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ON BOUNDED AND COMPACT COMPOSITION OPERATORS IN POLYDISCS

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Recently MacCluer and Shapiro [**6**] have characterized the compact composition operators in Hardy and weighted Bergman spaces of the disc, and MacCluer [**5**] has made an extensive study of these opertors in the unit ball of C^n. Angular derivatives and Carleson measures have played an essential role in these studies. In this article we study composition operators in polydiscs and characterize those operators which are bounded or compact in Hardy and weighted Bergman spaces. In addition to Carleson measure theorems resembling those of [**5**], [**6**], we give necessary and sufficient conditions satisfied by the maps inducing bounded or compact composition operators. We conclude by considering the role of angular derivatives on the compactness question explicitly.

1. Preliminaries. We adopt the notation described in [**2**], and [**3**]. We denote by U^n the unit polydisc in C^n, by T^n the distinguished boundary of U^n, by H^p the Hardy space of order p in U^n, by A^p_α the weighted Bergman spaces of order p with weights $\prod_{i=1}^n(1-|z_i|^2)^\alpha$, $\alpha > -1$, and by $\hat{A}^p_\alpha$ the weighted Bergman spaces with weights given by $(1-\|z\|^2)^{n\alpha}$, $n\alpha > -1$, where $\|.\|$ is the polydisc norm. We shall use m_n to denote the n-dimensional Lebesgue area measure on T^n, normalized so that $m_n(T^n) = 1$. By σ_n we shall denote the volume measure on $\overline{U^n}$, defined so that $\sigma_n(\overline{U^n}) = 1$, and by $\sigma_{n,\alpha}$ we shall denote the weighted measure on $\overline{U^n}$ given by $\prod_{i=1}^n(1-|z_i|^2)^\alpha\sigma_n$. We use R to describe rectangles on T^n, and use S(R) to denote the corona associated to these sets. In particular, if I is an interval on T of length δ centered at $e^{i(\theta_0+\frac{\delta}{2})}$, S(I) is

$$S(I) = \{z \in U : 1-\delta < r < 1, \theta_0 < \theta < \theta_0+\delta\}.$$

Then if $R = I_1 \times I_2 \times \cdots \times I_n \subset T^n$, with I_j intervals having length δ_j and having centers $e^{i(\theta_j^0+\frac{\delta_j}{2})}$, S(R) is given by

$$S(R) = S(I_1) \times S(I_2) \times \cdots \times S(I_n).$$

We use S to denote S(R) whenever convenient. If V is any open set in T^n we define

$$S(V) = \bigcup_\alpha S(R_\alpha)$$

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where $\{R_\alpha\}$ runs through all rectangles in V. A finite, nonnegative, Borel measure μ on $\overline{U^n}$ is said to be a (bounded) Carleson measure if

$$\mu(S(V)) \leq Cm_n(V) \quad \text{for all connected open sets} V \subset T^n.$$

μ is said to be a compact-Carleson measure if

$$\lim_{m_n(V)\to 0} \sup_{V\subset T^n} \frac{\mu(S(V))}{m_n(V)} = 0.$$

μ_α is said to be a (bounded) α-Carleson measure if

$$\mu_\alpha(S(R)) \leq C\prod_{i=1}^{n} \delta_i^{2+\alpha} \quad \text{for all } R \subset T^n,$$

and μ_α is a compact α-Carleson measure if

$$\lim_{\delta_i\to 0} \sup_{\theta\in T^n} \frac{\mu_\alpha(S(R))}{\prod_{i=1}^{n} \delta_i^{2+\alpha}} = 0.$$

We point out that Carleson measures as they are used in the literature are always bounded. The subscript α distinguishes between the α-Carleson measures, and the Carleson measures. One should also note that the definitions of these measures have nothing to do with Hardy or weighted Bergman spaces; however, Carleson measures are intimately related to Hardy spaces, and α-Carleson measures are related to weighted Bergman spaces A^p_α. Theorems A and B , given below, make this notion precise.

In [**3**] we discuss these spaces and measures in detail, and give various characterization and comparison theorems. These theorems which extend theorems of 'similar' type in the disc or the unit ball of C^n to polydiscs form the foundation on which this paper is based. For the sake of readability and to make this paper self-contained we shall state the main results of reference [**3**] here without proof.

THEOREM A. *Let A be the identity operator sending $H^p(U^n)$ into $L^p(\mu)$, $1 < p < \infty$. Then*
(i) A is bounded in $H^p(U^n)$ if and only if μ is a Carleson measure.
(ii) Assume that $H^p(U^n) \subset L^p(\mu)$. Then A is a compact operator if and only if μ is a compact Carleson measure.

THEOREM B. *Let I_α be the identity map from $A^p_\alpha(U^n)$ into $L^p(\mu)$, and $1 < p < \infty$, $\alpha > -1$. Then*
(i) I_α is a bounded operator if and only if μ is an α-Carleson measure.

(ii) I_α is a compact operator if and only if μ is a compact α-Carleson measure.

THEOREM C. *Suppose that $-1 < \alpha < \beta$ and I_α is the identity operator from $A^p_\alpha(U^n)$ into $L^p(\mu_\alpha)$. Then if I_α is a bounded (compact) operator, then so is I_β.*

We point out that condition $1 < p$ arises in the proofs of theorems A and B, given in [**3**], because of use of strong (p,p) type inequalities for maximal functions in their proof. We do not know if these theorems are also true for $p = 1$. The author wishes to thank his adviser Professor Walter Rudin for his many suggestions and helpful comments. He also thanks Professor Barbara D. MacCluer for informing him of the results of Sharma and Singh [**9**] and providing him with preprints of her work on composition operators.

2. Introduction. Let ϕ be a holomorphic map of U^n into U^n. We study the linear composition operator defined by

$$C_\phi f = f \circ \phi$$

for f belonging to Hardy or weighted Bergman spaces in U^n. Our basic goal is to determine in terms of properties of ϕ when C_ϕ is bounded and compact on these holomorphic function spaces of U^n. The idea of the following proposition dates back to Ryff [**8**]:

PROPOSITION 1. *Suppose $\phi(z_1, z_2, \ldots, z_n) = (\phi_1(z_1), \phi_2(z_2), \ldots, \phi_n(z_n))$ is a holomorphic map of U^n into U^n. Then ϕ induces a bounded composition operator on $H^p(U^n)$, for $1 \le p < \infty$. The same conclusion holds for $A^p_0(U^n)$.*

Combining this proposition with theorem 7.3.3 [**7**] implies:

COROLLARY 2. *Automorphisms of U^n induce bounded composition operators on $H^p(U^n)$ and $A^p_0(U^n)$, $1 \le p < \infty$.*

Proof. Note that the composition of Poisson kernel and ϕ is n-harmonic, and use the mean-value property for n-harmonic functions. ■
This proof also demonstrates that on H^p (or A^p_0)

$$\|C_\phi\| \le \left(\prod_{i=1}^{n} \frac{1 + |\phi_i(0)|}{1 - |\phi_i(0)|} \right)^{\frac{1}{p}}.$$

By proposition 1 the boundedness of C_ϕ on $H^p(U)$ is guaranteed. In fact it is easy to show that C_ϕ is bounded on H^p for $1 \le p < \infty$ for every $\phi : U^n \longrightarrow U$, ϕ holomorphic.

PROPOSITION 3. *Suppose $\phi : U^n \longrightarrow U$ is holomorphic. Then $C_\phi : H^p(U) \longrightarrow H^p(U^n)$ is bounded for $1 \leq p < \infty$.*

Proof. Since $f \in H^p(U)$ if and only if $|f|^p$ has a harmonic majorant there is a real harmonic function u(z) on the disc so that $|f|^p \leq u$. Since every harmonic function in the disc is the real part of a holomorphic function, and the real parts of holomorphic functions are n-harmonic, $u \circ \phi$ is n-harmonic, and

$$|f \circ \phi|^p \leq u \circ \phi.$$

Hence $f \circ \phi \in H^p(U^n)$. To complete the proof we need to show that if $C_\phi : H^p(U) \longrightarrow H^p(U^n)$ for $1 \leq p < \infty$, then C_ϕ is bounded. This follows from the closed graph theorem. That is, if $\|f_j - f\|_p \longrightarrow 0$ and $\|f_j \circ \phi - g\|_p \longrightarrow 0$ as $j \longrightarrow \infty$ then for every $z \in U$ $f_j(z) \longrightarrow f(z)$ and $f_j(\phi(z)) \longrightarrow g(z)$. In particular, $f_j(\phi(z)) \longrightarrow f(\phi(z))$. Hence $g = f \circ \phi$. ∎

The above proof uses the fact that evaluation at $z \in U^n$ is a bounded linear functional.

3. Main Results. Since a harmonic function in several variables is not necessarily the real part of a holomorphic function, the result of proposition 3 is not available when $\phi : U^n \longrightarrow U^n$. In fact, if $f \in H^2(U^2)$ and f_D is the restriction of f to the diagonal of U^2, then $f \longrightarrow f_D$ maps $H^2(U^2)$ $onto A_0^2(U^2)$. Hence $\phi(z_1, z_2) = (z_1, z_1)$ induces an unbounded composition operator on $H^2(U^2)$. Sharma and Singh **[9]** give a lower bound for the norm of C_ϕ on $H^2(U^2)$ which can be used to construct other examples of maps ϕ inducing unbounded composition operators. We postpone the precise statement of this result until we have given an application of theorems A, and B. This result will allow us to strengthen and complete the results of Sharma and Singh. We shall need the following lemma:

LEMMA 4. *Suppose $\phi : U^n \longrightarrow U^n$ is holomorphic and C_ϕ is bounded (compact) on a dense subset of $H^p(U^n)$ for $1 < p < \infty$. Then C_ϕ is bounded (compact).*

By compact on a dense subset E of H^p we mean that if f_j is a bounded sequence in E which converges to zero weakly, then $f_j \circ \phi$ converges to zero in the norm topology of H^p.

Proof. Suppose $E \subset H^p$ is dense and there is some $M < \infty$ such that $\|f \circ \phi\|_p \leq M\|f\|_p$ for all $f \in E$. Pick $h \in H^p$. Then there exists $f_k \in E$ so that $\|f_k - h\|_p \longrightarrow 0$, $\|f_k\|_p \leq C < \infty$, for some C. Hence

$$f_k(z) \longrightarrow h(z) \text{ for all } z \in U^n.$$

Also, for $r < 1$

$$\int_{T^n} |(f_k \circ \phi)_r|^p \, dm_n \leq \|f_k \circ \phi\|_p^p \leq M^p C^p.$$

By Fatou's lemma

$$\int_{T^n} |(h \circ \phi)_r|^p \, dm_n \le M^p C^p \text{ for all } r < 1.$$

Hence $h \circ \phi \in H^p$, $\|h \circ \phi\|_p \le MC$, where C can be taken as close to $\|h\|_p$ as desired.
Therefore, $\|h \circ \phi\|_p \le M\|h\|_p$.
To show compactness, as before suppose $E \subset H^p$ is dense and C_ϕ is compact on E. Let $f_j \subset H^p$, $\|f_j\|_p \le C < \infty$, and suppose that f_j converge to zero weakly. By lemma 1.4 [**3**], it suffices to prove $\|f_j \circ \phi\|_p \longrightarrow 0$. This follows from standard arguments. ∎

THEOREM 5. *Let $\phi : U^n \longrightarrow U^n$ be holomorphic, and $1 < p < \infty$. Put*

$$\phi^*(\zeta) = \lim_{r \to 1} \phi(r\zeta) \quad \zeta \in T^n$$

whenever this limit exists, and associate a measure μ to ϕ by setting

$$\mu(E) = m_n(\phi^*)^{-1}(E)) \quad (E \subset \overline{U^n}).$$

In other words μ is the measure that satisfies

$$(*) \qquad \int_{\overline{U^n}} h d\mu = \int_{T^n} (h \circ \phi^*) dm_n \quad (h \in C(\overline{U^n})).$$

Likewise, define μ_α to be

$$\mu_\alpha(E) = \sigma_{n,\alpha}(\phi^{-1}(E)) \quad (E \subset \overline{U^n}).$$

Then
(i) C_ϕ is bounded on H^p if and only if

$$(1) \qquad \mu(S(V)) \le Cm_n(V).$$

i.e. C_ϕ is bounded on H^p if and only if μ is a Carleson measure.
(ii) C_ϕ is compact if and only if

$$\lim_{m_n(V) \to 0} \sup_{V \subset T^n} \frac{\mu(S(V))}{m_n(V)} = 0$$

i.e. C_ϕ is compact if and only if μ is a compact Carleson measure.
(iii) C_ϕ is bounded (compact) on $A^p_\alpha(U^n), \alpha > -1$, if and only if μ_α is an (compact) α-Carleson measure.

We point out that this theorem states that if C_ϕ is a bounded (compact) operator for H^p or $A^p_\alpha(U^n)$, $1 < p < \infty$, for some p, then it is a bounded (compact) operator on those spaces for all p. This is analagous to the behavior of these operators observed by MacCluer and Shapiro for the disc [**6**] in dealing with the compactness question. As it is already pointed out the results here are stated only for $p > 1$. This is because theorems A and B, which are used in the proof of this theorem, are proved by using strong (p,p) type inequalities for maximal functions. [**3**]

Proof. This theorem follows from an application of theorems A and B. We note that (*) guarantees that μ, or μ_α, is a well-defined nonnegative measure on $\overline{U^n}$ since the right hand side is a bounded linear functional on $C(\overline{U^n})$.
Let $f \in H^p(U^n) \cap C(\overline{U^n})$. Then

$$\begin{aligned}\|C_\phi f\|_p^p &= \int_{T^n} |(f \circ \phi)^*|^p \, w dm_n \\ &= \int_{T^n} |f \circ (\phi^*)|^p \, dm_n = \quad (\text{since } f \in C(\overline{U^n})) \\ &= \int_{T^n} (|f|^p \circ \phi^*) \, dm_n = \\ &= \int_{\overline{U^n}} |f|^p dm_n \circ (\phi^*)^{-1}.\end{aligned}$$

Hence

$$\|C_\phi f\|_p^p = \int_{\overline{U^n}} |f|^p \, d\mu \text{ for all } f \in H^p \cap C(\overline{U^n}). \tag{3}$$

If μ is a Carleson measure then by (3) and theorem A (i) C_ϕ is bounded on $H^p \cap C(\overline{U^n})$. Since $H^p \cap C(\overline{U^n})$ is dense in H^p by lemma 4 we conclude that C_ϕ is bounded on H^p.
On the other hand if C_ϕ is bounded on $H^p(U^n) \cap C(\overline{U^n})$ then by lemma 4 it is bounded on H^p, and hence by theorem A (i) μ is a Carleson measure.
Conclusions (ii) and (iii) follow from theorems A (ii) and B in exactly the same manner. ■
Combining this theorem with theorem C we conclude:

COROLLARY 6. *Suppose $-1 < \alpha < \beta$ and C_ϕ is a bounded (compact) operator on $A^p_\alpha(U^n)$. Then the same is true on $A^p_\beta(U^n)$, $1 < p < \infty$.*

Proof. Recall that $\sigma_{n,\alpha} = \prod_{i=1}^n (1-|z_i|^2)^\alpha \sigma_n$, and put $\mu_\alpha = \sigma_{n,\alpha} \circ (\phi^*)^{-1}$ and $\mu_\beta = \sigma_{n,\beta} \circ (\phi^*)^{-1}$. Then by theorem C boundedness (compactness) of I_α implies boundedness (compactness) of I_β. Hence μ_β is a bounded (compact) β-Carleson measure. Using theorem 5 it follows that C_ϕ is bounded (compact) on $A^p_\beta(U^n)$. ■

Hence the boundedness (compactness) of composition operators on $A^p_\alpha(U^n)$ is somewhat independent of α. That is if C_ϕ is bounded (compact) on $A^2_\alpha(U^n)$ for some $-1 < \alpha$, then C_ϕ has the same property for $A^p_\beta(U^n)$ for all $\beta > \alpha$, and all $p \in (1, \infty)$. An immediate consequence of this corollary and corollary 2 is that

the automorphisms of U^n induce bounded composition operators on $A^p_\alpha(U^n)$ for all $\alpha > 0$.
Theorem 5 permits us to restrict our attention to $H^2(U^n)$ or $A^2_\alpha(U^n)$, where it is more convenient to work with these Hilbert spaces. The conclusions obtained for these spaces then carry to H^p and A^p_α without change.
To identify the unbounded composition operators on $H^p(U^n)$ we extend the following result of Sharma and Singh which they obtained for $H^2(U^2)$. We include its proof for completeness.

PROPOSITION 7. *C_ϕ induces an unbounded composition operator on $H^p(U^n)$ for $1 < p < \infty$ if*

$$\sup_{z\in U^n} \prod_{i=1}^{n} \frac{(1-|z_i|^2)}{(1-|\phi_i(z)|^2)} = \infty.$$

Proof. By theorem 5 it suffices to show

$$\|C_\phi\|_2^2 \geq \sup_{z\in U^n} \prod_{i=1}^{n} \frac{1-|z_i|^2}{1-|\phi_i(z)|^2}.$$

i.e. it suffices to consider the case for p=2. For $\zeta \in U^n$ define

$$g_z(\zeta) = \prod_{i=1}^{n} \frac{1}{1-\bar{\zeta}_i z_i}$$

and note that

$$\|g_z\|^2 = (g_z, g_z) = g_z(z) = \prod_{i=1}^{n} \frac{1}{1-|z_i|^2}.$$

For a fixed $z \in U^n$ we have

$$\|g_{\phi(z)}\|^2 = \|\prod_{i=1}^{n} \frac{1}{1-\bar{\zeta}_i\phi_i(z)}\|^2 = g_{\phi(z)}(\phi(z)) = \prod_{i=1}^{n} \frac{1}{1-|\phi_i(z)|^2}.$$

Since $|f(z)| \leq \|f\|_2 \prod_{i=1}^{n}(\frac{1}{1-|z_i|^2})^{\frac{1}{2}}$, we also have

$$\begin{aligned}\|g_{\phi(z)}\|^2 = g_{\phi(z)}(\phi(z)) &= (g_{\phi(z)}\circ\phi)(z) = (C_\phi g_{\phi(z)})(z)\\ &\leq \|C_\phi g_\phi(z)\|_2 \left(\prod_{i=1}^{n} \frac{1}{1-|z_i|^2}\right)^{\frac{1}{2}}.\end{aligned}$$

Hence for every $z \in U^n$

$$\|C_\phi\|_2^2 \geq \|g_{\phi(z)}\|_2^2 \prod_{i=1}^{n}(1-|z_i|^2) = \prod_{i=1}^{n} \frac{1-|z_i|^2}{1-|\phi_i(z)|^2}.$$

Taking supremum over all $z \in U^n$ it follows that

$$\|C_\phi\| \geq \sup_{z \in U^n} \left(\prod_{i=1}^{n} \frac{1 - |z_i|^2}{1 - |\phi_i|^2} \right)^{\frac{1}{2}}. \quad \blacksquare$$

Let us consider some examples of unbounded composition operators in $H^p(U^2)$, $1 < p < \infty$ provided by proposition 7. Note that for the diagonal map $\phi(z_1, z_2) = (z_1, z_1)$ we have

$$\|C_\phi\| \geq \sup_{z \in U^2} \frac{1 - |z_2|^2}{1 - |z_1|^2}.$$

Thus allowing $|z_1| \longrightarrow 1$ while keeping $|z_2| < 1$ it is easily observed that the composition operator induced by such a map will be unbounded. Since the class of bounded invertible composition operators are exactly those induced by the automorphisms of U^n (see Appendix A, theorem A.4 [**4**]) all maps $\phi(z_1, z_2) = (\phi_1(z_1), \phi_1(z_1))$ with $\phi_1(z_1) \in Aut(U)$ induce an unbounded composition operator on $H^p(U^2)$. To explain this point a little more, put $D = (z_1, z_1)$ and note that all these maps ϕ are obtained by taking $D \circ \psi$, where $\psi \in Aut(U^2)$. Thus $C_\phi = C_D C_\psi$. Since C_ψ is invertible, it has a bounded inverse, and since C_D is unbounded C_ϕ is unbounded. Furthermore if

$$\phi(z_1, z_2) = (z_1, f(z_1, z_2)),$$

where $|f(z_1, z_2)| < 1$, and f has the property that $|f| \longrightarrow 1$ as $z \longrightarrow \zeta$ for some $\zeta \in \partial U^2 \cap \{|z_2| < 1\}$. Then

$$\|C_\phi\| \geq \sup_{z \in U^2} \frac{1 - |z_2|^2}{1 - |f(z_1, z_2)|^2}.$$

Allowing $z \longrightarrow \zeta$ it is easy to see that the right hand side of the above expression tends to infinity. Hence the composition operators induced by all such maps will be unbounded.

Also, if C_ϕ is bounded on $H^p(U^n)$, $1 < p < \infty$, then

$$(*) \qquad \sup_{z \in U^n} \prod_{i=1}^{n} \left(\frac{1 - |z_i|^2}{1 - |\phi_i(z)|^2} \right) < \infty.$$

We point out that on the disc, if $\phi(0) = 0$, Schwarz's lemma implies that

$$\frac{1 - |z|^2}{1 - |\phi(z)|^2} \leq 1.$$

This is no longer so in polydiscs. However, by (*) for C_ϕ to be bounded $|\phi_i(z)|$ cannot approach 1 much faster than any of $|z_j|$. This result can be significantly strengthened by combining theorem 5 with the following proposition. Part (i) of this proposition for the case n=1 is given in [**1**, lemma 3.3, p. 239].

PROPOSITION 8. *Let μ be a nonnegative, Borel measure on $\overline{U^n}$. Then*
(i) μ is a Carleson measure if and only if

$$(1) \qquad \sup_{z_0 \in U^n} \int_{\overline{U^n}} \prod_{i=1}^{n} \frac{1-|(z_0)_i|^2}{|1-(\overline{z_0})_i z_i|^2}\, d\mu(z) \leq M < \infty$$

(ii) μ is a compact Carleson measure if and only if

$$(2) \qquad \lim \sup_{z_0 \in U^n} \int_{\overline{U^n}} \prod_{i=1}^{n} \frac{1-|(z_0)_i|^2}{|1-(\overline{z_0})_i z_i|^2}\, d\mu(z) = 0 \text{ as } \|z_0\| \longrightarrow 1.$$

Proof. First, suppose that (1) holds. We show that this implies that μ is a Carleson measure. Recall that R denotes

$$R = \left\{ (e^{i\theta_1}, \ldots, e^{i\theta_n}) \in T^n : \ |\theta_i - (\theta_0)_i| < \delta_i \right\},$$

and

$$S = S(R) = \left\{ (r_1 e^{i\theta_1}, \ldots, r_n e^{i\theta_n}) \in U^n : \ 1-\delta_i < r_i < 1\,,\ |\theta_i - (\theta_0)_i| < \delta_i \right\}.$$

Since if $z_0 = 0$ (1) implies that $\mu(\overline{U^n}) \leq M < \infty$ we can suppose that $\delta_i < \frac{1}{4}$ for all i. Take

$$(z_0)_i = (1 - \frac{\delta_i}{2}) e^{i(\theta_0)_i}.$$

Then for $z \in S$

$$\frac{1-|(z_0)_i|^2}{|1-(\overline{z_0})_i z_i|^2} \geq \frac{C}{1-|(z_0)_i|^2}.$$

Hence

$$\prod_{i=1}^{n} \frac{1-|(z_0)_i|^2}{|1-(\overline{z_0})_i z_i|^2} \geq \prod_{i=1}^{n} \frac{C}{1-|(z_0)_i|^2},$$

and

$$\mu(S) = \int_S d\mu = C \prod_{i=1}^{n} 1-|(z_0)_i|^2 \int_S \frac{C}{\prod_{i=1}^{n} 1-|(z_0)_i|^2}\, d\mu(z)$$
$$\leq 2C \prod_{i=1}^{n} \delta_i \int_S \prod_{i=1}^{n} \frac{1-|(z_0)_i|^2}{|1-(\overline{z_0})_i z_i|^2} d\mu(z) \leq CMm_n(R).$$

Hence $\mu(S) \le Cm_n(R)$, i.e. μ is a Carleson measure.
Conversely, suppose that μ is a Carleson measure and let $z_0 \in U^n$. If $\|z_0\| \le \frac{3}{4}$, it is easy to see that (1) holds, since the integrand can be bounded uniformly. Also, if $|z_i| < \frac{3}{4}$ the term corresponding to this i in the integrand in (1) can be bounded. So suppose $|(z_0)_i| > \frac{3}{4}$ for all i, and let

$$E_k = \left\{ z \in U^n : \max_i \frac{|z_i - \frac{(z_0)_i}{|(z_0)_i|}|}{(1 - |(z_0)_i|)} < 2^k \right\}.$$

By the Carleson condition

$$\mu(E_k) \le C \prod_{i=1}^n 2^k(1 - |(z_0)_i|).$$

Note that if $z \in E_1$ then

$$\prod_{i=1}^n \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} \le \prod_{i=1}^n \frac{C}{1 - |(z_0)_i|}$$

and for $k \ge 2$ if $z \in E_k - E_{k-1}$

$$\prod_{i=1}^n \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} \le \prod_{i=1}^n \frac{C}{2^{2k}(1 - |(z_0)_i|)}.$$

Hence

$$\begin{aligned}\int_{U^n} \prod_{i=1}^n \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2}\, d\mu(z) &\le \int_{E_1} + \sum_{k=2}^\infty \int_{E_k - E_{k-1}} \prod_{i=1}^n \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2}\, d\mu \\ &\le C \sum_{k=2}^\infty \frac{\mu(E_k - E_{k-1})}{\prod_{i=1}^n 2^{2k}(1 - |(z_0)_i|)} \\ &\le C \sum_{k=1}^\infty \frac{\mu(E_k)}{\prod_{i=1}^n 2^{2k}(1 - |(z_0)_i|)} \le C < \infty. i\end{aligned}$$

This finishes the proof of (i).
To prove (ii), first suppose that (2) holds. We show that μ is a compact Carleson measure. That is, if S and R are defined as before we need to show that

$$\lim_{|R| \to 0} \sup_{R \subset T^n} \frac{\mu(S)}{m_n(R)} = 0.$$

Making the same observations as in (i) for $z \in S$, we get

$$\frac{1}{m_n(R)} \int_S d\mu = \frac{C}{m_n(R)} \prod_{i=1}^n (1 - |(z_0)_i|^2) \int_S \frac{C}{\prod_{i=1}^n 1 - |(z_0)_i|^2} d\mu(z)$$

$$\leq C \int_{\overline{U^n}} \prod_{i=1}^{n} \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} d\mu(z).$$

By hypothesis the limit of right hand side tends to zero as $|R| \longrightarrow 0$. Hence

$$\lim_{|R| \to 0} \sup_{R \subset T^n} \frac{\mu(S)}{m_n(R)} = 0,$$

i.e. μ is a compact Carleson measure. Conversely, suppose that μ is a compact Carleson measure. We need to show that (2) holds. Proceeding as in the proof of the converse direction in (i), we get

$$\int_{\overline{U^n}} \prod_{i=1}^{n} \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} d\mu(z) \leq \int_{E_1} + \sum_{k=2}^{\infty} \int_{E_k - E_{k-1}} \prod_{i=1}^{n} \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} d\mu$$

$$\leq C \sum_{k=1}^{\infty} \frac{\mu(E_k)}{\prod_{i=1}^{n} 2^{2k}(1 - |(z_0)_i|)}.$$

Since μ is a compact Carleson measure $\frac{\mu(E_s)}{|E_s|} \longrightarrow 0$ as $|E_s| \longrightarrow 0$. Hence given $\epsilon > 0$ we can pick $\|z_0\|$ sufficiently close to 1, so that $\frac{\mu(E_s)}{\prod_{i=1}^{n} 1 - |(z_0)_i|} \leq \epsilon$. Then

$$\int_{\overline{U^n}} \prod_{i=1}^{n} \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} d\mu(z) \leq C\epsilon. \blacksquare$$

Combining this proposition with theorem 5 we get our main result for a map to induce bounded or compact composition operators on Hardy spaces of polydiscs.

THEOREM 9. *For $1 < p < \infty$*
(i) C_ϕ is bounded on H^p if and only if

$$\sup_{z_0 \in U^n} \int_{T^n} \prod_{i=1}^{n} \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i \phi_i^*|^2} dm_n < \infty.$$

(ii) C_ϕ is compact on H^p if and only if

$$\lim \sup_{z_0 \in U^n} \int_{T^n} \prod_{i=1}^{n} \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i \phi_i^*|^2} dm_n = 0 \quad as \ \|z_0\| \longrightarrow 1.$$

Proof. Let $\mu = m_n \circ (\phi^*)^{-1}$. Then by theorem 5 C_ϕ is a bounded (compact) operator on H^p if and only if μ is a bounded (compact) Carleson measure on $\overline{U^n}$. By the above proposition μ is a bounded (compact) Carleson measure if and only if (1) (or, for compactness (2)) hold. The assertion follows from these equations by setting $\mu = m_n \circ (\phi^*)^{-1}$ and making a change of variable, as done in theorem 5. $\blacksquare$

The following useful corollary can also be derived from theorem 5.

COROLLARY 10. *If C_ϕ is bounded in $H^p(U^n), 1 < p < \infty$, then ϕ^* cannot carry a set of positive measure on T^n into a set of measure zero on T^n.*

Proof. See [**5**] or [**4**, proposition 7.9]. ∎

Turning attention to the weighted Bergman spaces, proposition 7 can be modified to give a similar result for bounded composition operators on these spaces.

PROPOSITION 11. *If C_ϕ is bounded on $A^p_\alpha(U^n)$ then*

$$(1)\qquad \sup_{z\in U^n} \frac{\sum_{|s|=0}^{\infty}\prod_{i=1}^{n}(s_i+1)^{-1-\alpha}|\phi(z)|^{2s}}{\sum_{|s|=0}^{\infty}\prod_{i=1}^{n}(s_i+1)^{1+\alpha}|z|^{2s}} < \infty.$$

where the ususal multi-index notation is used, i.e. $|s| = s_1 + \cdots + s_n$, *and* $|z|^s = |z_1|^{s_1}\dots|z_n|^{s_n}$.

Proof. By theorem 5 it is sufficient to consider p=2. As in proposition 7 we show that

$$\|C_\phi\|_2^2 \geq (1).$$

Let $g_z^\alpha(\zeta)$ be the reproducing kernel for the weighted Bergman space A^2_α of polydisc U^n. Then as in proposition 7

$$\|g_z^\alpha\|_{2,\alpha}^2 = (g_z^\alpha, g_z^\alpha) = g_z^\alpha(z).$$

For $z \in U^n$ fixed, we have on the one hand

$$(2)\qquad \|g^\alpha_{\phi(z)}\|^2_{2,\alpha} = (g^\alpha_{\phi(z)}, g^\alpha_{\phi(z)}) = g^\alpha_{\phi(z)}(\phi(z)).$$

On the other hand, since by Cauchy-Schwarz inequality and proposition 1.1 [**3**]

$$(3)\qquad |f(z)|^2 \leq \|f\|^2_{2,\alpha}\left\{\sum_{|s|=0}^{\infty}\prod_{i=1}^{n}(s_i+1)^{1+\alpha}|z|^{2s}\right\}$$

$$g^\alpha_{\phi(z)}(\phi(z)) = |C_\phi g^\alpha_{\phi(z)}(z)| \leq \|C_\phi g^\alpha_{\phi(z)}\|^2_{2,\alpha}\left(\sum_{|s|=0}^{\infty}\prod_{i=1}^{n}(s_i+1)^{1+\alpha}|z|^{2s}\right).$$

Combining (2) and (3) gives:

$$(4)\qquad \|C_\phi\|^2_{2,\alpha} \geq \frac{g^\alpha_{\phi(z)}(\phi(z))}{\sum_{|s|=0}^{\infty}\prod_{i=1}^{n}(s_i+1)^{1+\alpha}|z|^{2s}}.$$

Taking supremum over all $z \in U^n$ and substituting

$$g_z^\alpha(\zeta) = \sum_{|s|=0}^{\infty}\prod_{i=1}^{n}(s_i+1)^{-1-\alpha}\overline{\zeta}^s z^s$$

into (4) gives the desired inequality in (1).

To complete the proof we need to show that g_z^α is in fact a reproducing kernel for $A_\alpha^2(U^n)$. This follows from the following computation. Let

$$c(s,\alpha) = \prod_{i=1}^{n}(s_i+1)^{-1-\alpha} = \left(\int_{\overline{U^n}} |\zeta|^{2s}\, d\sigma_{n,\alpha}(\zeta)\right)^{-1},$$

and note that if $f(z) = \sum_{|s|=0}^{\infty} a_s z^s$ then

$$I = \int_{\overline{U^n}} f(\zeta) g_z^\alpha(\zeta)\, d\sigma_{n,\alpha}(\zeta) =$$
$$= \int_{\overline{U^n}} \sum_{|s|=0}^{\infty} \sum_{|s'|=0}^{\infty} a_s \zeta^s c(s,\alpha) \overline{\zeta}^{s'} z^{s'}\, d\sigma_{n,\alpha}.$$

Interchanging the order of summations and integration and using orthogonality this becomes

$$I = \sum_{|s|=0}^{\infty} a_s c(s,\alpha) (\int_{\overline{U^n}} |\zeta|^{2s}\, d\sigma_{n,\alpha}) z^s,$$

which by the definition of $c(s,\alpha)$ is

$$I = \sum_{|s|=0}^{\infty} a_s z^s = f(z). \quad \blacksquare$$

Note that if $\alpha = -1$ proposition 11 duplicates the result of proposition 7. Parallel to the characterization of bounded and compact composition operators in Hardy spaces of polydiscs (proposition 8 and theorem 9), a similar proposition would lead to the characterization of these operators in weighted Bergman spaces of polydiscs. Similarity of the arguments involved points out that the conditions of theorems 9 and 13 are in some sense the correct characterizations of bounded and compact composition operators.

PROPOSITION 12. *Let μ_α be a nonnegative, Borel, measure in $\overline{U^n}$.*
(i) μ_α is an α-Carleson measure if and only if

$$(1) \qquad \sup_{z_0 \in U^n} \int_{\overline{U^n}} \prod_{i=1}^{n} \left[\frac{1-|(z_0)_i|^2}{|1-(\overline{z_0})_i z_i|^2}\right]^{2+\alpha} d\mu_\alpha \le M < \infty.$$

(ii) μ_α is a compact α-Carleson measure if and only if

$$(2) \qquad \lim \sup_{z_0 \in U^n} \int_{\overline{U^n}} \prod_{i=1}^{n} \left(\frac{1-|(z_0)_i|^2}{|1-(\overline{z_0})_i z_i|^2}\right)^{2+\alpha} d\mu_\alpha = 0 \textit{ as } \|z_0\| \to 1.$$

Proof. Let S and R be chosen as in the proof of proposition 8, and note that if $z_0 = 0$ (1) implies that $\mu_\alpha(\overline{U^n}) < \infty$. So, we can assume that $\delta_i < \frac{1}{4}$ for all i. Take $(z_0)_i = (1 - \frac{\delta_i}{2})e^{i(\theta_0)_i}$. Then for all $z \in S$

$$\prod_{i=1}^{n} \left(\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} \right)^{2+\alpha} \geq \prod_{i=1}^{n} \frac{C}{(1 - |(z_0)_i|^2)^{2+\alpha}}.$$

Hence

$$\begin{aligned}\mu_\alpha(S) &= \int_S d\mu_\alpha \\ &= C \prod_{i=1}^{n} (1 - |(z_0)_i|^2)^{2+\alpha} \int_S \prod_{i=1}^{n} \left[\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} \right]^{2+\alpha} d\mu_\alpha \\ &\leq 2CM \prod_{i=1}^{n} \delta_i^{2+\alpha}.\end{aligned}$$

Thus μ_α is an α-Carleson measure.
Conversely suppose that μ_α is an α-Carleson measure. Then proceeding exactly as in the proof of proposition 8 we get:

$$\begin{aligned}&\int_{\overline{U^n}} \prod_{i=1}^{n} \left(\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} \right)^{2+\alpha} d\mu_\alpha \\ &\quad \leq \int_{E_1} + \sum_{k=2}^{\infty} \int_{E_k - E_{k-1}} \left(\prod_{i=1}^{n} \frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i z_i|^2} \right)^{2+\alpha} d\mu_\alpha \\ &\quad \leq C \sum_{k=2}^{\infty} \frac{\mu_\alpha(E_k - E_{k-1})}{\prod_{i=1}^{n} 2^{2k} (1 - |(z_0)_i|)^{2+\alpha}} \\ &\quad \leq C \sum_{k=1}^{\infty} 2^{-2kn(2+\alpha)} \frac{\mu_\alpha(E_k)}{\prod_{i=1}^{n} \delta_i^{2+\alpha}}.\end{aligned}$$

Since μ_α is assumed to be an α-Carleson measure $\mu_\alpha(E_k) \leq C \prod_{i=1}^{n} \delta_i^{2+\alpha}$, the above integral is finite. The proof of (ii) follows proposition 8(ii) in exactly the same manner that (i) followed proposition 8(i). ■.
Combining this proposition with theorem 5 gives the following characterization of those maps that induce bounded or compact composition operators on weighted Bergman spaces of polydiscs.

THEOREM 13. *For $\alpha > -1$ and $1 < p < \infty$*
(i) C_ϕ is a bounded composition operator on $A^p_\alpha(U^n)$ if and only if

$$\sup_{z_0 \in U^n} \int_{U^n} \prod_{i=1}^{n} \left[\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i \phi_i|^2} \right]^{2+\alpha} d\sigma_{n,\alpha} \leq M < \infty.$$

(ii) C_ϕ is a compact composition operator on $A^p_\alpha(U^n)$ if and only if

$$\lim \sup_{z_0 \in U^n} \int_{U^n} \prod_{i=1}^{n} \left[\frac{1 - |(z_0)_i|^2}{|1 - (\overline{z_0})_i \phi_i|^2} \right]^{2+\alpha} d\sigma_{n,\alpha} = 0 \ \ as \ \|z_0\| \longrightarrow 1.$$

Proof. Let $\mu_\alpha = \sigma_{n,\alpha} \circ \phi^{-1}$ and use proposition 12, as in theorem 9. ■
Corollary 6 can be used to obtain a uniform condition for composition operators to be bounded or compact on A^p_α for all $\alpha > -1+ \in$ by replacing the exponents $2+\alpha$ in theorem 13 by $(-1+ \in)$. Also note that if $\alpha \longrightarrow -1$, theorem 13 gives the result of theorem 9, since as $\alpha \longrightarrow -1, A^2_\alpha \longrightarrow H^2$, by corollary 1.2 [**3**]. We conclude this section by giving the following analog of a theorem of MacCluer and Shapiro [theorem **6.4, 6**] for those composition operators on $\hat{A}^2_\alpha$ induced by the univalent holomorphic maps of polydiscs. The essential role played by the Schwarz's lemma in this proof makes results of this type not reproducible for $A^p_\alpha(U^n)$. This result also points out that from a function theory standpoint the spaces $\hat{A}^p_\alpha$ are the correct analogs of the weighted Bergman spaces in the disc.

THEOREM 14. *Suppose $\phi : U^n \longrightarrow U^n$ is holomorphic and univalent and that the Frechet derivative of ϕ^{-1} is bounded on $\phi(U^n)$. Then for every $\alpha > 0$ the composition operator C_ϕ is bounded on $\hat{A}^2_\alpha(U^n)$.*

Proof. See [**4**, theorem 7.12]. ■
By the equivalence established between the weighted Bergman spaces, $\hat{A}^p_\alpha$, and the Dirichlet spaces in part (iv) of corollary 5.4 [**4**] theorem 14 can be restated for the Dirichlet spaces of polydiscs. We also note that the hypotheses of theorems 9 and 13 can be restated in the context of theorem 4.1 [**2**] using the notion of angular derivative. However, these restatements would involve some weakening of the hypotheses of theorems 9 or 13, as they would put pointwise bounds on estimates of the integrands involved. We conclude by giving a description of some direct results (as in [**6**]) relating angular derivatives and compact composition operators in Hardy and weighted Bergman spaces of polydiscs. These reults make use and serve as an application of the results reported in [**2**].

4. Angular derivatives and composition operators in polydiscs. The main tools in our study will be theorem 5 and the Julia-Caratheodory theorems for polydiscs (4.1-4.3 [**2**]). We shall use the following basic lemma about compact operators. Suppose that ϕ is holomorphic map of U^n into U^n. Then on all function spaces of U^n in which the automorphisms induce bounded composition operators, we have:

LEMMA 15. *C_ϕ is compact if and only if $C_\phi C_\psi$ or $C_\psi C_\phi$ is compact for some $\psi \in Aut(U^n)$.*

Proof. Immediate. ■

The idea of the following theorem comes from [**10**]:

THEOREM 16. *Suppose $\phi : U^n \longrightarrow U^n$ is holomorphic, and $C_\phi : H^p(U^n) \longrightarrow H^p(U^n)$ is compact for $1 < p < \infty$. Then for each $\zeta \in T^n$ there is an i so that*

$$\liminf_{z \to \zeta} \frac{1 - |\phi_i|^2}{1 - \|z\|^2} = \infty.$$

Proof. To show a contradiction suppose otherwise. Then there exists a $\zeta \in T^n$ so that the conclusion of the theorem is not reached. By lemma 15 no loss of generality will occur if we suppose that $\zeta = (1, 1, \ldots, 1)$ and $\phi(0, 0, \ldots, 0) = 0$. Then for all i

$$\liminf_{z \to (1,1,\ldots,1)} \frac{1 - |\phi_i(z)|^2}{1 - \|z\|^2} \leq M < \infty.$$

By theorem 4.1(c) [**2**] for every

$$D_\alpha = \left\{ z : \max_i \frac{|1 - z_i|}{1 - |z_i|} < \alpha,\ \Re z_i > 0 \right\}$$

and all i, j

$$\left| \frac{1 - \phi_i(z)}{1 - z_j} \right| < M(\alpha) = M.$$

In these circumstances we prove the noncompactness of C_ϕ by exhibiting a sequence $f_j \subset H^p$ which converges to zero weakly, whereas $C_\phi f_j$ does not converge to zero in the norm topology of H^p (Lemma 1.4 [**3**]). Fix p and let

$$f_\delta(z_1, z_2, \ldots, z_n) = \left((1 - \delta)^n \prod_{i=1}^{n} (1 - z_i)^{-\delta} \right)^{\frac{1}{p}}.$$

f_δ forms a bounded family in H^p as $\delta \longrightarrow 0$. To see this, we need to show that there exists a δ_0 so that $\|f_\delta\| < M$ for all $\delta < \delta_0$. Since $f_\delta(z_1, z_2, \ldots, z_n) = f_\delta(z_1) \ldots f_\delta(z_n)$, it is sufficient to consider one term in the iterated integral for $\|f_\delta\|_p$. Let $z = re^{i\theta}$ and write

$$\begin{aligned} \|f_\delta\|_p^p &= \sup_{0<r<1} (1 - \delta)^p \left(\int_T |1 - z|^{-p\delta}\, dm(\theta) \right) \\ &= \sup_{0<r<1} (1 - \delta)^p \left(\int_T |1 - re^{i\theta}|^{-2\lambda}\, dm(\theta) \right), \end{aligned}$$

where $2\lambda = p\delta$. Since $|re^{i\theta}| < 1$, using binomial expansion and integrating, by orthogonality we get

$$\|f_\delta\|_p^p = (1-\delta)^p \sum_{k=1}^{\infty} \left(\frac{\Gamma(k+\lambda)}{k!\Gamma(\lambda)}\right)^2.$$

By Stirling's formula

$$\frac{\Gamma(\lambda+k)}{k!} \approx k^{-1+\lambda} = k^{\frac{p\delta}{2}-1}.$$

Hence

$$\|f_\delta\|_p^p \leq C(1-\delta)^p \sum_{k=1}^{\infty} \frac{1}{k^{2-p\delta}}.$$

Now it is clear that if δ_0 is chosen such that $p\delta_0 < 1$, then for all $\delta < \delta_0$ f_δ is a bounded family in H^p. Furthermore $f_\delta \longrightarrow 0$ as $\delta \longrightarrow 1^-$ on compact subsets of U^n. For $z \in D_\alpha$

$$|f_\delta(z_1, z_2, \ldots, z_n) \circ \phi|^p = |(1-\delta)^n \prod_{i=1}^{n} (1 - \phi_i(z_1, z_2, \ldots, z_n))^{-\delta}|$$

and since on D_α, $|\frac{1-\phi_i}{1-z_j}| < M$

$$\geq M^{-n\delta}(1-\delta)^n \prod_{i=1}^{n} |1 - z_i|^{-\delta}.$$

Hence

$$\|f_\delta \circ \phi\|_p^p \geq (1-\delta)^{-n} M^{-n\delta} \sup_{0<r<1} \int_{T^n} \prod_{i=1}^{n} |1 - r\zeta_i|^{-\delta}\, dm_n$$

$$= M^{-n\delta} \|f_\delta\|_p^p.$$

This inequality contradicts the compactness of C_ϕ. ■

Interpreting theorem 16 in the context of Julia-Caratheodory theorems [**2**] states that if C_ϕ is compact then for each $\zeta \in T^n$ some component of ϕ cannot have angular derivative at ζ. In fact one has:

PROPOSITION 17. *Let $\phi : U^n \longrightarrow U^n$ be holomorphic. If C_ϕ is compact on $H^p(U^n)$ for some $1 < p < \infty$ then $\phi^*(\zeta) \in U^n$ for almost every $\zeta \in T^n$. As usual* $\phi^*(\zeta) = \lim_{r\to 1} \phi(r\zeta)$.

Proof. Suppose that C_ϕ is compact on $H^p(U^n)$ for some p fixed. Let

$$E_i = \{\zeta \in T^n : |\phi_i^*(\zeta)| = 1\}$$

We show that $m_n(E_i) = 0$ for all i. Let $e_{k,i}(z) = z_i{}^k$. Then $e_{k,i} \in H^p(U^n)$, $\|e_{k,i}\|_p = 1$, and $e_{k,i}$ converge to zero weakly. By lemma 1.4 [**3**], since C_ϕ is compact, $C_\phi e_{k,i}$ converge to zero in the norm topology of H^p. Thus

$$m_n(E_i) \leq \int_{T^n} |\phi_i{}^*|^{kp} dm_n = \|(\phi_i{}^k)\|_p^p \longrightarrow 0$$

Since $\|\phi\| = 1$ if and only if $|\phi_i{}^*| = 1$ for some i

$$\begin{aligned} m_n\{\|\phi\| = 1\} &= m_n\{|\phi_i{}^*| = 1 \text{ for some } i\} \\ &\leq \sum_{i=1}^n m_n(E_i) = 0. \end{aligned}$$

Hence $\|\phi(\zeta)\| < 1$ for almost all $\zeta \in T^n$. ■

Theorem 16 and proposition 17 give two necessary conditions for compactness in $H^p(U^n)$, $1 < p < \infty$. For the disc MacCluer-Shapiro ask whether these two conditions provide a sufficiency result for compactness in $H^p(U)$. They answer negatively by constructing an example $\phi : U \longrightarrow U$ so that $|\phi| < 1$ on T, and ϕ has no angular derivatives on T, yet ϕ fails to induce a compact composition operator. It is easy to show that essentially the same construction works for polydiscs.

Example. Take $\phi(z_1, z_2, \ldots, z_n) = (\phi_1(z_1), \phi_2(z_2), \ldots, \phi_n(z_n))$, where $\phi_i(z_i)$ are the same as those given by MacCluer-Shapiro [**6**, Example 3.8]. To show that C_ϕ is noncompact, let $\{f_n(z_i)\} \subset H^p(U)$ be sequences converging to zero weakly, for which $C_{\phi_i} f_n$ does not converge to zero in the norm topology of H^p. Put

$$f_n(z_1, z_2, \ldots, z_n) = \prod_{i=1}^n f_n(z_i).$$

This sequence shows that C_ϕ is not compact.

As in the disc, compactness in $A^p_\alpha(U^n)$, $1 < p < \infty$, and $\alpha > -1$ can be described more precisely in terms of conditions related to angular derivatives. Define

$$R_{i,j} = \frac{1 - |\phi_i|^2}{1 - |z_j|^2}.$$

THEOREM 18. *(i) If C_ϕ is a compact composition operator on $A^p_\alpha(U^n)$, then for each $\zeta \in T^n$ there is an (i,j) so that*

$$\liminf_{z \to \zeta} R_{i,j} = \infty. \tag{1}$$

(ii) If for each $\zeta \in T^n$ there is an (i,j) so that ϕ satisfies condition (1), $R_{i,j}$ are bounded away from zero for all i and j, and if C_ϕ is bounded on $A^2_\beta(U^n)$ for some $-1 < \beta < \alpha$ then C_ϕ is compact on $A^p_\alpha(U^n)$.

Let us note that since for all j

$$\frac{1 - |\phi_i|^2}{1 - \|z\|^2} \geq \frac{1 - |\phi_i|^2}{1 - |z_j|^2} \tag{2}$$

condition (1) implies that for some i the greatest lower bound of the quotient on the left side of (2) is infinite as z tends to $\zeta \in T^n$.

Proof. (i) To show a contradiction suppose that (1) is false. Then by (2) there exists a $\zeta \in T^n$ so that for all (i,j)

$$\lim_{z\to\zeta} R_{i,j} \leq L < \infty.$$

In particular then, since $\frac{1-|\phi_i|^2}{1-\|z\|^2} = \max_j R_{i,j}$

$$\liminf_{z\to\zeta} \frac{1-|\phi_i|^2}{1-\|z\|^2} \leq L < \infty \text{ for all i.}$$

Then by part (a) of (theorem 4.2 [**2**]) the Julia-Caratheodory theorem, for all i

$$\frac{|1-\phi_i(z)|^2}{1-|\phi_i(z)|^2} \leq L \max_j \frac{|1-z_j|^2}{1-|z_j|^2},$$

i.e. for all $0 < c < \frac{1}{L}$

$$\phi(E_c) \subset E_{Lc}.$$

Thus

$$\phi^{-1}(E_{Lc}) \supset E_c.$$

Define $\mu_\alpha = \sigma_{n,\alpha} \circ \phi^{-1}$, where $\sigma_{n,\alpha}$ is the Lebsegue volume measure on U^n with weights $\prod_{i=1}^n (1-|z_i|^2)^\alpha$. By theorem 5, to show that C_ϕ is not compact on $A^p_\alpha(U^n)$ it suffices to show that for all $\delta > 0$ there exists $c > 0$ so that

$$\mu_\alpha(E_{Lc}) = \sigma_{n,\alpha} \circ \phi^{-1}(E_{Lc}) \geq \delta > 0.$$

Fix $0 < \delta < 1$ and choose $c > 0$ (depending on δ) sufficiently small so that

$$S = S(R(\zeta, \delta)) \supset E_c.$$

Hence by Julia's lemma (theorem 4.1, part a [**2**])

$$\phi^{-1}(S) \supset \frac{E_c}{L}.$$

Therefore

(3) $$\mu_\alpha(S) = \sigma_{n,\alpha} \circ \phi^{-1}(S) \geq \sigma_{n,\alpha}(E_{\frac{c}{L}})$$

$$\geq C(\delta^n)^{2+\alpha} = C\delta^{n(2+\alpha)}.$$

Taking $\delta_i = \delta$ in theorem B (i) or in theorem 5, we see that μ_α is a compact α-Carleson measure if and only if

$$\lim_{\delta\to 0} \sup_{E_{L^c}\subset U^n} \frac{\mu_\alpha(E_{L^c})}{\delta^{n(2+\alpha)}} = 0.$$

Therefore by (3), μ_α is not compact α-Carleson, and hence C_ϕ is not compact. From this contradiction, (i) follows.

(ii) Conversely, fix $\zeta \in T^n$ and suppose that condition (1) holds for some (i,j), viz. (i_0, j_0). We show that μ_α is a compact α-Carleson measure. By lemma 15 we may assume that $\phi(0) = 0$. Put

$$h_j(\delta) = \sup\left\{ \frac{1-|z_j|^2}{1-|\phi_{i_0}(z)|^2};\ z:\ 1-|z_j| < \delta \right\}. \tag{4}$$

By hypothesis

$$\lim_{\delta\to 0} h_{j_0}(\delta) = 0$$

and all h_j are bounded. Let $R(\zeta, \delta)$ be the rectangle on T^n centered at ζ and radius δ in each coordinate, and let $S = S(R)$ denote the corona associated with this region. Suppose that $\phi(z) \in S$. Then since $|1-\phi_i(z)| < 2\delta$

$$1 - |\phi_{i_0}|^2 < 4\delta$$

and by (4)

$$1-|z_j|^2 \leq (1-|\phi_{i_0}|^2)h_j(\delta) \leq 4\delta\, h_j(\delta).$$

So for $\gamma > 0$

$$\prod_{i=1}^n (1-|z_i|^2)^\gamma \leq \prod_{i=1}^n (4\delta\, h_i(\delta))^\gamma.$$

Recalling that

$$\mu_\alpha = \sigma_{n,\alpha} \circ \phi^{-1}$$

we have

$$\begin{aligned}
\mu_\alpha(S) = \sigma_{n,\alpha}\circ\phi^{-1}(S) &= \int_S \prod_{i=1}^n (1-|z_i|^2)^\alpha\, d\sigma_n \\
&\leq \prod_{i=1}^n (4\delta\, h_i(\delta))^{(\alpha-\beta)} \int_S \prod_{i=1}^n (1-|z_i|^2)^\beta\, d\sigma_n \\
&\leq \delta^{n(\alpha-\beta)} \prod_{i=1}^n (4h_i(\delta))^{(\alpha-\beta)} \int_S \prod_{i=1}^n (1-|z_i|^2)^\beta\, d\sigma_n \\
&= \delta^{n(\alpha-\beta)} \prod_{i=1}^n (4h_i(\delta))^{(\alpha-\beta)} \mu_\beta(S).
\end{aligned}$$

Let $\prod_{i=1}^{n}(4h_i(\delta))^{\alpha-\beta} = \epsilon(\delta)$. Then by hypothesis, $\epsilon(\delta) \longrightarrow 0$ as $\delta \longrightarrow 0$. Since also by hypothesis C_ϕ is bounded on $A^p_\beta(U^n)$, μ_β is a bounded β-Carleson measure. Hence

$$\begin{aligned}\mu_\alpha(S) &\leq C\,\epsilon(\delta)\delta^{n(\alpha-\beta)}\delta^{n(2+\beta)}\\ &= C\,\epsilon(\delta)\delta^{n(2+\alpha)}.\end{aligned}$$

Hence μ_α is a compact α-Carleson measure. Therefore by theorem 5 C_ϕ is a compact composition operator on $A^p_\alpha(U^n)$, $1 < p < \infty$, $\alpha > \beta > -1$.
Using our remark following the statement of theorem 18 we can restate the conditions of theorem 18 in terms of our definition of angular derivative given by theorem 4.1 [**2**]:

RESTATEMENT OF THEOREM 18. *(i) If C_ϕ is a compact composition operator on $A^p_\alpha(U^n)$ then for each $\zeta \in T^n$ some component of ϕ cannot have an angular derivative at ζ.*
(ii) If for each $\zeta \in T^n$ some component of ϕ fails to have an angular derivative at ζ, $R^{-1}_{i,j}$ are bounded for all i and j, and if C_ϕ is bounded on $A^p_\beta(U^n)$ for some $-1 < \beta < \alpha$ then C_ϕ is compact on $A^p_\alpha(U^n)$. ■

As it is pointed out by the referee the proofs of theorems 16 and 18 might also be approached through consideration of the adjoint of composition operators applied to appropriate kernel functions. Propositions 7 and 11 may also be stated in these terms. The author thanks the referee for pointing out this and several corrections.

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