MINIMAL TERM RANK OF A CLASS OF (0, 1)-MATRICES

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Let $\mathfrak{A}(R, S)$ be the class of $m \times n$ matrices all of whose entries are either 0 or 1 where every matrix A in the class $\mathfrak{A}(R, S)$ satisfies the conditions that row i of A has r_i ones, $i = 1, 2, \ldots, m$; and column j of A has s_j ones, $j = 1, 2, \ldots, n$. We let $R = (r_1, \ldots, r_m)$, $S = (s_1, \ldots, s_n)$, and assume that $r_1 \geqslant r_2 \geqslant \ldots \geqslant r_m \geqslant 0$; $s_1 \geqslant s_2 \geqslant \ldots \geqslant s_n \geqslant 0$. When this is the case we say the class $\mathfrak{A}(R, S)$ is *normalized.* We find a formula for the minimal term rank of $\mathfrak{A}(R, S)$ analogous to formulas for maximal term rank, minimal and maximal trace, and minimal column width already developed by Ryser and Fulkerson **(1, 3, 4, 5, 6)**. (For definitions and a more complete bibliography see **(6)**.)

1. A normal form for matrices in the class $\mathfrak{A}(R, S)$

THEOREM 1.1 *Consider the normalized class $\mathfrak{A}(R, S)$. Then for every e, f satisfying $1 \leqslant e \leqslant m$, $1 \leqslant f \leqslant n$, there exists an A in $\mathfrak{A}(R, S)$ which has one of the following two forms:*

$$\text{(a)} \left[\begin{array}{c|c|c|c} S & & S & \\ \hline & O & & O \\ \hline S & & & O \\ \hline & O & O & O \end{array}\right] \qquad \text{(b)} \left[\begin{array}{c|c|c} & S & \\ \hline S & S & \\ \hline & & O \end{array}\right]$$

Here the block

$$\left[\begin{array}{c|c} S & \\ \hline & O \end{array}\right]$$

in the upper left-hand corner of (a) *is a submatrix of size $e \times f$ and the one block in the upper left-hand corner of* (b) *is a submatrix of size $e \times f$. Here, as in what follows, S denotes a block of solid ones and O a block of solid zeros. Also any of the blocks in the decomposition may have one or both dimensions zero.*

The proof is by induction. For $e = f = 1$, if $\mathfrak{A}(R, S)$ has an invariant 1, then every A in $\mathfrak{A}(R, S)$ has form (b). (See **2** or **4** or **6**.) If there are no invariant ones, then there is a matrix A in $\mathfrak{A}(R, S)$ with a 0 in the 1–1 position and this matrix has form (a) with the S blocks not appearing and the O block in the lower right-hand corner of dimension 0×0. We assume the theorem true for $e, f - 1$ and prove it for e, f. Interchanging the roles of rows and

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columns will then imply that the theorem for $e-1, f$ will imply the theorem for e, f and will complete the proof.

We first note that degenerate forms of (b), namely

$$\text{(b}'\text{)} \qquad \left[\begin{array}{c|c|c} & S & \\ \hline & & O \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} & \\ \hline S & \\ \hline & O \end{array}\right],$$

are both of form (a), where in both matrices the block in the upper left-hand corner is a submatrix of size $e \times f$.

Suppose now that for $e, f-1$ there is a matrix A of form (b) which is not of the degenerate forms (b′). Then

$$A = \left[\begin{array}{c|c|c} B & S_1 & \\ \hline S_2 & S & \\ \hline & & O \end{array}\right],$$

where B is $e \times (f-1)$ and both S_1 and S_2 are blocks of ones with no dimension 0, and A already has the required form (b) for e, f.

We may now assume that there is a matrix A of form (a) for $e, f-1$ and show that this implies that there is a matrix A of form (a) or (b) for e, f to complete the proof.

The following diagram will prove helpful.

$$\text{(c)} \qquad Z = \left[\begin{array}{cc|c|c|c|c|c} \multicolumn{2}{c|}{A} & & A & A & A & \\ \hline L & & & D & E & F & \\ \cline{1-2}\cline{4-6} K & J & B & C & & G & B \\ \hline \multicolumn{2}{c|}{A} & & & & H & B \\ \hline P & N & & & & & \\ \cline{1-2} & M & B & B & B & B & B \end{array}\right]$$

Here the second A in the first row of the decomposition consists of one column and is in the fth column of Z. The bottom row of K is in the eth row of Z.

By induction we may assume the existence of a matrix of form (c) satisfying

$$\text{(1)} \qquad A = S, \qquad B = O.$$

(The A's and B's are not necessarily the same size but the A's are solid ones and the B's solid zeros.)

Of all matrices satisfying (1), let $\mathfrak{A}_1(R, S)$ be the subclass of $\mathfrak{A}(R, S)$ which has the maximum number of zeros in the leading $e \times f$ minor. We shall restrict our attention to $\mathfrak{A}_1(R, S)$.

Suppose that in all matrices of $\mathfrak{A}_1(R, S)$

$$\text{(2)} \qquad C = O.$$

We assume for the moment that C has at least one row and handle the contrary case subsequently.

Let $A_1 \in \mathfrak{A}_1(R, S)$ and suppose A_1 satisfies

(3) D has 1 in its last row.

(The case in which D has 0 rows would be trivial.)

Suppose that all matrices in $\mathfrak{A}_1(R, S)$ which satisfy (3) have

(4) $$E = S.$$

Let $A_2 \in \mathfrak{A}_1(R, S)$, satisfy (3), and satisfy

(5) F has a 0 in its first column, which we may assume to be in the last row of F.

Now suppose in A_2 that there is a 1 in G. We may assume this to be in the first column of G, but then an interchange (see 3) would give a matrix in $\mathfrak{A}_1(R, S)$ with $C \neq O$. Hence A_2 has

(6) $$G = O.$$

Suppose A_2 has a 1 in H. Then there is an interchange which will put a 1 in G and still satisfy all previous conditions, which is a contradiction. Thus A_2 has

(7) $$H = O.$$

We now consider the class $\mathfrak{A}_2(R, S)$ which consists of all matrices in $\mathfrak{A}_1(R, S)$ with conditions (2) to (7) holding. Suppose that in all matrices of $\mathfrak{A}_2(R, S)$

(8) $$J = O.$$

Let A_3 be a matrix in $\mathfrak{A}_2$ with a 1 in the last column of K. We may assume this 1 to be in the first row of K. Thus we assume that A_3 satisfies

(9) K has a 1 in its first row, last column.

If A_3 has a 0 in L, by interchanges we can get the 0 in the last row, last column of L. (This may put the 0 in the first column of F outside the last row.) Now an interchange gives a matrix in $\mathfrak{A}_1$ with $C \neq O$. Hence we assume that A_3 has

(10) $$L = S.$$

Let $\mathfrak{A}_3(R, S)$ be the subclass of $\mathfrak{A}_2(R, S)$ satisfying (8), (9), (10). Suppose that all matrices of $\mathfrak{A}_3$ have

(11) $$M = O.$$

Let A_4 be a matrix in $\mathfrak{A}_3$ which satisfies

(12) N has a 1 in its last row, which we may assume to be in the first column of N.

Suppose that in A_4, P has a 0. Then by interchanges we get this 0 into the

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last row, last column of P. Then an interchange gives a matrix in $\mathfrak{A}_2$ with $J \neq O$. Hence

$$P = S. \tag{13}$$

We now have a matrix of form (a) for e, f.

We return now to the case when C has 0 rows. The matrices in the class $\mathfrak{A}_1$ have the form

$$W = \left[\begin{array}{c|c|c|c|c|c}
S & & S & \multicolumn{2}{c|}{S} & \\ \hline
 & O & D & E & F & O \\ \hline
S & U & Q & T & G & O \\ \cline{2-5}
 & & R & & H & \\ \hline
 & O & O & \multicolumn{2}{c|}{O} & O
\end{array}\right],$$

where the last row of D is in the eth row of W, and D consists of one column, which is in the fth column of W.

We may choose $A_5 \in \mathfrak{A}_1$ such that

$$D \text{ has a 1 in its last row.} \tag{14}$$

Again suppose that all matrices in $\mathfrak{A}_1$ with (14) holding have

$$E = S. \tag{15}$$

There is a matrix A_6 in $\mathfrak{A}_1$ with (14) holding and

$$F \text{ has a zero in its first column, last row.} \tag{16}$$

Interchange if necessary to get rows with all zeros below any other rows in the block made up of G and H. Then we assume A_5 is such that

$$H = O, \qquad G \text{ has a 1 in every row.} \tag{17}$$

Then we must have

$$Q = S, \tag{18}$$

for otherwise an interchange removes a 1 from D to give a matrix satisfying (1) with more 0's in the leading $e \times f$ minor than those in $\mathfrak{A}_1$. Also

$$T = S, \tag{19}$$

for otherwise we could remove a 1 from E. Also

$$U = S, \tag{20}$$

for otherwise an interchange would put a 0 in Q. Hence we have found a matrix of form (b) for e, f. Our proof is now complete.

2. A formula for $\tilde{\rho}$

THEOREM 2.1. *If $\mathfrak{A}(R, S)$ has minimal term rank $\tilde{\rho}$, then there is a matrix*

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A in $\mathfrak{A}(R, S)$ where the leading e rows and leading f columns of A will exhaust all 1's of A with $e + f = \tilde{\rho}$.

This has been shown in **(2)**.

Let $t_{ef} = ef + (r_{e+1} + r_{e+2} + \ldots + r_m) - (s_1 + \ldots + s_f)$ for $1 \leqslant e \leqslant m$, $1 \leqslant f \leqslant n$. Let A in $\mathfrak{A}(R, S)$ be of the form

$$\text{(d)} \qquad A = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

with W of size $e \times f$. Then t_{ef} is the number of zeros in W plus the number of 1's in Z. (See **4** or **6**.)

Let $\psi(e,f) = \min t_{ij}$, where min is over $e \leqslant i \leqslant m$, $f \leqslant j \leqslant n$. Let $\phi(e,f) = \min(t_{ik} + t_{qj} + (e - i)(f - j))$ where min is over

$$1 \leqslant i \leqslant e, \quad f \leqslant k \leqslant n, \quad 1 \leqslant j \leqslant f, \quad e \leqslant q \leqslant m.$$

Let $H(e,f)$ be the maximum number of 0's which any matrix in $\mathfrak{A}(R, S)$ can contain in its leading $e \times f$ minor. Then

THEOREM 2.2. $H(e,f) = \min[\psi(e,f), \phi(e,f)]$, $1 \leqslant e \leqslant m$, $1 \leqslant f \leqslant n$.

Clearly $H(e,f) \leqslant \psi(e,f)$ and $H(e,f) \leqslant \phi(e,f)$, but the normal form (a) of § 1 gives a matrix with $H(e,f) = \phi(e,f)$ and the normal form (b) gives a matrix with $H(e,f) = \psi(e,f)$.

THEOREM 2.3. $\tilde{\rho} = \min(\min(e + f), m, n)$, *where the second* min *is over those* e, f *for which* $t_{ef} = H(e,f)$.

This is clear, for if A is of the form (d) (with $t_{ef} = H(e,f)$) and W has $H(e,f)$ ones, then $Z = O$, and the first e rows and f columns of A exhaust all 1's of A. Because of Theorem 2.1, the proof is complete.

We remark that neither $\phi(e,f)$ nor $\psi(e,f)$ is enough by itself to give $H(e,f)$. We also remark that the computations here are not excessive. The t_{ef} are easily computed by a recursion formula **(4)** and one knows precisely for what e, $\min_i t_{iq} = t_{eq}$.

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