multipliers, if $F(\theta_1, \cdots, \theta_n)$ has a maximum or minimum at a point $P_0 = (\overline{\theta}_1, \cdots, \overline{\theta}_n)$, then

$$\nabla F(\overline{\theta}_1, \cdots, \overline{\theta}_n) = \lambda \cdot \nabla g(\overline{\theta}_1, \cdots, \overline{\theta}_n),$$

for some constant λ, where $g(\theta_1, \cdots, \theta_n) = \sum_{j=1}^{n} \theta_j$, and ∇f is the gradient of the function f, hence

$$2L_n f'(\theta_j) - c_n\{[f'(\theta_j)]^2 + f(\theta_j) \cdot f''(\theta_j)\} = \lambda, \tag{2}$$

for $j = 1, \cdots, n$.

Now, we shall proceed to verify that there exists a unique critical point $P_0 = (ml/n, \cdots, ml/n)$. In fact, if $P_0 = (\overline{\theta}_1, \cdots, \overline{\theta}_n)$ is a critical point, then from (2) we have

$$\begin{aligned} &2L_n[f'(\overline{\theta}_j) - f'(\overline{\theta}_i)] \\ &\quad = c_n\{[f'(\overline{\theta}_j)]^2 + f(\overline{\theta}_j) \cdot f''(\overline{\theta}_j) - [f'(\overline{\theta}_i)]^2 - f(\overline{\theta}_i) \cdot f''(\overline{\theta}_i)\}. \end{aligned}$$

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By using $f(\overline{\theta}_j)\cdot f''(\overline{\theta}_j) = [f'(\overline{\theta}_j)]^2 - \mu$, we have

$$2L_n[f'(\overline{\theta}_j) - f'(\overline{\theta}_i)] = 2c_n\{[f'(\overline{\theta}_j)]^2 - [f'(\overline{\theta}_i)]^2\}.$$

As $f''(\theta) = \{[f'(\theta)]^2 - \mu\}/f(\theta) < 0$ on $0 < \theta < l$, $f'(\theta)$ is strictly decreasing over $(0, l)$, and hence $f'(\theta)$ is one-to-one over $(0, l)$. Thus, if $\overline{\theta}_i \neq \overline{\theta}_j$, we have

$$L_n = c_n[f'(\overline{\theta}_j) + f'(\overline{\theta}_i)]. \tag{3}$$

If there are at least three distinct $\overline{\theta}_i$'s, say, $\overline{\theta}_1$, $\overline{\theta}_2$ and $\overline{\theta}_3$, then from (3), one has

$$f'(\overline{\theta}_1) + f'(\overline{\theta}_2) = f'(\overline{\theta}_1) + f'(\overline{\theta}_3),$$

that is, $f'(\overline{\theta}_2) = f'(\overline{\theta}_3)$, which is impossible. It remains to show that we cannot have exactly two distinct values for $\overline{\theta}_k$'s. If not, let us assume that

$$\overline{\theta}_1 = \cdots = \overline{\theta}_k = \varphi < \overline{\theta}_{k+1} = \cdots = \overline{\theta}_n = \psi.$$

It follows from (3) that

$$kf(\varphi) + (n-k)f(\psi) = c_n[f'(\varphi) + f'(\psi)], \tag{4}$$

since $k\varphi + (n-k)\psi = ml$, $\psi = (ml - k\varphi)/(n-k) < 1$, and $0 \leqslant \varphi \leqslant ml/n < l$. Set

$$H(\theta) = kf(\theta) + (n-k)f\left(\frac{ml - k\theta}{n-k}\right),$$

and

$$K(\theta) = c_n\left[f'(\theta) + f'\left(\frac{ml - k\theta}{n-k}\right)\right].$$

Let us first study the function H. Clearly

$$H'(\theta) = k\left[f'(\theta) - f'\left(\frac{ml - k\theta}{n-k}\right)\right],$$

and since $f''(\theta) < 0$, we have

$$H''(\theta) = k\left[f''(\theta) + \frac{k}{n-k}f''\left(\frac{ml - k\theta}{n-k}\right)\right] < 0, \quad \text{for } 0 < \theta < l,$$

Hence $H'(\theta)$ is strictly decreasing over $(0, l)$.

Observe that

$$H'(0) = k\left[f'(0) - f'\left(\frac{ml}{n-k}\right)\right] > 0,$$

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because $f'(\theta)$ is strictly decreasing, and

$$H'(ml/n) = k[f'(ml/n) - f'(ml/n)] = 0.$$

This implies that $H'(\theta) > 0$ over $(0, ml/n)$; hence H is increasing, concave down over the interval $[0, ml/n]$, and

$$H(ml/n) = nf(ml/n).$$

Next, let us examine the function K. As $f(\theta)$ is a C^2-function, $f'''(\theta)$ exists, and

$$\begin{aligned} f'''(\theta) &= \frac{2f'(\theta)f''(\theta)f(\theta) - \{[f'(\theta)]^2 - \mu\} \cdot f'(\theta)}{[f(\theta)]^2} \\ &= \frac{f'(\theta)f''(\theta)}{f(\theta)} < 0 \end{aligned}$$

because $f(\theta) > 0$, $f'(\theta) > 0$ and $f''(\theta) < 0$ over $(0, l)$. Thus

$$K'(\theta) = c_n\left[f''(\theta) + \left(\frac{-k}{n-k}\right)f''\left(\frac{ml - k\theta}{n-k}\right)\right],$$

and

$$K''(\theta) = c_n\left[f'''(\theta) + \left(\frac{k}{n-k}\right)^2 \cdot f'''\left(\frac{ml - k\theta}{n-k}\right)\right] < 0.$$

Hence $K(\theta)$ is also concave down over $[0, ml/n]$, and

$$\begin{aligned} K(0) &= c_n\left[f'(0) + f'\left(\frac{ml}{n-k}\right)\right] \\ &= \frac{nf(ml/n)}{f'(ml/n)}\left[f'(0) + f'\left(\frac{ml}{n-k}\right)\right] \\ &> nf(ml/n) = H(ml/n), \end{aligned}$$

since $[f'(0) + f'(ml/(n-k))]f'(ml/n) > 1$, $K(ml/n) = c_n[f'(ml/n) + f'(ml/n)] = 2n\,f(ml/n) > H(ml/n)$. Thus by comparing the graphs of both functions H and K over $[0, ml/n]$, we can see that $H(\theta) < K(\theta)$. This implies

$$kf(\varphi) + (n-k)f(\psi) < c_n[f'(\varphi) + f'(\psi)],$$

which contradicts (4). This proves that we have a unique critical point $P_0 = (ml/n, \cdots, ml/n)$.

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Finally, we prove that $F(P_0) = 0$ is the minimum. To prove this, observe that we have $\partial g/\partial\theta_i = 1$, for $i = 1, \cdots, n$, and

$$\frac{\partial^2 F}{\partial\theta_i^2}\Big|_{P_0} = 2\left[f'\left(\frac{ml}{n}\right)\right]^2 - 2n\,f\left(\frac{ml}{n}\right)\cdot f''\left(\frac{ml}{n}\right),$$

and
$$\frac{\partial^2 F}{\partial\theta_i\partial\theta_j}\Big|_{P_0} = 2\left[f'\left(\frac{ml}{n}\right)\right]^2, \quad \text{for } i \neq j$$

because
$$\frac{\partial^2 F}{\partial\theta_i^2} = 2\,[f'(\theta_i)]^2 + 2L_n\cdot[f''(\theta_i)] - c_n\cdot[4f'(\theta_i)f''(\theta_i)],$$

and
$$\frac{\partial^2 F}{\partial\theta_i\partial\theta_j} = 2f'(\theta_i)f'(\theta_j), \quad \text{for } i \neq j.$$

Set $\partial^2 F/\partial\theta_i^2\,|_{P_0} = a$ and $\partial^2 F/\partial\theta_i\partial\theta_j\,|_{P_0} = b$. Then $a > b$ because $f(ml/n) > 0$ and $f''(ml/n) < 0$. A simple computation will show that for $2 \leqslant r \leqslant n$ we have at P_0

$$\begin{vmatrix} 0 & \partial g/\partial\theta_1 & \cdots & \partial g/\partial\theta_r \\ \partial g/\partial\theta_1 & \partial^2 F/\partial\theta_1^2 & \cdots & \partial^2 F/\partial\theta_1\partial\theta_r \\ \vdots & \vdots & \ddots & \vdots \\ \partial g/\partial\theta_r & \partial^2 F/\partial\theta_r\partial\theta_1 & \cdots & \partial^2 F/\partial\theta_r^2 \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 & \cdots & 1 \\ 1 & a & b & \cdots & b \\ 1 & b & a & \ddots & b \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & b & \cdots & b & a \end{vmatrix}$$

which has the value $-r(a-b)^{r-1} < 0$. Therefore $F(P_0)$ is a local minimum (see [3]). Since this is the only critical value, it is the global minimum. This completes the proof of the theorem. ∎

We have some variations of Theorem 1. The proofs of them are almost identical with that of Theorem 1. So we just state one of them without proof.

Theorem 1A. *Let $f(\theta)$ be a C^2-function over $[0, l]$, such that $f(\theta) > 0$ and $f'(\theta) < 0$ over $(0, l)$. Suppose $0 < \theta_i < l$, for $i = 1, \cdots, n$; $\sum_{i=1}^{n}\theta_i = ml$, for some positive constant m with $m < n$, and*

$$[f'(\theta)]^2 - f(\theta)\cdot f''(\theta) = \mu,$$

for $0 < \theta < l$, where μ is a positive constant and $\mu > [f'(\theta)]^2$. Then

$$\left(\sum_{i=1}^{n} f(\theta_i)\right)^2 \geqslant c_n\sum_{j=1}^{n} f(\theta_j)\cdot f'(\theta_j),$$

where $c_n = n\cdot f(ml/n)/f'(ml/n)$. Equality holds if and only if $\theta_1 = \cdots = \theta_n = ml/n$.

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2. Isoperimetric-type inequalities

As an application of Theorem 1, we shall give a proof of inequality (1), that is,

$$L_n^2 \geqslant 4d_nA_n,\ d_n = n\cdot\tan(\pi/n). \tag{5}$$

From plane geometry, the n-sided polygon inscribed in a circle has maximal area among all n-sided polygons with given n sides. (see [1]). So it suffices to prove the inequality just for polygons inscribed in a circle.

Let P_n be a polygon inscribed in a circle of radius r. Let a_i be the length of the ith side of P_n and θ_i the half of the centre angle subtended by the ith side of P_n respectively. Then

$$L_n = \sum_{i=1}^n a_i = \sum_{i=1}^n 2r\sin\theta_i = 2r\sum_{i=1}^n \sin\theta_i,$$

and

$$A_n = \sum_{i=1}^n 1/2a_i\cdot r\cdot\cos\theta_i = \sum_{i=1}^n r^2\sin\theta_i\cdot\cos\theta_i.$$

The proof of inequality (5) is reduced to the inequality in the following Proposition 1.

Proposition 1. *Let* $\sum_{i=1}^n \theta_i = \pi$. $0 < \theta_i < \pi/2$, $i = 1, \cdots, n$. *Then*

$$\left(\sum_{i=1}^n \sin\theta_i\right)^2 \geqslant d_n\sum_{i=1}^n \sin\theta_i\cdot\cos\theta_i. \tag{6}$$

where $d_n = n\cdot\tan(\pi/n)$, *with equality if and only if* $\theta_1 = \cdots = \theta_n = \pi/n$.

Proof: Set $f(\theta) = \sin\theta$ and $l = \pi/2$. Then the function f satisfies all the conditions in Theorem 1 with $\mu = 1$, $m = 2$ and $c_n = n\cdot\tan(\pi/n)$. This proves the proposition by Theorem 1. □

Proposition 1 can be generalised. In fact, we have

Proposition 2. *The inequality (6) remains true if* $\sum_{i=1}^n \theta_i = k\pi$, $0 < k < n/2$.

The proof of Proposition 2 is identical with that of Proposition 1.

The geometrical meaning of the inequality of Proposition 2 is interesting. In fact, this inequality will give us the following new isoperimetric-type inequality.

Theorem 2. *Among all polygons* $OA_1\cdots A_{n+1}$ *(see figure below) with* $\theta_1 + \cdots + \theta_n = k\pi$, $0 < k < 1$, *the one which encloses the largest area is the polygon with* $\theta_1 = \cdots = \theta_n = k\pi/n$.

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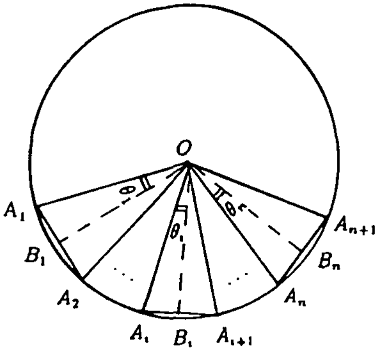

Theorem 2 can be generalised. Let P_n be any polygon $OA_1 \cdots A_{n+1}$ with A_i lying on a sphere S^2 in Euclidean three space (or, more generally, on an m-sphere S^m in Euclidean $m+1$-space), with O the centre of the sphere. Let $\theta_1 + \cdots + \theta_n = k\pi$, $0 < k < n/2$, where θ_i is half of the centre angle subtended by the ith side. Then the same conclusion holds. Here the area of P_n is defined to be the sum of the areas of the triangles $\triangle OA_iA_{i+1}$, $1 \leqslant i \leqslant n$.

Finally, let us observe that if we interchange $\sin\theta_i$ and $\cos\theta_i$ in Proposition 1, then from Theorem 1A we have the following:

Proposition 3. *If* $\sum_{i=1}^{n} \theta_i = k\pi$, $0 \leqslant k \leqslant 1$ *and* $0 < \theta_i < \pi/2$, *then*

$$\left(\sum_{i=1}^{n} \cos\theta_i\right)^2 \geqslant d_n \sum_{i=1}^{n} \sin\theta_i \cdot \cos\theta_i, \tag{7}$$

where $d_n = n \cdot \cot(k\pi/n)$. *Equality holds if and only if* $\theta_1 = \cdots = \theta_n = k\pi/n$.

Proposition 3 also has a geometric interpretation. If we use the notation of Theorem 2, the length of OB_i is equal to $r\cos\theta_i$, $1 \leqslant i \leqslant n$. Hence we obtain another isoperimetric-type inequality.

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