ON SYSTEMS OF STANDARDS*

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Rigorous formulations closely following current practice, are proposed for the absolute and the relative estimation of certain target quantities. A system of standards is defined as a set of estimates that has certain properties. This is followed by a discussion of the meaning of "reducing estimates to a specified system" and certain procedures are suggested for making this process less ambiguous. It is finally shown that relative estimates can be employed for replacing a system which is based on absolute measurements alone by another, more accurate system.

1. ABSOLUTE AND RELATIVE ESTIMATES

There are in principle two ways in which data can be estimated. One of these makes no use of previously existing estimates of like data and leads to *absolute estimates*, the second uses previously existing estimates of like data for calibrating the new data, thus leading to *relative estimates*.

Absolute estimates are obtained in the following way:

Let the *target variable* a have a unique numerical value for each element A of the *object set* $\{A\}$ (if $\{A\}$ were the set of all stars, a might be, e.g., the luminosity.) Consider a set $\{\vec{x}^*\}$ of measurements (that is, estimates which may be regarded as unbiased without restricting generality) of quantities whose true values are $\vec{x}$. Let $\{\vec{p}\}$ be a set of unknown parameters and assume the existence of a model function Φ, that is, a set of functional relationships

$$\Phi(\vec{a}, \vec{x}, \vec{p})=\vec{0} \qquad (1)$$

between the vectors $\vec{a}$, $\vec{x}$, $\vec{p}$. Estimates $\vec{a}^*$ are called *absolute* if it is possible to find estimates $\vec{p}^*$ for $\vec{p}$ simultaneously with $\vec{a}^*$ by solving

$$\Phi(\vec{a}, \vec{x}^*, \vec{p})=\vec{0} \qquad (1a)$$

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for $\vec{a}$ and $\vec{p}$.

Estimates are called *relative* if the pattern for their determination is as follows. Let $\vec{a}_R$ be a vector composed of some components of $\vec{a}$ and let $\vec{a}_R{}^\dagger$ be a previously determined estimate of $\vec{a}_R$. Estimates $\vec{a}^*$ of $\vec{a}$ and $\vec{p}^*$ of $\vec{p}$ are now obtained from solving the relationships

$$\Psi(\vec{a},\ \vec{a}_R{}^\dagger,\ \vec{x}^*,\ \vec{p})=\vec{0} \tag{2}$$

for $\vec{a}$ and $\vec{p}$, where the functions Ψ must be such that the variables represented by $\vec{a}_R{}^\dagger$ cannot be eliminated from them such that $\vec{a}$ and $\vec{p}$ could then still be calculated from the remaining system. (Otherwise, the $\vec{a}^*$ resulting from this calculation would be absolute estimates.)

Very often the estimation of $\vec{a}^*$ and $\vec{p}^*$ can be carried out in steps, as follows. Assuming a relationship of type (1) between $\vec{a}$, $\vec{x}$ and $\vec{p}$, we first find $\vec{p}^*$ from solving

$$\Phi(\vec{a}_R{}^\dagger,\ \vec{x}^*,\ \vec{p})=\vec{0} \tag{3}$$

for $\vec{p}$, and then calculate $\vec{a}^*$ from solving

$$\Phi(\vec{a},\ \vec{x}^*,\ \vec{p}^*)=\vec{0} \tag{1b}$$

for $\vec{a}$.

2. DEFINITION OF A SYSTEM OF STANDARDS

Consider a subset of $\{A\}$, the *standard set* $\{A_S\}$, such that for all of its elements, a set of estimates $\{\vec{a}_S{}^\dagger\}$ of the target variable is available. These estimates will be called a *System of Standards* (or a *system*) if the following conditions are satisfied. There exist non-empty subsets $\{A_{SR}\}$ of $\{A_S\}$ such that the corresponding subsets $\{\vec{a}_{SR}{}^\dagger\}$, of $\{\vec{a}_S{}^\dagger\}$ used in equations of the type (2), will suffice for a nonambiguous estimation of $\vec{a}$ and $\vec{p}$. (This process may yield new estimates also for those elements of $\vec{a}$ for which the estimates $\vec{a}_{SR}^*$ were available.) The estimates a_S^* which one obtains in this process (for any elements of $\{A\}$) are considered to be "on the system $\{a_S{}^\dagger\}$". This definition attempts to give a precise description of current usage. It is clear that the same measurements $\vec{x}^*$ will lead to different sets of estimates $\{\vec{a}_S^*\}$ when the model function Ψ (or Φ), delete or the way in which the subset $\{\vec{a}_{SR}{}^\dagger\}$ is selected from $\{\vec{a}_S{}^\dagger\}$, or both, are open to choice.

3. RESTRICTING THE AMBIGUITIES

One might think that there should be little question about ambiguity in the model function, since relationships between the variables are determined by the physical and geometrical circumstances of the

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situation, including the number and kind of relevant parameters. Remember, however, that any model function is in reality always only a more or less accurate representation of the actual physics and geometry of any situation; witness the fact that linear (or quadratic) interpolation is based on the assumption that in certain restricted circumstances, a linear (or quadratic) relationship between the relevant parameters is for the purpose at hand a sufficiently accurate representation of their relationship. Thus, different - equally defensible - choices for the model function will then lead, - even if nothing else is changed - to different sets of $\vec{a}_S*$. The only way to relieve this undesirable ambiguity is for all investigators of any particular situation to agree on the same model function. This is obviously not likely to happen. Even more so, there will be different sets of parameters $\vec{p}$ associated with different models. The relationships between the sets of parameters belonging to different models is difficult.

Also, - even for identical models - the set(s) of estimates $\{\vec{a}_S*\}$ (and $\{\vec{p}*\}$) which are obtained as the result of the investigation will apparently differ for different sets $\{\vec{a}_{SR}^\dagger\}$. The $\{\vec{a}_S*\}$ and the $\vec{p}*$ will generally be functions of other, fixed parameters which are not adjusted in the course of the calculations. A set of ranges of these parameters defines a region, such that only such components of $\vec{a}_{SR}^\dagger$ whose fixed parameters are within the region in consideration are chosen for a particular adjustment. Changes of the region that become necessary will thus lead to discontinuties in the $\vec{a}_{SR}*$ and the $\vec{p}*$, unless a weighting function is applied in such a way that at the edge of a region, an element of the reference set enters (or leaves) the adjustment with weight zero at the boundary of the region and that, once a region is defined, all elements of $\{\vec{a}_S^\dagger\}$ whose parameters fall within it are included in the adjustment.

Such weighting could, for example, be accomplished as follows. Let q be one of the parameters which defines the region that determines the set $\{A_{SR}\}$, such that all elements of $\{A_S\}$ for which q satisfies $q_0 \leqq q \leqq q_1$ are included in $\{A_{SR}\}$. The weighting function with which an element for which the parameter in question has the value q enters the adjustment could, for example, be

$$w(q)=\left(\frac{4(q-q_0)(q-q_1)}{(q_0-q_1)^2}\right)^2 \qquad (4)$$

which vanishes together with its derivatives at the boundaries ($q=q_0$ and $q=q_1$), and reaches a maximum of 1 at midinterval ($q=\frac{1}{2}(q_0+q_1)$). If the weight of an element of $\{\vec{a}_{SR}^\dagger\}$ in the adjustment within the region which determines $\{A_{RS}\}$ were specified by Eq. (4) (the generalization to more than one variable q is obvious), the estimates $\vec{a}_S*$ for the target variable and $\vec{p}*$ for the reduction parameters would vary continuously and smoothly when this region is changed by a continuous change of its boundaries. If investigators in various areas could agree on standardizing modeling functions and sampling regions, and would adopt one of the several feasible weight functions, the presently existing ambiguities in establishing data estimates on a particular system would be removed. *Until this happens, there can be no precise meaning of the unqualified statement that certain data are on a certain system.*

4. REDUCTION UNCERTAINTIES

Any system consists of a set of estimates. The process by which any estimates are obtained yields, at least in principle, also their covariance matrix Q. (Unfortunately, Q is usually not communicated.) The variance of any function $\phi(\vec{a}^*)$ of the set $\{\vec{a}^*\}$ of estimates is given by $\sigma_\phi{}^2=(\partial\phi/\partial\vec{a}^*)^T Q(\partial\phi/\partial\vec{a}^*)$, that of a function $\psi(\vec{a}^*, \vec{y}^*)$ of these estimates $\vec{a}^*$ and another, with them *uncorrelated* set $\{\vec{y}^*\}$ of statistical variates (e.g., measurements) whose covariance matrix is Y is given by $\sigma_\psi{}^2=\sigma_a{}^2+\sigma_y{}^2$ with $\sigma_a{}^2=(\partial\psi/\partial\vec{a}^*)^T Q(\partial\psi/\partial\vec{a}^*)$ and $\sigma_y{}^2=(\partial\psi/\partial\vec{y}^*)^T Y(\partial\psi/\partial\vec{y}^*)$. The standard error σ_ψ is a certain fraction of the most likely amount by which the function ψ, evaluated at its arguments, will deviate from the "truth"; it is, in fact, inversely proportional to the "average" accuracy of the function ψ. (This fraction depends on the distribution function of the errors involved, for a normal distribution it is about 3/2).

Common usage regards relative estimates, $\vec{a}^*$ obtained from solving a set of equations of the type (2) with respect to $\vec{a}$ (and $\vec{p}$), to be "*on the system*" of $\{\vec{a}_R{}^\dagger\}$. This somehow implies that there should be no systematic differences between two different sets $\{\vec{a}_1{}^*\}$ and $\{\vec{a}_2{}^*\}$ of estimates of the same set of quantities as long as $\{\vec{a}_1{}^*\}$ and $\{\vec{a}_2{}^*\}$ were obtained with the use of the same $\{\vec{a}_R{}^\dagger\}$. Such differences are, however, in fact unavoidable. In order to demonstrate this, we regard two different sets $\{\vec{x}_1{}^*\}$ and $\{\vec{x}_2{}^*\}$ of measurements[1], not necessarily of the same quantities. These are used for the estimation of two sets of estimates $\{\vec{a}_1{}^*\}$ and $\{\vec{a}_2{}^*\}$ of $\{\vec{a}\}$ from adjusting for $\{\vec{a}_1{}^*\}$ and $\{\vec{p}_1{}^*\}$, and $\{\vec{a}_2{}^*\}$ and $\{\vec{p}_2{}^*\}$, respectively, the equations

$$\Phi(\vec{a}, \vec{a}_R{}^\dagger, \vec{x}_1{}^*, \vec{p}_1)=\vec{0}$$

and

$$\Psi(\vec{a}, \vec{a}_R{}^\dagger, \vec{x}_2{}^*, \vec{p}_2)=\vec{0}$$

respectively. From the adjustment based on $\Phi=\vec{0}$ one will get Q_1 and from that based on $\Psi=\vec{0}$, we get Q_2 as the covariance matrices of $\{\vec{a}_1{}^*, \vec{p}_1{}^*\}$ and $\{\vec{a}_2{}^*, \vec{p}_2{}^*\}$, respectively. The covariance matrices P_1 and P_2, respectively, of $\{\vec{a}_1{}^*\}$ and $\{\vec{a}_2{}^*\}$, respectively, are, of course, minors of the corresponding Q_V. The systematic difference $|\phi(\vec{a}_1{}^*, \vec{y}^*) - \phi(\vec{a}_2{}^*, \vec{y}^*)|$ can always be directly computed. Its most likely value is the above-mentioned fraction of the dispersion of this difference (as far as it depends on the differences between the different estimates for the same $\vec{a}$) which is

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$$[(\partial\phi/\partial\vec{a})^T(P_1+P_2)(\partial\phi/\partial\vec{a})]^{\frac{1}{2}},$$

clearly not zero. This shows that in fact, somehow, the two different estimates $\vec{a}_1$* and $\vec{a}_2$*, in spite of entrenched terminology, do not represent the same system. Since one of them ($\vec{a}_1$*, say) could be $\vec{a}_R^\dagger$ (in this case, $\vec{a}_2$* would be simply a reestimation - possibly for the intended purpose of improving precision and accuracy - of the values $\vec{a}$ for the reference set) we see immediately that the "*preservation of the system*", is in the sense a system was defined in this paper as *impossible* a task as is the construction of a perpetuum mobile. This fact should cause investigators to reexamine the often followed practice of carefully "reducing different sets of observations first to the same system" and leaving the determination of the errors (i.e., estimation of the differences against the true values) to a common, later step. The assumption that the system values and the reduced values have (by and large) the same "systematic errors" may well turn out to be an illusion.

5. CONCERNING ACCURACY

Relative estimates of a certain set of target variables can actually be more accurate than those which constitute the reference system to which they were adjusted. In order to show this, we suppose the elements of a system $\{\vec{a}_R^\dagger\}$ were estimated absolutely by solving for $\vec{a}$ (and $\vec{p}$) the equations

$$\Phi(\vec{a},\ \vec{x}^*,\ \vec{p})=\vec{0}. \qquad (5)$$

Let Q_α be the covariance matrix of $\{\vec{a}_R^\dagger\}$ which resulted from this adjustment. Now obtain a set of relative estimates $\{\vec{a}^*\}$ of the same variables from solving for $\vec{a}$ (and $\vec{q}$) the equations

$$\Psi(\vec{a},\ \vec{a}_R^\dagger,\ \vec{y}_1^*\ \vec{q})=\vec{0} \qquad (5)$$

where $\vec{y}^*$ are certain measurements and $\vec{q}$ certain parameters. Consider a function $\phi(\vec{a})$. Denote the variances of $\phi(\vec{a}_R^\dagger)$ and $\phi(\vec{a}^*)$ by $\sigma_\dagger^2$ and σ_*^2, respectively. It is now quite feasible that, given such a pair $\{\vec{a}_R^\dagger\}$ and $\{\vec{a}^*\}$, $\sigma_\dagger^2>\sigma_*^2$ for most ϕ, indicating that the estimates $\{\vec{a}^*\}$ are more accurate (there is no question that they can easily be more precise) that the $\{\vec{a}_R^\dagger\}$. This seems strange at first, because one is tempted to argue that the errors introduced by the measurements $\vec{y}^*$ are added, in a way, to the errors originating from the $\vec{a}_R^\dagger$ so that the errors of the a* must be at least as large as those of the $\vec{a}_R^\dagger$. Consider, however, that Eq. (6) may, - while incapable of determining $\vec{a}$ without $\vec{a}_R^\dagger$ - subject the $\vec{a}$ to constraints not enforcable by Eq. (5) because of the way in which the $\vec{x}^*$ were obtained. Consider, for example, a set $\vec{a}_R^\dagger$ which are available just as a series of pairs of numbers, while new measurements $\vec{y}^*$ allow one to enforce the constraint that the number pairs which constitute the $\vec{a}$, if interpreted as coordinates in a plane, must lie on a straight line. This powerful new constraint may well make the relative estimates $\{\vec{a}^*\}$ "on the system of the $\{\vec{a}_R^{\dagger}\}$" more accurate than the elements of the system. Depending on the circumstances, there may or may not be significant systematic differences between $(\vec{a}_R^{\dagger})$ and $(\vec{a}^*)$; examples for both these possibilities could be quoted from the literature. In any case, however, these considerations indicate that the imposition of newly available constraints through additional sets of pertinent measurements may well generate a new system consisting of relative estimates which is more accurate and precise than the original system of absolute estimates to which the new system was reduced.

Which of two concrete systems is more accurate can be decided only by calculating the covariance matrices of the estimates which constitute the systems. This is usually connected with a large computational effort and seldom, if ever, properly done. Therefore, one finds frequently misjudgments on the basis of incomplete information.

NOTES

1. One might also define a measurement as an estimate that is directly obtained by reading the scale of a measuring device without any calculation involving the estimation of additional parameters. Note that according to this definition, individual readings on different settings of an experimental setup on the same object, which will differ from each other only because of the measuring errors, are measurements, while their average is not, because of the measuring errors, are measurements, while their average is not, because it is an estimate computed by taking the mean of different estimates of the same quantity.

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