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HARDY-TYPE INEQUALITIES FOR MEANS

Zsolt Páles and Lars-Erik Persson

In this paper we consider inequalities of the form

$$\sum_{n=1}^{\infty} M(x_1, \dots, x_n) \leqslant C \sum_{n=1}^{\infty} x_n,$$

where M is a mean. The main results of the paper offer sufficient conditions on M so that the above inequality holds with a finite constant C. The results obtained extend Hardy's and Carleman's classical inequalities together with their various generalisations in a new direction.

1. Introduction

Hardy's celebrated inequality states that, for $p > 1$,

$$\sum_{n=1}^{\infty} \Big(\frac{x_1 + \cdots + x_n}{n}\Big)^p \leqslant \Big(\frac{p}{p-1}\Big)^p \sum_{n=1}^{\infty} x_n^p, \tag{1}$$

for all nonnegative sequences (x_n). In integral form it was stated and proved in [**9**] but it was also pointed out that this discrete form follows from the integral version. Hardy's original motivation was to get a simple proof of Hilbert's inequality, see the special cases proved in [**7, 8**]. It is almost impossible to summarise the enormous literature concerning the generalisations and extensions of this inequality. We recommend the books [**12, 14**], and [**16**] and the historical survey paper [**13**] on this subject for the interested readers.

In this paper, we follow a new approach in generalising Hardy's inequality. The main idea is to rewrite (1) in terms of power means and then to replace them by more general means.

Replacing x_n by $x_n^{1/p}$ and p by $1/p$ in (1), we get that

$$\sum_{n=1}^{\infty} \Big(\frac{x_1^p + \cdots + x_n^p}{n}\Big)^{1/p} \leqslant \Big(\frac{1}{1-p}\Big)^{1/p} \sum_{n=1}^{\infty} x_n \tag{2}$$

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for $0 < p < 1$. As it was proved by Knopp [11], this inequality is also valid for $p < 0$. Taking the limit $p \to 0$, Carleman's inequality can also be derived:

$$\sum_{n=1}^{\infty} \sqrt[n]{x_1 \cdots x_n} \leqslant e \sum_{n=1}^{\infty} x_n. \tag{3}$$

For some further developments and historical remarks concerning (3), we refer to [20].

Define for $p \in \mathbb{R}$ the pth power (or Hölder) mean of the positive numbers $x_1, \ldots, x_n$ by

$$\mathcal{P}_p(x_1, \ldots, x_n) := \begin{cases} \left(\frac{x_1^p + \cdots + x_n^p}{n}\right)^{1/p} & \text{if } p \neq 0, \\ \sqrt[n]{x_1 \cdots x_n} & \text{if } p = 0. \end{cases} \tag{4}$$

The power mean $\mathcal{P}_1$ is the arithmetic mean which will also be denoted by $\mathcal{A}$ in the sequel.

Now observe that (2) and (3) are particular cases of the inequality

$$\sum_{n=1}^{\infty} M(x_1, \ldots, x_n) \leqslant C \sum_{n=1}^{\infty} x_n, \tag{5}$$

if M is the pth power mean $\mathcal{P}_p$ with parameter $p < 1$ and C is the constant $c_p := (1-p)^{-1/p}$ if $p \neq 0$ and $c_0 := e$ if $p = 0$. The aim of this paper is to study (5) for means M that are more general than power means. A function M is said to be a *mean on* $\mathbb{R}_+$ if it is a real valued function defined on the set $\bigcup_{n=1}^{\infty} \mathbb{R}_+^n$ such that, for all $n \in \mathbb{N}$, $x_1, \ldots, x_n > 0$,

$$\min(x_1, \ldots, x_n) \leqslant M(x_1, \ldots, x_n) \leqslant \max(x_1, \ldots, x_n)$$

holds. In the sequel, a mean M will be called a *Hardy mean* if there exists a positive real constant C such that (5) holds for all positive sequences $x = (x_n)$. Due to the Hardy, Carleman, and Knopp inequalities, the pth power mean is a Hardy mean if $p < 1$. One can easily see that the arithmetic mean is not a Hardy mean, therefore the the following result holds.

THEOREM A. *Let $p \in \mathbb{R}$. Then, the power mean $\mathcal{P}_p$ is a Hardy mean if and only if $p < 1$.*

The notion of power means is generalised by the notion of *quasi-arithmetic means* (see [10]): If $f : \mathbb{R}_+ \to \mathbb{R}$ is continuous strictly monotonic function then the quasi-arithmetic mean $\mathcal{M}_f$ is defined by

$$\mathcal{M}_f(x_1, \ldots, x_n) := f^{-1}\left(\frac{f(x_1) + \cdots + f(x_n)}{n}\right). \tag{6}$$

By taking f as a power function or a logarithmic function, the resulting quasi-arithmetic mean will be a power mean. More surprisingly, the power means can be characterised as the only homogeneous quasi-arithmetic means (see [**10**, **19**]). The characterisation of Hardy means among quasi-arithmetic means is due to Mulholland [**15**].

THEOREM B. *Let $f : \mathbb{R}_+ \to \mathbb{R}$ be a continuous strictly monotonic function. Then, the quasiarithmetic mean $\mathcal{M}_f$ is a Hardy mean if and only if there exists $p < 1$ and a constant C such that*

$$\mathcal{M}_f(x_1, \dots, x_n) \leqslant C\mathcal{P}_p(x_1, \dots, x_n)$$

for all $n \in \mathbb{N}$ and $x_1, \dots, x_n > 0$.

A further generalisation of quasi-arithmetic means can be obtained in terms of two arbitrary functions. These means — called *quasi-arithmetic means with weight-function* — were introduced by Bajraktarević [**2**, **3**]. Let $f : \mathbb{R}_+ \to \mathbb{R}$ be continuous strictly monotonic function and let $w : \mathbb{R}_+ \to \mathbb{R}$ be a positive function. Now define the function $\mathcal{M}_{f,w}$ by

$$\mathcal{M}_{f,w}(x_1, \dots, x_n) := f^{-1}\Big(\frac{w(x_1)f(x_1) + \dots + w(x_n)f(x_n)}{w(x_1) + \dots + w(x_n)}\Big). \tag{7}$$

It is easy to check that $\mathcal{M}_{f,w}$ is a mean on $\mathbb{R}_+$. In the particular case $w \equiv 1$, the mean $\mathcal{M}_{f,w}$ reduces to $\mathcal{M}_f$, that is, the class of means $\mathcal{M}_{f,w}$ is more general than that of the quasi-arithmetic means.

It is a remarkable result of Aczél and Daróczy [**1**] that the homogenous means among the $\mathcal{M}_{f,w}$ means are exactly the Gini means [**6**] that form a two-parameter class of means including most of the classical homogeneous means. For $p, q \in \mathbb{R}$, the Gini mean $\mathcal{G}_{p,q}$ mean of the variables $x_1, \dots, x_n > 0$ is defined as follows:

$$\mathcal{G}_{p,q}(x_1, \dots, x_n) := \begin{cases} \Big(\dfrac{x_1^p + \dots + x_n^p}{x_1^q + \dots + x_n^q}\Big)^{1/(p-q)} & \text{if } p \neq q, \\ \exp\Big(\dfrac{x_1^p \ln(x_1) + \dots + x_n^p \ln(x_n)}{x_1^p + \dots + x_n^p}\Big) & \text{if } p = q. \end{cases} \tag{8}$$

Clearly, in the particular case $q = 0$, the mean $\mathcal{G}_{p,q}$ reduces to the pth power mean $\mathcal{P}_p$. It is also obvious that $\mathcal{G}_{p,q} = \mathcal{G}_{q,p}$, therefore, we may restrict our attention only to the case $p \leqslant q$ in our investigations.

Finally, we recall the concept of the most general means considered in this paper, the concept of deviation mean introduced by Daróczy [**4**]. For, a function $E : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ will be called a *deviation function* on $\mathbb{R}_+$ if $E(x, x) = 0$ for all $x > 0$ and the function $y \mapsto E(x, y)$ is continuous and strictly decreasing on $\mathbb{R}_+$ for each fixed $x > 0$. The *E-deviation mean* of the values $x_1, \dots, x_n > 0$ is now defined as the unique solution y of the equation

$$E(x_1, y) + \dots + E(x_n, y) = 0 \tag{9}$$

and is denoted by $\mathcal{M}_E(x_1, \dots, x_n)$.

In the particular case $E(x,y) := f(x) - f(y)$ the solution of (9) is the value $y = \mathcal{M}_f(x_1, \dots, x_n)$ defined in (6). Similarly, by taking $E(x,y) := w(x)\big(f(x) - f(y)\big)$, the resulting deviation mean will be $\mathcal{M}_{f,w}$. Thus, all the means defined previously (power means, quasi-arithmetic means, quasi-arithmetic means with weight-function, and Gini means) are particular deviation means.

The main results of this paper offer necessary as well as sufficient conditions for (5) if M is a deviation mean. This result is then translated to the case of Gini means as well.

2. Main Results

Our first main result offers necessary as well as sufficient conditions in order that a deviation mean be a Hardy mean.

Theorem 1. *Let $E : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ be a deviation on $\mathbb{R}_+$. If $M = \mathcal{M}_E$ is a Hardy mean, then*

$$\mathcal{M}_E(x_1, \dots, x_n) \leqslant C\mathcal{P}_p(x_1, \dots, x_n) \tag{10}$$

holds with $p = 1$ for all $n \in \mathbb{N}$ and $x_1, \dots, x_n > 0$ and there is no positive constant constant C^* such that

$$C^*\mathcal{P}_1(x_1, \dots, x_n) \leqslant \mathcal{M}_E(x_1, \dots, x_n) \tag{11}$$

is valid on the same domain. Conversely, if (10) is satisfied with a parameter $p < 1$, then $\mathcal{M}_E$ is a Hardy mean.

Proof: Assume first that $\mathcal{M}_E$ is a Hardy mean, that is,

$$\sum_{n=1}^{\infty} \mathcal{M}_E(x_1, \dots, x_n) \leqslant C \sum_{n=1}^{\infty} x_n$$

for all positive sequences (x_n). The left hand side of this inequality contains positive terms, therefore, omitting those for $n \geqslant k$, we get that, for each $k \in \mathbb{N}$,

$$\sum_{n=1}^{k} \mathcal{M}_E(x_1, \dots, x_n) \leqslant C \sum_{n=1}^{\infty} x_n.$$

Now, for all fixed $n \geqslant k$, letting $x_n \to 0$ on the right hand side, we get that

$$\sum_{n=1}^{k} \mathcal{M}_E(x_1, \dots, x_n) \leqslant C \sum_{n=1}^{k} x_n \tag{12}$$

for all $k \in \mathbb{N}$ and $x_1, \dots, x_k > 0$.

As a consequence of the characterisation theorem of (quasi)deviation means by Páles [17], the mean $\mathcal{M}_E$ is internal, that is, if $u_1, \dots, u_n, v_1, \dots, v_m > 0$ and $\mathcal{M}_E(u_1, \dots, u_n) \leqslant \mathcal{M}_E(v_1, \dots, v_m)$, then

$$\mathcal{M}_E(u_1, \dots, u_n) \leqslant \mathcal{M}_E(u_1, \dots, u_n, v_1, \dots, v_m) \leqslant \mathcal{M}_E(v_1, \dots, v_m).$$

Thus, if $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k > 0$ then, for $i = 1, \ldots, k-1$,

$$\mathcal{M}_E(x_1, \ldots, x_i) \geqslant \min(x_1, \ldots, x_i) = x_i \geqslant x_{i+1} = \mathcal{M}_E(x_{i+1})$$

implies that

$$\mathcal{M}_E(x_1, \ldots, x_i) \geqslant \mathcal{M}_E(x_1, \ldots, x_i, x_{i+1}).$$

Therefore, we get

$$\mathcal{M}_E(x_1) \geqslant \mathcal{M}_E(x_1, x_2) \geqslant \cdots \geqslant \mathcal{M}_E(x_1, x_2, \ldots, x_k).$$

Applying these inequalities, it follows from (12), that

$$k\mathcal{M}_E(x_1, \ldots, x_k) \leqslant C(x_1 + \cdots + x_k) \tag{13}$$

for $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k > 0$. The left and right hand sides being symmetric with respect to $x_1, \ldots, x_k$, we get that (13) is valid for all $x_1, \ldots, x_k > 0$. Dividing both sides by k, we infer that the inequality (10) (with $p = 1$) holds on the domain indicated.

If (11) were satisfied with some positive constant D, then the arithmetic mean would be a Hardy mean. Since this is not true according to Theorem A, we conclude that (11) cannot hold.

Now assume that (10) holds with $p < 1$ and with a finite constant C. Then (10) combined with Theorem A yields

$$\sum_{n=1}^{\infty} \mathcal{M}_E(x_1, \ldots, x_n) \leqslant C \sum_{n=1}^{\infty} \mathcal{P}_p(x_1, \ldots, x_n) \leqslant C c_p \sum_{n=1}^{\infty} x_n.$$

Thus the proof is complete. □

REMARK 1. There is a gap between the necessary and the sufficient conditions of Theorem 1. In view of Theorem B, we conjecture that the necessary and sufficient condition is analogous to that of for quasi-arithmetic means. Thus, we formulate the following:

OPEN PROBLEM 1. Prove or disprove that in order that a deviation mean $\mathcal{M}_E$ is a Hardy mean it is necessary (and also sufficient) that there exist a power $p < 1$ and a real constant C such that (10) be valid on the domain indicated.

Now we turn our attention to Gini means. The result contained in the theorem below offers a sharper condition than what follows directly from Theorem 1.

THEOREM 2. *Let $p, q \in \mathbb{R}$. If $\mathcal{G}_{p,q}$ is a Hardy mean, then*

$$\min(p, q) \leqslant 0 \quad \text{and} \quad \max(p, q) \leqslant 1. \tag{14}$$

Conversely, if

$$\min(p, q) \leqslant 0 \quad \text{and} \quad \max(p, q) < 1, \tag{15}$$

then $\mathcal{G}_{p,q}$ is a Hardy mean.

PROOF: As we pointed out in the introduction, Gini means are deviation means. Therefore, if $\mathcal{G}_{p,q}$ is a Hardy mean, then, by Theorem 1, $\mathcal{G}_{p,q}$ cannot be larger than the arithmetic mean and there exists a constant C such that

$$\mathcal{G}_{p,q}(x_1,\ldots,x_n) \leqslant C\mathcal{P}_1(x_1,\ldots,x_n) = C\mathcal{G}_{1,0}(x_1,\ldots,x_n) \tag{16}$$

for all $n \in \mathbb{N}$ and $x_1,\ldots,x_n > 0$. We are going to show that the constant C here can be replaced by 1, that is, $\mathcal{G}_{p,q} \leqslant \mathcal{G}_{1,0}$. By the Comparison Theorem of Gini means due to Daróczy and Losonczi [**5**], this inequality holds if and only if (14) is satisfied. Now, assume the contrary of (14), that is, assume that at least one of the inequalities in (14) is not valid. Since $\mathcal{G}_{p,q}$ is not larger than $\mathcal{A} = \mathcal{G}_{1,0}$, we conclude that only one inequality in (14) can be violated. Thus, either

$$\min(p,q) > 0 \qquad \text{and} \qquad \max(p,q) < 1, \tag{17}$$

or

$$\min(p,q) < 0 \qquad \text{and} \qquad \max(p,q) > 1. \tag{18}$$

Suppose first that $p < q$. Then (16) can be rewritten in terms of power means as follows:

$$\left(\frac{\mathcal{P}_q^q(x_1,\ldots,x_n)}{\mathcal{P}_p^p(x_1,\ldots,x_n)}\right)^{1/(q-p)} \leqslant C\mathcal{P}_1(x_1,\ldots,x_n)$$

or, equivalently,

$$\mathcal{P}_q^q(x_1,\ldots,x_n) \leqslant C^{q-p}\mathcal{P}_p^p(x_1,\ldots,x_n)\mathcal{P}_1^{q-p}(x_1,\ldots,x_n). \tag{19}$$

Define the moment space of the three power means $\mathcal{P}_p$, $\mathcal{P}_q$, $\mathcal{P}_1$ as follows

$$\begin{aligned}&\sigma_{\mathbb{R}_+}(\mathcal{P}_p,\mathcal{P}_q,\mathcal{P}_1)\\ &\quad := \mathbb{R}_+^3 \cap \mathrm{cl}\left\{\big(\mathcal{P}_p(x_1,\ldots,x_n),\mathcal{P}_q(x_1,\ldots,x_n),\mathcal{P}_1(x_1,\ldots,x_n)\big) \mid n \in \mathbb{N},\ x_1,\ldots,x_n > 0\right\}.\end{aligned}$$

Then, clearly, (19) implies that, for all $(u,v,w) \in \sigma_{\mathbb{R}_+}(\mathcal{P}_p,\mathcal{P}_q,\mathcal{P}_1)$,

$$v^q \leqslant C^{q-p}u^p w^{q-p}. \tag{20}$$

On the other hand, the moment space $\sigma_{\mathbb{R}_+}(\mathcal{P}_p,\mathcal{P}_q,\mathcal{P}_1)$ is completely described by the results of Páles [**18**, Theorems 2, 3]. Using this description, we show that (20) cannot hold if (14) is not satisfied.

We consider the two possibilities described by (17) and (18). If (17) holds, then $0 < p < q < 1$. In this case, by [**18**, Theorem 3], we have

$$\sigma_{\mathbb{R}_+}(\mathcal{P}_p,\mathcal{P}_q,\mathcal{P}_1) = \left\{(u,v,w) \mid 0 < u \leqslant v,\ v^{q(1-p)/(q-p)}u^{p(q-1)/(q-p)} \leqslant w\right\}.$$

Putting the minimal value $w = v^{q(1-p)/(q-p)}u^{p(q-1)/(q-p)}$ into (20), after simplifications, we get that

$$v^{pq} \leqslant C^{q-p}u^{pq}$$

for all $0 < u \leqslant v$. Taking the limit $u \to 0$, we get an obvious contradiction.

If (18) is valid, then $p < 0 < 1 < q$. In this case, by [**18**, Theorem 2], we have

$$\sigma_{\mathbb{R}_+}(\mathcal{P}_p, \mathcal{P}_q, \mathcal{P}_1) = \{(u, v, w) \mid 0 < u \leqslant w \leqslant v\}.$$

Putting $w = u$ into (20), we get that

$$v^q \leqslant C^{q-p}u^q,$$

thus, by taking the limit $u \to 0$, we again get a contradiction.

Thus, we have proved that $\mathcal{G}_{p,q} \leqslant \mathcal{G}_{1,0}$. From this, by using the Daróczy-Losonczi comparison theorem, it follows that (14) holds (in the case $p < q$). The same must hold for the case $p > q$ because of the symmetry property $\mathcal{G}_{p,q} = \mathcal{G}_{q,p}$. If $p = q$, that is, $\mathcal{G}_{p,p}$ is a Hardy mean then $\mathcal{G}_{p-(1/n),p} \leqslant \mathcal{G}_{p,p}$ is also a Hardy mean for all $n \in \mathbb{N}$. Thus, from what we have proved, it follows that $p - (1/n) \leqslant 0$ and $p \leqslant 1$ for all n. Taking the limit $n \to \infty$, we get that $p \leqslant 0$. Therefore, (14) holds in this case as well.

To prove the sufficiency of (15), define $r := \max(p, q, 0)$. Then, again by the comparison theorem of Gini means [**5**], we have that $\mathcal{G}_{p,q} \leqslant \mathcal{G}_{r,0} = \mathcal{P}_r$. Therefore, by the Hardy-Carleman-Knopp inequality,

$$\sum_{n=1}^{\infty} \mathcal{G}_{p,q}(x_1, \dots, x_n) \leqslant \sum_{n=1}^{\infty} \mathcal{P}_r(x_1, \dots, x_n) \leqslant c_r \sum_{n=1}^{\infty} x_n.$$

Thus the proof is complete. □

REMARK 2. There is a gap in the condition of Theorem 2, namely in the case $\min(p, q) \leqslant 0$ and $\max(p, q) = 1$ the Gini mean $\mathcal{G}_{p,q}$ is not characterised from the point if it is a Hardy mean or not. Nevertheless, we conjecture that the condition (15) is not only sufficient but it is also necessary. Thus, we can formulate the following two problems.

OPEN PROBLEM 2. Prove or disprove that in order that a Gini mean $\mathcal{G}_{p,q}$ be a Hardy mean it is necessary (and also sufficient) that (15) be valid.

OPEN PROBLEM 3. If the Gini mean $M = \mathcal{G}_{p,q}$ is a Hardy mean, then determine the smallest possible value $C = c_{p,q}$ such that (5) is satisfied.

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Institute of Mathematics and Informatics
University of Debrecen
H-4010 Debrecen, Pf. 12
Hungary
e-mail: pales@math.klte.hu

Department of Mathematics
Luleå University of Technology
Luleå
Sweden
e-mail: larserik@sm.luth.se