

TORSION IN TENSOR POWERS OF MODULES

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Abstract. Tensor products usually have nonzero torsion. This is a central theme of Auslander’s 1961 paper; the theme continues in the work of Huneke and Wiegand in the 1990s. The main focus in this article is on tensor powers of a finitely generated module over a local ring. Also, we study torsion-free modules N with the property that $M \otimes_R N$ has nonzero torsion unless M is very special. An important example of such a module N is the Frobenius power ${}^e R$ over a complete intersection domain R of characteristic $p > 0$.

§1. Introduction

In a 1961 paper, Auslander [1] studied torsion in tensor products of nonzero finitely generated modules M and N over unramified regular local rings R . Under the assumption that $M \otimes_R N$ is torsion-free, he proved that

- (1) M and N must be torsion-free, and
- (2) M and N are Tor independent; that is, $\mathrm{Tor}_i^R(M, N) = 0$ for all $i \geq 1$.

The two conclusions are cleverly intertwined in his proof, which we revisit in Section 3 of the present paper. We show, over a reduced complete intersection ring R of positive characteristic p , that $M \otimes_R {}^e R$ is torsion-free if and only if M is torsion-free and of finite projective dimension, in which case $\mathrm{Tor}_i^R(M, {}^e R) = 0$ for all $i \geq 1$. (Here $\varphi : R \rightarrow R$ is the Frobenius endomorphism and ${}^e R$ is the module obtained from R by restriction of scalars along φ^e .) When R is F-finite, we obtain a criterion for regularity: R is regular if and only if $(\varphi^e M) \otimes_R {}^e R$ is torsion-free for some (equivalently, every) nonzero finitely generated R -module M .

Our main results are in Section 2, where we study torsion in tensor powers. We obtain detailed information on annihilators of elements in $\otimes_R^n M$ and draw several conclusions. Suppose, for example, that $\underline{r} = r_1, \dots, r_d$ is a

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regular sequence in R and that M is the cokernel of the $d \times 1$ matrix $[\underline{r}]^t$. We show in Theorem 2.5 that $\otimes_R^t M$ is torsion-free if and only if $t \leq d$. This result should be compared with Auslander’s observation in [1, p. 638] that the same holds when M is the $(d - 1)$ st syzygy of a module of projective dimension d over a d -dimensional regular local ring (see also [7, Proposition 3.1]). If R is local, the “only if” direction holds much more generally: we show in Theorem 2.7 that, if we write M as the cokernel of an $m \times n$ matrix θ with entries in the maximal ideal of R , and if some entry of θ is a nonzerodivisor, then $\otimes_R^t M$ has nonzero torsion for every $t \geq m$.

Throughout this article, R is a commutative, Noetherian ring.

§2. Torsion in tensor powers

In this section we establish results on annihilators of elements in tensor powers of modules.

NOTATION 2.1. Given elements $\underline{m} := m_1, \dots, m_d$ in an R -module M , we consider the element in $\otimes_R^d M$ defined by

$$\tau(\underline{m}) := \sum_{\sigma \in S_d} \text{sign}(\sigma) m_{\sigma(1)} \otimes \cdots \otimes m_{\sigma(d)}.$$

PROPOSITION 2.2. *Let M be an R -module. If elements m_1, \dots, m_d in M and r_1, \dots, r_d in R satisfy*

$$(2.2.1) \quad r_1 m_1 + \cdots + r_d m_d = 0,$$

then $(r_1, \dots, r_d) \cdot \tau(\underline{m}) = 0$ in $\otimes_R^d M$.

Proof. The twisted shuffle product gives the graded R -algebra $\bigoplus_{n \geq 0} \otimes_R^n M$ a strictly skew-commutative structure (see [11, Chapter X, (12.4)]. Strictly skew-commutative means that for any $a \in \otimes_R^i M$ and $b \in \otimes_R^j M$ there are equalities

$$a \star b = (-1)^{ij} b \star a, \quad \text{and} \quad a \star a = 0 \quad \text{when } i \text{ is odd.}$$

By definition of the shuffle product, $\tau(\underline{m}) = m_1 \star \cdots \star m_d$. Thus, for each j we have

$$\begin{aligned} r_j \cdot \tau(\underline{m}) &= m_1 \star \cdots \star m_{j-1} \star r_j m_j \star m_{j+1} \star \cdots \star m_n \\ &= - \sum_{i \neq j} r_i (m_1 \star \cdots \star m_{j-1} \star m_i \star m_{j+1} \star \cdots \star m_n) = - \sum_{i \neq j} r_i 0 = 0. \quad \square \end{aligned}$$

There is a “universal” source for the element $\tau(\underline{m})$ in the following sense.

REMARK 2.3. Consider the polynomial ring $\mathbb{Z}[\underline{x}]$ on indeterminates $\underline{x} := x_1, \dots, x_d$, and let U be the $\mathbb{Z}[\underline{x}]$ -module with presentation

$$0 \rightarrow \mathbb{Z}[\underline{x}] \xrightarrow{[x_1, \dots, x_d]^t} \mathbb{Z}[\underline{x}]^d \rightarrow U \rightarrow 0.$$

Let u_1, \dots, u_d be the generators of U corresponding to the standard basis for $\mathbb{Z}[\underline{x}]^d$, so that $x_1u_1 + \dots + x_du_d = 0$; that is, \underline{x} and \underline{u} satisfy (2.2.1). Then $\text{ann}_{\mathbb{Z}[\underline{x}]} \tau(\underline{u}) \supseteq (\underline{x})$ by Proposition 2.2; we will see, in Theorem 2.5 below, that in fact $\text{ann}_{\mathbb{Z}[\underline{x}]} \tau(\underline{u}) = (\underline{x})$.

Given any R -module M with a syzygy relation (2.2.1), consider the ring homomorphism $\mathbb{Z}[\underline{x}] \rightarrow R$ taking x_i to r_i , for each i , and extending the structure homomorphism $\mathbb{Z} \rightarrow R$. The hypothesis on M implies that there is a homomorphism of $\mathbb{Z}[\underline{x}]$ -modules

$$f: U \rightarrow M \quad \text{with } f(u_i) = m_i \text{ for } i = 1, \dots, d.$$

Under the induced map $\otimes^d f: \otimes_{\mathbb{Z}[\underline{x}]}^d U \rightarrow \otimes_R^d M$, the element $\tau(\underline{u})$ maps to $\tau(\underline{m})$.

This remark prompts the discussion below, culminating in Theorem 2.5. First we review some notions regarding depth. For details, see [4, Chapter 1].

2.1. Depth

Let M be a finitely generated R -module, and let I be an ideal of R satisfying $IM \neq M$. The I -depth of M is the number

$$\text{depth}_R(I, M) = \inf \{ n \geq 0 \mid \text{Ext}_R^n(R/I, M) \neq 0 \}.$$

The I -depth of M is always finite and is equal to the length of every maximal M -regular sequence in I .

If $\underline{x} := x_1, \dots, x_d$ is a sequence of elements in R , and if K is the Koszul complex on \underline{x} , then the (\underline{x}) -depth of M may be computed from its Koszul homology:

$$\text{depth}_R((\underline{x}), M) = d - \sup \{ i \geq 0 \mid H_i(K \otimes_R M) \neq 0 \}.$$

This is the *depth sensitivity* of the Koszul complex.

Suppose now that \underline{x} is R -regular. Then K is a free resolution of $R/(\underline{x})$, and hence $H_*(K \otimes_R M) \cong \text{Tor}_*^R(R/(\underline{x}), M)$. In this case, we have

$$(2.3.1) \quad \text{depth}_R((\underline{x}), M) = d - \sup \{ i \geq 0 \mid \text{Tor}_i^R(R/(\underline{x}), M) \neq 0 \}.$$

If R is local with maximal ideal \mathfrak{m} , we write $\text{depth}_R M$ for the \mathfrak{m} -depth of M and call it the *depth* of M .

2.2. A Koszul syzygy module

Let R be a Noetherian ring, and let $\underline{r} := r_1, \dots, r_d$ be a regular sequence in R with $(\underline{r}) \neq R$. Consider the complex

$$F := 0 \rightarrow R \xrightarrow{[r_1, \dots, r_d]^t} R^d \rightarrow 0$$

concentrated in degrees 0 and 1. Set $M = H_0(F)$; as r_1 is a nonzerodivisor, F is a free resolution of M .

LEMMA 2.4. *Let M , d , and F be as in Section 2.2. For each $n = 1, \dots, d$, the following statements hold:*

- (1) M and $\otimes_R^{n-1} M$ are Tor independent; and
- (2) $\otimes_R^n F$ is a free resolution of $\otimes_R^n M$, and $\text{pd}_R(\otimes_R^n M) = n$.

Proof. The base case is $n = 1$, and then (1) and (2) are clear. Fix an integer n with $2 \leq n \leq d$, and assume that these statements hold for all integers $\leq n - 1$. Set $I = (\underline{r})$. Since $\otimes_R^{n-1} F$ is a free resolution of $\otimes_R^{n-1} M$, we have

$$\text{Tor}_*^R(R/I, \otimes_R^{n-1} M) = H_*((R/I) \otimes_R (\otimes_R^{n-1} F)) \cong (\otimes_R^{n-1} ((R/I) \otimes_R F))_*$$

where the last isomorphism holds because the complex in question has zero differential. In particular, $\text{Tor}_{n-1}^R(R/I, \otimes_R^{n-1} M) \cong R/I \neq 0$, so that

$$(2.4.1) \quad \sup\{i \geq 0 \mid \text{Tor}_i^R(R/(\underline{r}), \otimes_R^{n-1} M) \neq 0\} = n - 1.$$

We can now complete the induction step.

(1) The induction hypothesis implies that $\otimes_R^{n-1} F$ is a free resolution of $\otimes_R^{n-1} M$, so (2.4.1) and (2.3.1) show that

$$(2.4.2) \quad \text{depth}_R(I, \otimes_R^{n-1} M) = d - (n - 1) \geq 1.$$

Moreover, $\text{Tor}_*^R(M, \otimes_R^{n-1} M)$ is the homology of the complex

$$F \otimes_R (\otimes_R^{n-1} M) : 0 \rightarrow \otimes_R^{n-1} M \xrightarrow{[\underline{r}]^t} (\otimes_R^{n-1} M)^d \rightarrow 0$$

(concentrated in degrees 0 and 1). By (2.4.2), some r_i is a nonzerodivisor on $\otimes_R^{n-1} M$, and it follows that M and $\otimes_R^{n-1} M$ are Tor independent.

(2) By hypothesis, F and $\otimes_R^{n-1} F$ are free resolutions of M and $\otimes_R^{n-1} M$, respectively. We have already proved, in (1), that these modules are Tor independent, so the complex $F \otimes_R (\otimes_R^{n-1} F)$, that is, $\otimes_R^n F$, is a free resolution of $\otimes_R^n M$. In particular, $\text{pd}_R(\otimes_R^n M) \leq n$; that the equality holds follows from (2.4.1). □

2.3. Torsion submodule

Let $Q(R)$ be the total quotient ring of R . The *torsion submodule* $\top_R M$ of M is the kernel of the natural homomorphism $M \rightarrow Q(R) \otimes_R M$. The inclusion $\top_R M \subseteq M$ gives rise to an exact sequence

$$(2.4.3) \quad 0 \rightarrow \top_R M \rightarrow M \rightarrow \perp_R M \rightarrow 0.$$

The module M is *torsion* if $\top_R M = M$ (i.e., $M_{\mathfrak{p}} = 0$ for each $\mathfrak{p} \in \text{Ass}(R)$), and M is *torsion-free* if $\top_R M = 0$. Thus, M is torsion-free if and only if $\bigcup \text{Ass } M \subseteq \bigcup \text{Ass } R$. The stronger condition, that $\text{Ass } M \subseteq \text{Ass } R$, is therefore a sufficient condition for M to be torsion-free. We will invoke this criterion twice in the proof of the next theorem.

Part (1) of the next result is reminiscent of Auslander’s discussion in [1, p. 638] (see also [7, Proposition 3.1]).

THEOREM 2.5. *Let M and \underline{r} be as in Section 2.2. The following statements hold:*

- (1) $\otimes_R^n M$ is torsion-free if and only if $n \leq d - 1$;
- (2) the element $\tau(\underline{m})$ in $\otimes_R^d M$ satisfies $\text{ann}_R \tau(\underline{m}) = (\underline{r})$; and
- (3) the map $R/(\underline{r}) \rightarrow \otimes_R^d M$ of R -modules with $1 \mapsto \tau(\underline{m})$ induces a splitting

$$\otimes_R^d M \cong (R/(\underline{r})) \oplus W,$$

where W is torsion-free; in particular, we have

$$\text{Hom}_R(R/(\underline{r}), \otimes_R^d M) = R\tau(\underline{m}) \neq 0.$$

Proof. Set $I = (\underline{r})$, let $n \leq d - 1$, and fix a prime $\mathfrak{p} \in \text{Ass}(\otimes_R^n M)$. If $I \subseteq \mathfrak{p}$, it follows from Lemma 2.4 that $(\otimes_R^n F)_{\mathfrak{p}}$ is a minimal free resolution of $(\otimes_R^n M)_{\mathfrak{p}}$; therefore,

$$\text{depth}_{R_{\mathfrak{p}}}(\otimes_R^n M)_{\mathfrak{p}} = \text{depth } R_{\mathfrak{p}} - n \geq d - n \geq 1,$$

which is a contradiction. Thus, $I \not\subseteq \mathfrak{p}$, and then the $R_{\mathfrak{p}}$ -module $M_{\mathfrak{p}}$ is a nonzero free module; hence, so is $(\otimes_R^n M)_{\mathfrak{p}}$. Therefore, $\text{depth } R_{\mathfrak{p}} = \text{depth}_{R_{\mathfrak{p}}}(\otimes_R^n M)_{\mathfrak{p}}$. We have shown that $\text{Ass}(\otimes_R^n M) \subseteq \text{Ass } R$, and hence that $\otimes_R^n M$ is torsion-free. The “only if” direction of (1) will follow from (3).

For (2) and (3), by construction $r_1 m_1 + \dots + r_d m_d = 0$, so Proposition 2.2 gives an inclusion $I \subseteq \text{ann}_R \tau(\underline{m})$. The reverse inclusion will follow, once we ascertain that the map in (3) splits. Consider the homomorphisms of R -modules

$$\begin{aligned} \otimes_R^d(F_0) &\twoheadrightarrow \otimes_R^d M \twoheadrightarrow (\otimes_R^d M) \otimes_R R/I \cong H_0((\otimes_R^d F) \otimes_R R/I) \\ &= \otimes_R^d(F_0 \otimes_R R/I), \end{aligned}$$

where the surjections are the natural ones; the isomorphism holds because $\otimes_R^d F$ is a free resolution of $\otimes_R^d M$, and the equality holds because the differential on F has its image in IF . Let $\underline{e} = e_1, \dots, e_d$ be the standard basis for $F_0 = R^d$, in Section 2.2, and let \underline{e}' be the induced basis of the free R/I -module $F_0 \otimes_R R/I$. Under the composite map, the element $\tau(\underline{e})$ maps to $\tau(\underline{e}')$, and $\{\tau(\underline{e}')\}$ extends to a basis of the R/I -module $\otimes_R^d(F_0 \otimes_R (R/I))$. Since $\tau(\underline{e})$ maps to $\tau(\underline{m})$ in $\otimes_R^d M$, the map in (2) splits and gives a decomposition

$$\otimes_R^d M \cong (R/I) \oplus W.$$

It remains to verify that W is torsion-free; given the decomposition above, the other parts of (3) are a consequence of this fact.

For $\mathfrak{p} \in \text{Spec } R$ with $I \not\subseteq \mathfrak{p}$, the $R_{\mathfrak{p}}$ -module is $M_{\mathfrak{p}}$ free, and hence so is $W_{\mathfrak{p}}$. Assume now that $I \subseteq \mathfrak{p}$. The Koszul complex on \underline{r} , viewed as elements in $R_{\mathfrak{p}}$, is a minimal resolution of $(R/I)_{\mathfrak{p}}$, and so it is a direct summand of $(\otimes_R^n F)_{\mathfrak{p}}$, the minimal free resolution of $(\otimes_R^n M)_{\mathfrak{p}}$. The ranks of the free modules in the top degree, d , of these complexes coincide (and equal 1), whence $\text{pd}_{R_{\mathfrak{p}}} W_{\mathfrak{p}} \leq d - 1$ and

$$\text{depth}_{R_{\mathfrak{p}}} W_{\mathfrak{p}} = \text{depth } R_{\mathfrak{p}} - \text{pd}_{R_{\mathfrak{p}}} W_{\mathfrak{p}} \geq 1.$$

These observations show that $\text{Ass } W \subseteq \text{Ass } R$, so W is torsion-free as claimed. □

2.4. Local rings

Next we focus on local rings, where the preceding results can be strengthened to some extent.

LEMMA 2.6. *Let M be a finitely generated module over a local ring (R, \mathfrak{m}) , and let $m_1, \dots, m_d \in M$. If the images of $\{m_1, \dots, m_d\}$ in $M/\mathfrak{m}M$ are linearly independent, then $\tau(\underline{m})$ is not in $\mathfrak{m}(\otimes_R^d M)$.*

Proof. Let m'_i be the image of m_i in the k -vector space $M/\mathfrak{m}M$. Since $\{m'_1, \dots, m'_d\}$ is linearly independent, $\tau(\underline{m}') \neq 0$. Hence, $\tau(\underline{m}) \notin \mathfrak{m}(\otimes_R^d M)$. □

Given an R -module M , we write $I(M)$ for the ideal (r_{ij}) defined by the entries in a matrix in some minimal presentation

$$R^\mu \xrightarrow{[r_{ij}]} R^\nu \rightarrow M \rightarrow 0 \quad \text{where } \nu = \nu_R(M).$$

This ideal is independent of the presentation. Moreover, $I(M)$ contains a nonzerodivisor if and only if, over $Q(R)$, the total quotient ring of R , the module $Q(R) \otimes_R M$ can be generated by fewer than ν elements. To see this we note that, since $Q(R)$ is semilocal, the module $Q(R) \otimes_R M$ needs ν generators if and only if $\nu_{R_p} M_p = \nu$ for some $\mathfrak{p} \in \text{Ass } R$; moreover, $\nu_{R_p} M_p = \nu$ if and only if the presentation remains minimal when localized at \mathfrak{p} , that is, if and only if $I(M) \subseteq \mathfrak{p}$. Thus, $Q(R) \otimes_R M$ needs ν generators if and only if $I(M) \subseteq \mathfrak{p}$ for some $\mathfrak{p} \in \text{Ass } R$, that is to say, if and only if $I(M)$ consists of zerodivisors.

Recall that M is said to have rank r if $Q(R) \otimes_R M$ is free over $Q(R)$ of rank r (see [4, Proposition 1.4.3] for different characterizations of this property).

THEOREM 2.7. *Let R be a local ring, and let M be a nonzero finitely generated R -module satisfying one of the following conditions:*

- (1) $I(M)$ contains a nonzerodivisor; in this case, set $b = \nu_R(M)$; or
- (2) M has rank; in this case, set $b = \text{rank}_R(M) + 1$.

If M is not free, then for each nonzero finitely generated R -module N one has

$$\text{Tr}((\otimes_R^n M) \otimes_R N) \neq 0 \quad \text{for each } n \geq b.$$

Proof. It suffices to prove the statement for $n = b$, since

$$(\otimes_R^n M) \otimes_R N \cong (\otimes_R^b M) \otimes_R ((\otimes_R^{n-b} M) \otimes_R N),$$

and $N \neq 0$ implies that $(\otimes_R^i M) \otimes_R N \neq 0$ for each $i \geq 0$, by Nakayama's lemma.

(1) Let m_1, \dots, m_b be a minimal generating set for the R -module M . The element $\tau(\underline{m})$ in $\otimes_R^b M$ is annihilated by $I(M)$, by Proposition 2.2, and is not in $\mathfrak{m}(\otimes_R^b M)$, by Lemma 2.6. It follows that, for each x in $N \setminus \mathfrak{m}N$, the element $\tau(\underline{m}) \otimes x$ in $(\otimes_R^b M) \otimes_R N$ is nonzero and is annihilated by $I(M)$ and hence is in the torsion submodule; this is where the hypothesis that $I(M)$ contains a nonzerodivisor is used.

(2) We claim that there exists a syzygy relation (2.2.1) with \underline{m} a minimal generating set for M , $(\underline{r}) \subseteq \mathfrak{m}$, and some r_i a nonzerodivisor.

Indeed, $\nu_R(M) \geq b$ since M is not free. Choose elements m_1, \dots, m_b that form part of a minimal generating set for M and such that m_1, \dots, m_{b-1} form a basis for $Q(R) \otimes_R M$ over $Q(R)$. Then there is a syzygy relation as in (2.2.1) in which r_b is a nonzerodivisor.

The element $\tau(\underline{m})$ in $\otimes_R^b M$ is annihilated by (\underline{r}) , by Proposition 2.2, and is not in $\mathfrak{m}(\otimes_R^b M)$, by Lemma 2.6. Since (\underline{r}) has a nonzerodivisor, it follows as in (1) that the torsion submodule of $(\otimes_R^b M) \otimes_R N$ is nonzero. \square

We learned recently that in 2011, in response to a query on MathOverflow, David Speyer gave a proof in [13] of (1) that is quite similar to ours when R is a domain.

One cannot always expect torsion in tensor powers of nonfree modules, as the following shows.

EXAMPLE 2.8. Let $R = k[[x, y]]/(xy)$, where k is a field. The torsion-free R -module $M := R/(x)$ is not free; however, $\otimes_R^n M$ is isomorphic to $R/(x)$ for every $n \geq 1$ and hence is torsion-free.

The preceding results bring to the fore the following question:

QUESTION 2.9. Let R be a local domain. Is there an integer b , depending only on R , such that $\otimes_R^n M$ has torsion for every finitely generated nonfree R -module M and every integer $n \geq b$?

The condition that R be a domain is to avoid the situation of Example 2.8. When R is regular, one can take $b = \dim R$, by results of Auslander [1, Theorem 3.2] and Lichtenbaum [10, Corollary 3].

§3. Torsion “carriers”

Some modules, even though they are torsion-free, usually generate torsion in tensor products. For example, over a local ring (R, \mathfrak{m}, k) of positive depth, the maximal ideal \mathfrak{m} is such a module: for any finitely generated nonfree R -module M , the tensor product $\mathfrak{m} \otimes_R M$ has torsion. To see this, observe that the short exact sequence

$$0 \rightarrow \mathfrak{m} \rightarrow R \rightarrow k \rightarrow 0$$

yields an injection from the torsion module $\text{Tor}_1^R(k, M)$ into $\mathfrak{m} \otimes_R M$; moreover, $\text{Tor}_1^R(k, M) \neq 0$ because M is not free.

We give two more examples of torsion carriers: the integral closure \overline{R} of a 1-dimensional analytically unramified ring R , and the Frobenius powers

$\varphi^e R$ of a complete intersection R of characteristic p . Recall that a local ring is *analytically unramified* provided that its completion is reduced. If R is 1-dimensional, an equivalent condition is that R be Cohen–Macaulay with finitely generated integral closure \overline{R} (see [9, Theorem 4.6]).

THEOREM 3.1. *Let R be a 1-dimensional analytically unramified local ring, and let \overline{R} be the integral closure of R in its total quotient ring. If M is a finitely generated R -module for which $\overline{R} \otimes_R M$ is torsion-free, then M is free.*

Proof. Let $\mathfrak{p}_1, \dots, \mathfrak{p}_s$ be the minimal prime ideals of R , and for each i let r_i be the dimension of the $R_{\mathfrak{p}_i}$ -vector space $M_{\mathfrak{p}_i}$. Put $n = \nu_R M$, the minimal number of generators of the R -module M , and choose an exact sequence

$$0 \rightarrow K \rightarrow R^{(n)} \rightarrow M \rightarrow 0.$$

If we can show that $r_i = n$ for each i , we will know that K is torsion and hence zero, and we will be done.

Put $D_i = \overline{R/\mathfrak{p}_i}$, the integral closure of the domain R/\mathfrak{p}_i . Since R is reduced, we have inclusions

$$R \hookrightarrow \prod_{i=1}^s R/\mathfrak{p}_i \hookrightarrow \prod_{i=1}^s D_i \hookrightarrow \prod_{i=1}^s Q(R/\mathfrak{p}_i) = Q(R).$$

We see that $\overline{R} = \prod_{i=1}^s D_i$; moreover, each D_i is a semilocal Dedekind domain and therefore a principal ideal domain. Since $\overline{R} \otimes_R M$ is torsion-free, it is projective, in fact free of rank r_i on the component D_i . Therefore, setting $e_i = \nu_R D_i$, we have the equations

$$r_1 e_1 + \dots + r_s e_s = \nu_R(\overline{R} \otimes_R M) = (\nu_R \overline{R}) \cdot (\nu_R M) = (e_1 + \dots + e_s)n.$$

Since $r_i \leq n$ for each i , it follows from these equations that $r_i = n$ for each i . □

Let R be a Noetherian ring of positive characteristic p , and let $\varphi: R \rightarrow R$ be the Frobenius endomorphism $r \mapsto r^p$. Given an R -module M and a positive integer e , we write $\varphi^e M$ for the R -module obtained from M by restriction of scalars along φ^e ; thus, $r \cdot m = r^{p^e} m$ for $r \in R$ and $m \in M$. Observe that M is torsion-free if and only if $\varphi^e M$ is torsion-free for some (equivalently, all) $e \geq 1$. Following [12], we write $F^e(M)$ for the tensor product $M \otimes_R \varphi^e R$. One views $F^e(M)$ as a right R -module: the action of R on $F^e(M)$

comes from the right (ordinary) action of R on $\varphi^e R$. Thus, $F^e(R) \cong R$ as R -modules, and it follows that $F^e(M)$ is finitely generated if M is finitely generated.

THEOREM 3.2 ([12, Corollaire 1.10]). *Let R be a local ring of characteristic p , and let M be a finitely generated R -module. If M has finite projective dimension, then $\text{Tor}_i^R(M, \varphi^e R) = 0$ for all $e \geq 1$ and all $i \geq 1$.*

The converse of Theorem 3.2 is true and was proved by Herzog [5, Theorem 3.1]. For complete intersections, the following strong converse was proved by Avramov and Miller.

THEOREM 3.3 ([3, Main Theorem]). *Let (R, \mathfrak{m}) be a complete intersection of characteristic p , and let M be a finitely generated R -module. If $\text{Tor}_i^R(M, \varphi^e R) = 0$ for some $e \geq 1$ and some $i \geq 1$, then M has finite projective dimension.*

The proof that (1) \implies (2) in the next theorem follows many of the same steps Auslander used in [1, proof of Lemma 3.1]. The main differences are that we have to allow for the possibility that $\varphi^e R$ is not finitely generated and that we appeal to Theorems 3.2 and 3.3 for a replacement of rigidity of Tor over regular local rings. Recall that a module M is generically free provided that $M_{\mathfrak{p}}$ is a free $R_{\mathfrak{p}}$ -module for each $\mathfrak{p} \in \text{Ass } R$.

THEOREM 3.4. *Let (R, \mathfrak{m}) be a complete intersection of characteristic p , and let M be a finitely generated, generically free R -module. Fix a positive integer e . The following conditions are equivalent:*

- (1) $F^e(M)$ is torsion-free, and
- (2) M is torsion-free and of finite projective dimension.

Proof. Suppose that (1) holds, and apply $-\otimes_R \varphi^e R$ to the short exact sequence (2.4.3), getting an exact sequence

$$F^e(\tau_R M) \xrightarrow{\alpha} F^e(M) \xrightarrow{\beta} F^e(\perp_R M) \rightarrow 0.$$

Since $F^e(\tau_R M)$ is torsion and $F^e(M)$ is torsion-free, we see that $\alpha = 0$, whence β is an isomorphism. In particular, $F^e(\perp_R M)$ is torsion-free. Next, consider the universal pushforward (see [6, Section 1]):

$$(3.4.1) \quad 0 \rightarrow \perp_R M \rightarrow R^{(m)} \rightarrow N \rightarrow 0.$$

Applying $-\otimes_R \varphi^e R$ to this sequence, we obtain an injection

$$\text{Tor}_1^R(N, \varphi^e R) \hookrightarrow F^e(\perp_R M).$$

Now $\perp_R M$ is clearly generically free, and from the construction of the universal pushforward in [6, Section 1], one checks that N is generically free as well. It follows that $\text{Tor}_1^R(N, \varphi^e R)$ is torsion. Since $F^e(\perp_R M)$ is torsion-free, we have $\text{Tor}_1^R(N, \varphi^e R) = 0$. Now we invoke Theorems 3.2 and 3.3 to see that

$$\text{Tor}_i^R(N, \varphi^e R) = 0 \quad \text{for all } i \geq 1$$

and, moreover, that N has finite projective dimension. From (3.4.1) it follows that $\text{Tor}_i^R(\perp_R M, \varphi^e R) = 0$ for all $i \geq 1$ and that $\perp_R M$ has finite projective dimension. Therefore, we will have (2) once we show that $\top_R M = 0$. For this, we apply $-\otimes_R \varphi^e R$ once again to (2.4.3), to get an injection

$$F^e(\top_R M) \hookrightarrow F^e(M).$$

Since $F^e(\top_R M)$ is torsion and $F^e(M)$ is torsion-free, we have $F^e(\top_R M) = 0$. If $\top_R M$ were nonzero, there would be a surjection $\top_R M \twoheadrightarrow R/\mathfrak{m}$. But then $F^e(R/\mathfrak{m}) = 0$, that is, $\mathfrak{m}^{\varphi^e R} = \varphi^e R$, an obvious contradiction, since $\mathfrak{m}^{\varphi^e R} \subseteq \mathfrak{m}$. Thus $\top_R M = 0$, and the proof that (1) \implies (2) is complete.

Now assume that (2) holds. Since M is torsion-free, we can build the universal pushforward (see [6, Section 1]):

$$0 \rightarrow M \rightarrow R^{(\nu)} \rightarrow N \rightarrow 0,$$

where $\nu = \nu_R M^*$. Then N has finite projective dimension. Now Theorem 3.2 implies that $\text{Tor}_i^R(N, \varphi^e R) = 0$ for all $i \geq 1$. Therefore, $\text{Tor}_1^R(M, \varphi^e R) = 0$, and we get an injection $F^e(M) \hookrightarrow (\varphi^e R)^{(\nu)}$, whence $F^e(M)$ is torsion-free. \square

From Theorem 3.2 (alternatively, from the proof of Theorem 3.4), we get Tor independence (item (2) in the Introduction), as follows.

COROLLARY 3.5. *If R and M satisfy the equivalent conditions of Theorem 3.4, then $\text{Tor}_i^R(M, \varphi^{e'} R) = 0$ for every $i \geq 1$ and every $e' \geq 1$.*

Of course, if M is torsion-free, the converse of Corollary 3.5 holds, by Theorem 3.3. In fact, it suffices to check that $\text{Tor}_i^R(M, \varphi^{e'} R) = 0$ for a single e' and a single i .

Recall that R is F -finite provided that φ is a finite map, that is, that R is module-finite over $\varphi(R)$. In this case, φ^e is a finite map for each $e \geq 1$. Note that the action of R on the module $(\varphi^e M)$ in items (1) and (2) below is the Frobenius action $m \cdot r = mr^{p^e}$.

COROLLARY 3.6. *Assume that (R, \mathfrak{m}) is a reduced local ring, is F -finite, and is a complete intersection. The following conditions are equivalent:*

- (1) $F^e(\varphi^{e'} M)$ is torsion-free for every torsion-free R -module M and every pair e, e' of positive integers;
- (2) $F^e(\varphi^{e'} M)$ is torsion-free for some nonzero finitely generated R -module M and some pair e, e' of positive integers; and
- (3) R is regular.

Proof. Obviously (1) \implies (2), and the implication (3) \implies (1) holds by Kunz's theorem [8, Theorem 2.1] that the R -module $\varphi^e R$ is flat when R is regular.

To prove that (2) \implies (3), we note that $\varphi^{e'} M$ is a finitely generated R -module, by F -finiteness. Also, $\varphi^{e'} M$ is generically free because R is reduced. By Theorem 3.4, $\varphi^{e'} M$ has finite projective dimension, and now [2, Theorem 1.1] implies that R is regular. \square

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