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Independence and almost automorphy of higher order

JIAHAO QIU

Wu Wen-Tsun Key Laboratory of Mathematics, USTC,
Chinese Academy of Sciences and School of Mathematics,
University of Science and Technology of China, Hefei, Anhui 230026, P. R. China
(e-mail: qiujh@mail.ustc.edu.cn)

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Abstract. In this paper it is shown that for a minimal system (X, T) and $d, k \in \mathbb{N}$, if (x, x_i) is regionally proximal of order d for $1 \leq i \leq k$, then $(x, x_1, \ldots, x_k)$ is $(k+1)$-regionally proximal of order d. Meanwhile, we introduce the notion of $\mathrm{IN}^{[d]}$-pair: for a dynamical system (X, T) and $d \in \mathbb{N}$, a pair $(x_0, x_1) \in X \times X$ is called an $\mathrm{IN}^{[d]}$-pair if for any $k \in \mathbb{N}$ and any neighborhoods U_0, U_1 of x_0 and x_1 respectively, there exist different $(p_1^{(i)}, \ldots, p_d^{(i)}) \in \mathbb{N}^d$, $1 \leq i \leq k$, such that

$$\bigcup_{i=1}^{k} \{p_1^{(i)}\epsilon(1) + \cdots + p_d^{(i)}\epsilon(d) : \epsilon(j) \in \{0, 1\}, 1 \leq j \leq d\} \backslash \{0\} \in \mathrm{Ind}(U_0, U_1),$$

where $\mathrm{Ind}(U_0, U_1)$ denotes the collection of all independence sets for (U_0, U_1). It turns out that for a minimal system, if it does not contain any non-trivial $\mathrm{IN}^{[d]}$-pair, then it is an almost one-to-one extension of its maximal factor of order d.

Key words: independence, almost automorphy of higher order
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1. *Introduction*

By a *topological dynamical system* or just a *dynamical system*, we mean a pair (X, T), where X is a compact metric space with a metric ρ and $T : X \to X$ is a homeomorphism.

In recent years, the study of the dynamics of rotations on nilmanifolds and inverse limits of this kind of dynamics has drawn much interest, since it relates to many dynamical properties and has important applications in number theory. We refer to [12] and the references therein for a systematic treatment on the subject.

In a pioneer work, Host, Kra and Maass [13] introduced the notion of *regionally proximal relation of order d* for a dynamical system (X, T), denoted by $\mathbf{RP}^{[d]}(X)$. For

$d \in \mathbb{N}$, we say that a minimal system is a *system of order d* if $\mathbf{RP}^{[d]}(X) = \Delta$, and this is equivalent for (X, T) to an inverse limit of nilrotations on d-step nilsystems (see [13, Theorem 2.8]). For a minimal distal system (X, T), it was proved that $\mathbf{RP}^{[d]}(X)$ is an equivalence relation and $X/\mathbf{RP}^{[d]}(X)$ is the maximal factor of order d [13]. Then Shao and Ye [22] showed that in fact for any minimal system, $\mathbf{RP}^{[d]}(X)$ is an equivalence relation and $\mathbf{RP}^{[d]}(X)$ has the so-called lifting property.

The notion of *k-regional proximal relation* was introduced in [16]. It was shown that for a minimal system (X, T) and $k \geq 2$, if (x, x_i) is regionally proximal for all $1 \leq i \leq k$, then $(x_1, \ldots, x_k)$ is k-regionally proximal, that is, for every $\delta > 0$, there exist $x_i' \in X$, $1 \leq i \leq k$, and $n \in \mathbb{Z}$ such that $\rho(x_i, x_i') < \delta$ and $\rho(T^n x_1', T^n x_i') < \delta$, $1 \leq i \leq k$. In this paper we extend this result to higher order (Theorem 3.2).

Following the *local entropy theory* (for a survey see [11]), each dynamical system admits a maximal zero topological entropy factor, and this factor is induced by the smallest closed invariant equivalence relation containing *entropy pairs* [3]. In [19], entropy pairs are characterized as those pairs that admit an *interpolating set* of positive density. Later on, the notions of *sequence entropy pairs* [15] and *untame pairs* (called *scrambled pairs* in [14]) were introduced. In [20] the concept of *independence* was extensively studied and used to unify the aforementioned notions. Let (X, T) be a dynamical system and $\mathcal{A} = (U_0, U_1, \ldots, U_k)$ be a tuple of subsets of X. We say that a subset $F \subset \mathbb{Z}$ is an *independence set* for $\mathcal{A}$ if for any non-empty finite subset $J \subset F$ and any $s = (s(j) : j \in J) \in \{0, 1, \ldots, k\}^J$ we have $\bigcap_{j \in J} T^{-j} U_{s(j)} \neq \emptyset$. It was shown that a pair of points x_0, x_1 in X is a sequence entropy pair if and only if each $\mathcal{A} = (U_0, U_1)$, where U_0 and U_1 are neighborhoods of x_0 and x_1 respectively, has arbitrarily long finite independence sets. Also, the pair is an untame pair if and only if each $\mathcal{A} = (U_0, U_1)$ as before has infinite independence sets. It was shown [8, 15, 20] that a minimal null (respectively, tame) system is an almost one-to-one extension of its maximal equicontinuous factor and is uniquely ergodic.

For $d \in \mathbb{N}$ and $p_1, \ldots, p_d \in \mathbb{Z}$, we call the set $\{p_1\epsilon(1) + \cdots + p_d\epsilon(d) : \epsilon(j) \in \{0, 1\}, 1 \leq j \leq d\} \backslash \{0\}$ an IP_d-set. The notion of Ind_{fip}-pair was studied in [4]: a pair of points x_0, x_1 in X is an Ind_{fip}-pair if and only if the independence sets for each $\mathcal{A} = (U_0, U_1)$ as before contain an IP_d-set for any $d \in \mathbb{N}$. It was shown that a minimal system without any non-trivial Ind_{fip}-pair is an almost one-to-one extension of its maximal factor of order ∞.

So, it is natural to ask: can we give a finer classification of almost automorphy of higher order (see Definition 2.7) using independence?

In this paper we introduce the notion of $IN^{[d]}$-pair. A pair of points x_0, x_1 in X is an $\mathrm{IN}^{[d]}$-pair if and only if the independence sets for each $\mathcal{A} = (U_0, U_1)$ as before contain a union of arbitrarily finitely many IP_d-sets. Using dynamical cubespaces, we first provide a characterization of $\mathrm{IN}^{[d]}$-pairs for minimal systems (Lemma 2.12). By [12, Ch. 6], the dynamical cubespaces of minimal nilsystems can also be viewed as nilsystems. Following this, it is shown that for minimal nilsystems, non-trivial regionally proximal of order d pairs are $\mathrm{IN}^{[d]}$-pairs (Theorem 4.5). Moreover, this property also holds for inverse limits of minimal nilsystems.

For a minimal system and $d \in \mathbb{N}$, by reducing to the maximal factor of order ∞ which is an inverse limit of minimal nilsystems [4], we can show that any non-trivial regionally proximal of order d pair is an $\mathrm{IN}^{[d]}$-pair if it is minimal in the product system (Lemma 5.4). Among other things, it turns out that for a minimal system, if it does not contain any non-trivial $\mathrm{IN}^{[d]}$-pair, then it is an almost one-to-one extension of its maximal factor of order d (Theorem 5.7).

This paper is organized as follows. In §2 the basic notions used in the paper are introduced. In §3 we discuss the k-regionally proximal relation of higher order (Theorem 3.2). In §4 it is shown that for a minimal nilsystem any regionally proximal of order d pair is an $\mathrm{IN}^{[d]}$-pair (Theorem 4.5). In §5, among other things, we show that for any minimal system, if it does not contain any non-trivial $\mathrm{IN}^{[d]}$-pair, then it is an almost one-to-one extension of its maximal factor of order d (Theorem 5.7). In the final section we construct a minimal system with trivial $\mathrm{IN}^{[d]}$-pairs, and a non-trivial regionally proximal relation of order d (Example 6.1).

2. *Preliminaries*

In this section we gather definitions and preliminary results that will be necessary later on. Let $\mathbb{N}$ and $\mathbb{Z}$ be the sets of all positive integers and integers, respectively.

2.1. *Topological dynamical systems.*

A *topological dynamical system* (or *dynamical system*) is a pair (X, T), where X is a compact metric space with a metric ρ and $T : X \to X$ is a homeomorphism. If A is a non-empty closed subset of X and $TA \subset A$, then $(A, T|_A)$ is called a *subsystem* of (X, T), where $T|_A$ is the restriction of T on A. If there is no ambiguity, we use the notation T instead of $T|_A$. For $x \in X$, $\mathcal{O}(x, T) = \{T^n x : n \in \mathbb{Z}\}$ denotes the *orbit* of x. A dynamical system (X, T) is called *minimal* if every point has dense orbit in X. A subset Y of X is called *minimal* if (Y, T) is a minimal subsystem of (X, T). A point $x \in X$ is called *minimal* if it is contained in a minimal set Y or, equivalently, if the subsystem $(\overline{\mathcal{O}(x, T)}, T)$ is minimal.

A *homomorphism* between the dynamical systems (X, T) and (Y, T) is a continuous onto map $\pi : X \to Y$ which intertwines the actions; one says that (Y, T) is a *factor* of (X, T) and that (X, T) is an *extension* of (Y, T). One also refers to π as a *factor map* or an *extension* and one uses the notation $\pi : (X, T) \to (Y, T)$. The systems are said to be *conjugate* if π is a bijection. An extension π is determined by the corresponding closed invariant equivalence relation $R_\pi = \{(x, x') \in X \times X : \pi(x) = \pi(x')\}$. An extension $\pi : (X, T) \to (Y, T)$ is *almost one-to-one* if the G_δ set $X_0 = \{x \in X : \pi^{-1}(\pi(x)) = \{x\}\}$ is dense.

2.2. *Discrete cubes and faces.*

Let X be a set and let $d \geq 1$ be an integer. We view the element $\epsilon \in \{0, 1\}^d$ as a sequence $\epsilon = (\epsilon(1), \ldots, \epsilon(d))$, where $\epsilon(i) \in \{0, 1\}$, $1 \leq i \leq d$. If $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $\epsilon \in \{0, 1\}^d$, we define

$$\vec{n} \cdot \epsilon = \sum_{i=1}^{d} n_i \epsilon(i).$$

We denote the set of maps $\{0, 1\}^d \to X$ by $X^{[d]}$. For $\epsilon \in \{0, 1\}^d$ and $\mathbf{x} \in X^{[d]}$, $\mathbf{x}(\epsilon)$ will be used to denote the ϵ-component of $\mathbf{x}$. For $x \in X$, write $x^{[d]} = (x, x, \ldots, x) \in X^{[d]}$.

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The *diagonal* of $X^{[d]}$ is $\Delta^{[d]} = \Delta^{[d]}(X) = \{x^{[d]} : x \in X\}$. Usually, when $d = 1$, we denote the diagonal by Δ_X or Δ instead of $\Delta^{[1]}$. We can isolate the first coordinate, writing $X_*^{[d]} = X^{2^d-1}$ and writing $\mathbf{x} \in X^{[d]}$ as $\mathbf{x} = (\mathbf{x}(\vec{0}), \mathbf{x}_*)$, where $\mathbf{x}_* = (\mathbf{x}(\epsilon) : \epsilon \in \{0,1\}^d \backslash \{\vec{0}\}) \in X_*^{[d]}$.

Identifying $\{0,1\}^d$ with the set of vertices of the Euclidean unit cube, a Euclidean isometry of the unit cube permutes the vertices of the cube and thus the coordinates of a point $\mathbf{x} \in X^{[d]}$. These permutations are the *Euclidean permutations* of $X^{[d]}$.

A set of the form

$$F = \{\epsilon \in \{0,1\}^d : \epsilon(i_1) = a_1, \ldots, \epsilon(i_k) = a_k\} \tag{1}$$

for some $k \geq 0$, $1 \leq i_1 < \cdots < i_k \leq d$ and $a_i \in \{0,1\}$ is called a *face* of *codimension* k of the discrete cube $\{0,1\}^d$. (The case $k = 0$ corresponds to $\{0,1\}^d$.) A face of codimension 1 is called a *hyperface*. If all $a_i = 1$ we say that the face is *upper*. Note all upper faces contain $\vec{1}$ and there are exactly 2^d upper faces.

For $\epsilon, \epsilon' \in \{0,1\}^d$, we say that $\epsilon \geq \epsilon'$ if $\epsilon(i) \geq \epsilon'(i)$ for all $1 \leq i \leq d$. Let F be a face of $\{0,1\}^d$. The *smallest element* of the face F is defined by $\min F$, meaning that $\min F \in F$ and $\epsilon \geq \min F$ for all $\epsilon \in F$. Indeed, if a face F has form (1), then $\min F(i_j) = a_j$ for $1 \leq j \leq k$, and $\min F(i) = 0$ for $i \in \{1, \ldots, d\} \backslash \{i_1, \ldots, i_k\}$.

2.3. *Dynamical cubespaces.* Let (X, T) be a dynamical system and $d \in \mathbb{N}$. We define $\mathbf{Q}^{[d]}(X)$ to be the closure in $X^{[d]}$ of elements of the form

$$(T^{\vec{n}\cdot\epsilon}x = T^{n_1\epsilon(1)+\cdots+n_d\epsilon(d)}x : \epsilon \in \{0,1\}^d),$$

where $\vec{n} = (n_1, \ldots, n_d) \in \mathbb{Z}^d$ and $x \in X$. We call this set the *dynamical cubespace of dimension d* of the system.

It is important to note that $\mathbf{Q}^{[d]}(X)$ is invariant under the Euclidean permutations of $X^{[d]}$.

Definition 2.1. Face transformations are defined inductively as follows. Let $T_1^{[1]} = \mathrm{id} \times T$. If $\{T_j^{[d-1]}\}_{j=1}^{d-1}$ is already defined, then set

$$T_j^{[d]} = T_j^{[d-1]} \times T_j^{[d-1]}, \quad 1 \leq j \leq d-1,$$
$$T_d^{[d]} = \mathrm{id}^{[d-1]} \times T^{[d-1]}.$$

It is easy to see that for $1 \leq j \leq d$, the face transformation $T_j^{[d]} : X^{[d]} \to X^{[d]}$ can be defined, for every $\mathbf{x} \in X^{[d]}$ and $\epsilon \in \{0,1\}^d$, by

$$T_j^{[d]}\mathbf{x} = \begin{cases} (T_j^{[d]}\mathbf{x})(\epsilon) = T\mathbf{x}(\epsilon), & \epsilon(j) = 1, \\ (T_j^{[d]}\mathbf{x})(\epsilon) = \mathbf{x}(\epsilon), & \epsilon(j) = 0. \end{cases}$$

The *face group* of dimension d is the group $\mathcal{F}^{[d]}(X)$ of transformations of $X^{[d]}$ spanned by the face transformations. The *parallelepiped group* of dimension d is the group $\mathcal{G}^{[d]}(X)$ spanned by the diagonal transformation and the face transformations. We often write $\mathcal{F}^{[d]}$ and $\mathcal{G}^{[d]}$ instead of $\mathcal{F}^{[d]}(X)$ and $\mathcal{G}^{[d]}(X)$, respectively. For convenience, we denote the orbit closure of $\mathbf{x} \in X^{[d]}$ under $\mathcal{F}^{[d]}$ by $\overline{\mathcal{F}^{[d]}}(\mathbf{x})$, instead of $\overline{\mathcal{O}(\mathbf{x}, \mathcal{F}^{[d]})}$. Let $\mathbf{Q}_x^{[d]}(X) = \mathbf{Q}^{[d]}(X) \cap (\{x\} \times X^{2^d-1})$.

THEOREM 2.2. [22] *Let (X, T) be a minimal system and $d \in \mathbb{N}$. Then:*
(1) *$(\mathbf{Q}^{[d]}(X), \mathcal{G}^{[d]})$ is a minimal system;*
(2) *$(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal for all $x \in X$;*
(3) *$\overline{\mathcal{F}^{[d]}}(x^{[d]})$ is the unique $\mathcal{F}^{[d]}$-minimal subset in $\mathbf{Q}_x^{[d]}(X)$ for all $x \in X$.*

2.4. *Proximality and regional proximality of higher order.* Let (X, T) be a dynamical system. A pair $(x, y) \in X \times X$ is *proximal* if

$$\inf_{n \in \mathbb{Z}} \rho(T^n x, T^n y) = 0,$$

and *distal* if it is not proximal. Denote by $\mathbf{P}(X)$ the set of all proximal pairs of X. The dynamical system (X, T) is *distal* if (x, y) is a distal pair whenever $x, y \in X$ are distinct.

An extension $\pi : (X, T) \to (Y, T)$ is *proximal* if $R_\pi \subset \mathbf{P}(X)$.

Definition 2.3. Let (X, T) be a dynamical system and $d \in \mathbb{N}$. The *regionally proximal relation of order d* is the relation $\mathbf{RP}^{[d]}(X)$ defined by: $(x, y) \in \mathbf{RP}^{[d]}(X)$ if and only if for every $\delta > 0$, there exist $x', y' \in X$ and $\vec{n} \in \mathbb{N}^d$ such that $\rho(x, x') < \delta$, $\rho(y, y') < \delta$, and

$$\rho(T^{\vec{n}\cdot\epsilon} x', T^{\vec{n}\cdot\epsilon} y') < \delta \quad \text{for all } \epsilon \in \{0, 1\}^d \backslash \{\vec{0}\}.$$

We say that (X, T) is a *system of order d* if $\mathbf{RP}^{[d]}(X)$ is trivial.

It follows from [22, Lemma 3.5] that

$$\mathbf{P}(X) \subset \cdots \subset \mathbf{RP}^{[d+1]}(X) \subset \mathbf{RP}^{[d]}(X) \subset \cdots \subset \mathbf{RP}^{[2]}(X) \subset \mathbf{RP}^{[1]}(X). \tag{2}$$

THEOREM 2.4. [22] *Let (X, T) be a minimal system and $d \in \mathbb{N}$. Then:*
(1) *$(x, y) \in \mathbf{RP}^{[d]}(X)$ if and only if $(x, y, \ldots, y) = (x, y_*^{[d+1]}) \in \mathbf{Q}^{[d+1]}(X)$ if and only if $(x, y, \ldots, y) = (x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$;*
(2) *$\mathbf{RP}^{[d]}(X)$ is an equivalence relation.*

The regionally proximal relation of order d allows us to construct the maximal factor of order d of a minimal system. That is, any factor of order d factorizes through this system.

THEOREM 2.5. [22] *Let $\pi : (X, T) \to (Y, T)$ be the factor map between minimal systems and $d \in \mathbb{N}$. Then:*
(1) *$(\pi \times \pi)\mathbf{RP}^{[d]}(X) = \mathbf{RP}^{[d]}(Y)$;*
(2) *(Y, T) is a system of order d if and only if $\mathbf{RP}^{[d]}(X) \subset R_\pi$.*

In particular, the quotient of (X, T) under $\mathbf{RP}^{[d]}(X)$ is the maximal factor of order d of X.

It follows that for any minimal system (X, T),

$$\mathbf{RP}^{[\infty]}(X) = \bigcap_{d \geq 1} \mathbf{RP}^{[d]}(X)$$

is a closed invariant equivalence relation.

We now formulate the definition of systems of order ∞.

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Definition 2.6. A minimal system (X, T) is a *system of order* ∞ if the equivalence relation $\mathbf{RP}^{[\infty]}(X)$ is trivial, that is, coincides with the diagonal.

Let (X, T) be a minimal system and $d \in \mathbb{N} \cup \{\infty\}$. Set

$$\mathbf{RP}^{[d]}[x] = \{y \in X : (x, y) \in \mathbf{RP}^{[d]}(X)\}.$$

Definition 2.7. Let (X, T) be a minimal system and $d \in \mathbb{N} \cup \{\infty\}$. A point $x \in X$ is called a *d-step almost automorphic point* if $\mathbf{RP}^{[d]}[x] = \{x\}$.

A minimal system (X, T) is called *d-step almost automorphic* if it has a d-step almost automorphic point.

Almost automorphic systems of higher order were studied systematically in [18]. In particular, we have the following proposition.

PROPOSITION 2.8. [18, Theorem 8.13] *Let (X, T) be a minimal system. Then (X, T) is a d-step almost automorphic system for some $d \in \mathbb{N} \cup \{\infty\}$ if and only if it is an almost one-to-one extension of its maximal factor of order d.*

2.5. *Independence.* The notion of *independence* was firstly introduced and studied in [20]. It corresponds to a modification of the notion of *interpolating set* studied in [10, 19].

Definition 2.9. Let (X, T) be a dynamical system. Given a tuple $\mathcal{A} = (U_0, U_1, \ldots, U_k)$ of subsets of X, we say that a subset $F \subset \mathbb{Z}$ is an *independence set* for $\mathcal{A}$ if for any non-empty finite subset $J \subset F$ and any $s = (s(j) : j \in J) \in \{0, 1, \ldots, k\}^J$ we have

$$\bigcap_{j \in J} T^{-j} U_{s(j)} \neq \emptyset.$$

We shall denote the collection of all independence sets for $\mathcal{A}$ by $\mathrm{Ind}(U_0, U_1, \ldots, U_k)$.

We now define $\mathrm{IN}^{[d]}$-pairs.

Definition 2.10. Let (X, T) be a dynamical system and $d \in \mathbb{N}$. A pair $(x_0, x_1) \in X \times X$ is called an $\mathrm{IN}^{[d]}$-pair if for any $k \in \mathbb{N}$ and any neighborhoods U_0, U_1 of x_0 and x_1 respectively, there exist different $(p_1^{(i)}, \ldots, p_d^{(i)}) \in \mathbb{N}^d$, $1 \leq i \leq k$, such that

$$\bigcup_{i=1}^{k} \{p_1^{(i)} \epsilon(1) + \cdots + p_d^{(i)} \epsilon(d) : \epsilon \in \{0, 1\}^d\} \backslash \{0\} \in \mathrm{Ind}(U_0, U_1).$$

Denote by $\mathrm{IN}^{[d]}(X)$ the set of all $\mathrm{IN}^{[d]}$-pairs of (X, T).

Remark 2.11. It is easy to see that for a dynamical system, any $\mathrm{IN}^{[d]}$-pair is regionally proximal of order d, sequence entropy pairs coincide with $\mathrm{IN}^{[1]}$-pairs and any Ind_{fip}-pair is an $\mathrm{IN}^{[d]}$-pair for every $d \in \mathbb{N}$.

2.6. *A criterion for being an* $\mathrm{IN}^{[d]}$*-pair.* We characterize $\mathrm{IN}^{[d]}$-pairs using dynamical cubespaces.

Let $d, k \in \mathbb{N}$. We fix an enumeration $\omega_1, \ldots, \omega_{2^d-1}$ of all elements of $\{0,1\}^d \backslash \{\vec{0}\}$. For $1 \le i \le k$, $1 \le j \le 2^d - 1$, let

$$F_{ij} = \left\{ \epsilon \in \{0,1\}^{k(2^d+d)} : \begin{array}{c} \epsilon(k(i-1)+j) = 1, \text{ and} \\ \epsilon(k2^d + d(i-1)+s) = \omega_j(s), \ 1 \le s \le d \end{array} \right\}.$$

For $t_j \in \{0,1\}^{2^d-1}$, $1 \le j \le k$, let $\hat{\theta} = \hat{\theta}(t_1, \ldots, t_k) \in \{0,1\}^{k(2^d+d)}$ such that

$$\hat{\theta}(n) = \begin{cases} t_i(j), & n = k(i-1)+j, \ 1 \le i \le k, 1 \le j \le 2^d - 1, \\ 0 & \text{otherwise.} \end{cases}$$

For $1 \le a \le k$, $1 \le b \le 2^d - 1$, let $\theta = \theta(t_1, \ldots, t_k, a, b) \in \{0,1\}^{k(2^d+d)}$ such that

$$\theta(n) = \begin{cases} t_i(j), & n = k(i-1)+j, \ 1 \le i \le k, 1 \le j \le 2^d - 1, \\ \omega_b(s), & n = k2^d + d(a-1)+s, \ 1 \le s \le d, \\ 0 & \text{otherwise.} \end{cases}$$

Let

$$\Theta_{k,d} = \left\{ \theta = \theta(t_1, \ldots, t_k, a, b) : \begin{array}{c} 1 \le a \le k, 1 \le b \le 2^d - 1, \\ t_j \in \{0,1\}^{2^d-1}, 1 \le j \le k \end{array} \right\}.$$

It is easy to check that $\theta = \theta(t_1, \ldots, t_k, a, b) \in F_{ij}$ if and only if $a = i, b = j$ and $t_a(b) = 1$.

LEMMA 2.12. *Let (X, T) be a minimal system and $d \in \mathbb{N}$, $x_0, x_1 \in X$ with $x_0 \neq x_1$. For any $k \in \mathbb{N}$, if there is some $\mathbf{x} \in \mathbf{Q}^{[k(2^d+d)]}(X)$ such that $\mathbf{x}(\theta) = x_{t_a(b)}$ for any $\theta \in \Theta_{k,d}$, then (x_0, x_1) is an $\mathrm{IN}^{[d]}$-pair.*

Proof. For $i = 0, 1$, let U_i be a neighborhood of x_i and choose $\delta > 0$ with $B(x_i, \delta) = \{y \in X : \rho(x_i, y) < \delta\} \subset U_i$.

Let $k \in \mathbb{N}$ and let $\mathbf{x} \in \mathbf{Q}^{[k(2^d+d)]}(X)$ such that $\mathbf{x}(\theta) = x_{t_a(b)}$ for any $\theta \in \Theta_{k,d}$.

By Theorem 2.2, there exist

$$\vec{n} = (n_1, \ldots, n_{k2^d}, m_1^{(1)}, \ldots, m_d^{(1)}, \ldots, m_1^{(k)}, \ldots, m_d^{(k)}) \in \mathbb{N}^{k(2^d+d)},$$

$n \in \mathbb{N}$ and $x \in X$ such that

$$\rho(T^{n+\vec{n}\cdot\epsilon}x, \mathbf{x}(\epsilon)) < \delta \quad \text{for all } \epsilon \in \{0,1\}^{k(2^d+d)}. \tag{3}$$

For $1 \le i \le k$, set $\vec{m}_i = (m_1^{(i)}, \ldots, m_d^{(i)})$. Recall that $\mathbf{x}(\theta) = x_{t_a(b)}$ and $\vec{n} \cdot \theta = \vec{n} \cdot \hat{\theta} + \vec{m}_a \cdot \omega_b$. Thus by (3) we get that

$$T^{n+\vec{n}\cdot\hat{\theta}}x \in T^{-\vec{m}_a\cdot\omega_b} U_{t_a(b)}.$$

Moreover,

$$T^{n+\vec{n}\cdot\hat{\theta}}x \in \bigcap_{i=1}^{k} \bigcap_{j=1}^{2^d-1} T^{-\vec{m}_i\cdot\omega_j} U_{t_i(j)},$$

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which implies that

$$\bigcup_{i=1}^{k}\{\vec{m}_i\cdot\epsilon:\epsilon\in\{0,1\}^d\}\backslash\{0\}\in\mathrm{Ind}(U_0,U_1).$$

As k is arbitrary, we conclude that (x_0, x_1) is an $\mathrm{IN}^{[d]}$-pair. □

2.7. *Nilpotent groups, nilmanifolds and nilsystems.* Let L be a group. For $g, h \in L$, we write $[g, h] = ghg^{-1}h^{-1}$ for the commutator of g and h. We write $[A, B]$ for the subgroup spanned by $\{[a, b] : a \in A, b \in B\}$. The commutator subgroups L_j, $j \geq 1$, are defined inductively by setting $L_1 = L$ and $L_{j+1} = [L_j, L]$. Let $k \geq 1$ be an integer. We say that L is *k-step nilpotent* if L_{k+1} is the trivial subgroup.

Let L be a k-step nilpotent Lie group and Γ a discrete cocompact subgroup of L. The compact manifold $X = L/\Gamma$ is called a *k-step nilmanifold*. The group L acts on X by left translations, and we write this action as $(g, x) \mapsto gx$. Let $\tau \in L$ and T be the transformation $x \mapsto \tau x$ of X. Then (X, T) is called a *k-step nilsystem*.

We also make use of inverse limits of nilsystems, and so we recall the definition of an inverse limit of systems (restricting ourselves to the case of sequential inverse limits). If $\{(X_i, T_i)\}_{i\in\mathbb{N}}$ are systems with $\mathrm{diam}(X_i) \leq 1$ and $\phi_i : X_{i+1} \to X_i$ are factor maps, the *inverse limit* of the systems is defined to be the compact subset of $\prod_{i\in\mathbb{N}} X_i$ given by $\{(x_i)_{i\in\mathbb{N}} : \phi_i(x_{i+1}) = x_i, i \in \mathbb{N}\}$, which is denoted by $\lim_{\longleftarrow}\{X_i\}_{i\in\mathbb{N}}$. It is a compact metric space endowed with the distance $\rho(x, y) = \sum_{i\in\mathbb{N}} 1/2^i \rho_i(x_i, y_i)$. We note that the maps $\{T_i\}$ induce a transformation T on the inverse limit.

The following structure theorem characterizes inverse limits of nilsystems using dynamical cubespaces.

THEOREM 2.13. (Host, Kra and Maass [13, Theorem 1.2]) *Assume that (X, T) is a minimal system and let $d \geq 2$ be an integer. The following properties are equivalent.*

(1) *If $\mathbf{x}, \mathbf{y} \in \mathbf{Q}^{[d]}(X)$ have $2^d - 1$ coordinates in common, then $\mathbf{x} = \mathbf{y}$.*
(2) *If $x, y \in X$ are such that $(x, y, \ldots, y) \in \mathbf{Q}^{[d]}(X)$, then $x = y$.*
(3) *The system (X, T) is an inverse limit of $(d-1)$-step minimal nilsystems.*

This result shows that a minimal system is a system of order d if and only if it is an inverse limit of minimal d-step nilsystems.

THEOREM 2.14. [4, Theorem 3.6] *A minimal system (X, T) is a system of order ∞ if and only if it is an inverse limit of minimal nilsystems.*

3. *k-regionally proximal relation of higher order*

In this section we discuss the k-regionally proximal relation of higher order.

Definition 3.1. Let (X, T) be a dynamical system and $d \in \mathbb{N}$. For $k \geq 2$, a k-tuple $(x_1, \ldots, x_k) \in X^k$ is said to be *k-regionally proximal of order d* if for any $\delta > 0$, there exist $x_i' \in X$, $1 \leq i \leq k$, and $\vec{n} \in \mathbb{N}^d$ such that $\rho(x_i, x_i') < \delta$, $1 \leq i \leq k$, and

$$\max_{1\leq i<j\leq k}\rho(T^{\vec{n}\cdot\epsilon}x_i', T^{\vec{n}\cdot\epsilon}x_j') < \delta \quad \text{for all } \epsilon\in\{0,1\}^d\backslash\{\vec{0}\}.$$

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In the proof of the following theorem, we will use enveloping semigroups in abstract topological dynamical systems. For more details, see Appendix A.

THEOREM 3.2. *Let (X, T) be a minimal system and let $d, k \in \mathbb{N}$ with $k \geq 2$. For points $x, x_i \in X$, $1 \leq i \leq k$, if (x, x_i) is regionally proximal of order d for all i, then $(x_1, \ldots, x_k)$ is k-regionally proximal of order d.*

Proof. Let $d, k \in \mathbb{N}$ with $k \geq 2$. Fix $x \in X$ and let $x_i \in \mathbf{RP}^{[d]}[x]$, $1 \leq i \leq k$.

We will show that $(x_1, \ldots, x_k)$ is k-regionally proximal of order d.

CLAIM 1. *Let $y \in \mathbf{RP}^{[d]}[x]$ and let $\mathbf{y} \in X^{[d+1]}$ such that*

$$\mathbf{y}(\epsilon) = \begin{cases} y, & \epsilon = (0, \ldots, 0, 1), \\ x & \textit{otherwise}. \end{cases}$$

Then $\mathbf{y} \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$.

Proof of Claim 1. As $(x, y) \in \mathbf{RP}^{[d]}(X) \subset \mathbf{RP}^{[d-1]}(X)$, we have $(x, y_*^{[d]}) \in \overline{\mathcal{F}^{[d]}}(x^{[d]})$ by Theorem 2.4. Notice that $(\overline{\mathcal{F}^{[d]}}(x^{[d]}), \mathcal{F}^{[d]})$ is minimal by Theorem 2.2. Then there is some sequence $\{\vec{n}_j\}_{j \in \mathbb{N}} \subset \mathbb{Z}^d$ such that

$$(T^{\vec{n}_j \cdot \epsilon} : \epsilon \in \{0, 1\}^d)(x, y_*^{[d]}) \to x^{[d]}, \tag{4}$$

as $j \to \infty$. Let σ be the map from $\mathbb{Z}^d$ to $\mathbb{Z}^{d+1}$ such that

$$\vec{n} = (n_1, \ldots, n_d) \mapsto \sigma(\vec{n}) = (n_1, \ldots, n_d, 0).$$

Again by Theorem 2.4, $(x, y_*^{[d+1]}) \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$. Then by (4) we have

$$(T^{\sigma(\vec{n}_j) \cdot \omega} : \omega \in \{0, 1\}^{d+1})(x, y_*^{[d+1]}) \to \mathbf{y},$$

as $j \to \infty$, which implies that $\mathbf{y} \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$. □

For $1 \leq i \leq k$ and $s = 0, 1$, let

$$F_i^s = \{\epsilon \in \{0, 1\}^{d+k} : \epsilon(j) = 0, 1 \leq j \leq d,\ \epsilon(d+i) = s\}.$$

CLAIM 2. *For every $1 \leq i \leq k$, there is $\mathbf{p}_i \in E(\overline{\mathcal{F}^{[d+k]}}(x^{[d+k]}), \mathcal{F}^{[d+k]})$ such that:*

(1) $\mathbf{p}_i(\epsilon) = \mathrm{id}$, $\epsilon \in F_i^0$;

(2) $\mathbf{p}_i(\epsilon)x = x_i$, $\epsilon \in F_i^1$;

(3) $\mathbf{p}_i(\epsilon)x = x$, $\epsilon \in \{0, 1\}^{d+k} \setminus F_i^1$.

Proof of Claim 2. Let $i \in \{1, \ldots, k\}$ and let $\mathbf{a}_i \in X^{[d+1]}$ such that

$$\mathbf{a}_i(\epsilon) = \begin{cases} x_i, & \epsilon = (0, \ldots, 0, 1), \\ x & \text{otherwise}. \end{cases}$$

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Then $\mathbf{a}_i \in \overline{\mathcal{F}^{[d+1]}}(x^{[d+1]})$ by Claim 1. Notice that $(\overline{\mathcal{F}^{[d+1]}}(x^{[d+1]}), \mathcal{F}^{[d+1]})$ is minimal. Then there is some sequence $\{\vec{n}^{(l)} = (n_1^{(l)}, \ldots, n_{d+1}^{(l)})\}_{l\in\mathbb{N}} \subset \mathbb{Z}^{d+1}$ such that

$$(T^{\vec{n}^{(l)}\cdot\epsilon}x : \epsilon \in \{0,1\}^{d+1}) \to \mathbf{a}_i, \tag{5}$$

as $l \to \infty$. For $l \in \mathbb{N}$, let $\vec{m}^{(l)} = (m_1^{(l)}, \ldots, m_{d+k}^{(l)}) \in \mathbb{Z}^{d+k}$ such that

$$m_j^{(l)} = \begin{cases} n_j^{(l)}, & j = 1, \ldots, d, \\ n_{d+1}^{(l)}, & j = d+i, \\ 0 & \text{otherwise.} \end{cases}$$

Then by (5) we have that:

(1) for $\epsilon \in F_i^0$, $T^{\vec{m}^{(l)}\cdot\epsilon} = T^0 = \mathrm{id}$;

(2) for $\epsilon \in F_i^1$, $T^{\vec{m}^{(l)}\cdot\epsilon}x = T^{n_{d+1}^{(l)}}x \to x_i$, as $l \to \infty$;

(3) for $\epsilon \in \{0,1\}^{d+k}\backslash F_i^1$, $T^{\vec{m}^{(l)}\cdot\epsilon}x = T^{\vec{n}^{(l)}\cdot\tilde{\epsilon}}x \to x$, as $l \to \infty$, where $\tilde{\epsilon} \in \{0,1\}^{d+1}$ with $\tilde{\epsilon}(i) = \epsilon(i)$, $1 \le i \le d$, and $\tilde{\epsilon}(d+1) = \epsilon(d+i)$.

Now assume that

$$(T^{\vec{m}^{(l)}\cdot\epsilon} : \epsilon \in \{0,1\}^{d+k}) \to \mathbf{p}_i$$

in $E(\overline{\mathcal{F}^{[d+k]}}(x^{[d+k]}), \mathcal{F}^{[d+k]})$ pointwise. It is easy to check that $\mathbf{p}_i$ meets the requirement. □

Now let $\mathbf{y} = \mathbf{p}_k \cdots \mathbf{p}_1 x^{[d+k]}$. For $1 \le i \le k$, let $\omega_i = \min F_i^1$ and let

$$F = \bigg\{\epsilon \in \{0,1\}^{d+k} : \sum_{j=1}^{d} \epsilon(j) > 0\bigg\}.$$

CLAIM 3. $\mathbf{y} \in \overline{\mathcal{F}^{[d+k]}}(x^{[d+k]})$ *and:*

(1) $\mathbf{y}(\omega_i) = x_i,\ 1 \le i \le k;$

(2) $\mathbf{y}(\epsilon) = x, \epsilon \in F.$

Proof of Claim 3. Notice that $\omega_i \in F_j^0$ for any $i \neq j$. Thus $\mathbf{p}_j(\omega_i) = \mathrm{id}$ by property (1) of Claim 2. By property (2) of Claim 2, we have $\mathbf{p}_i(\omega_i)x = x_i$. This shows that

$$\mathbf{y}(\omega_i) = \mathbf{p}_k(\omega_i)\cdots\mathbf{p}_1(\omega_i)x = \mathbf{p}_i(\omega_i)x = x_i.$$

Let $\epsilon \in F$. Then $\epsilon \notin \bigcup_{i=1}^k F_i^1$. By property (3) of Claim 2, $\mathbf{p}_i(\epsilon)x = x$ for every i and thus we get that

$$\mathbf{y}(\epsilon) = \mathbf{p}_k(\epsilon)\cdots\mathbf{p}_1(\epsilon)x = x.$$

This shows Claim 3. □

Fix $\delta > 0$. As $\mathbf{y} \in \overline{\mathcal{F}^{[d+k]}}(x^{[d+k]})$, there is some $\vec{m} = (m_1, \ldots, m_{d+k}) \in \mathbb{N}^{d+k}$ such that for all $\epsilon \in \{0,1\}^{d+k}$,

$$\rho(T^{\vec{m}\cdot\epsilon}x, \mathbf{y}(\epsilon)) < \delta. \tag{6}$$

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Let $x_i' = T^{\vec{m}\cdot\omega_i}x,\ 1 \le i \le k$, and $\vec{n} = (m_1, \ldots, m_d)$. By (6) and property (1) of Claim 3, for $1 \le i \le k$, we have

$$\rho(x_i', x_i) = \rho(T^{\vec{m}\cdot\omega_i}x, \mathbf{y}(\omega_i)) < \delta.$$

For $\epsilon \in \{0, 1\}^d \backslash \{\vec{0}\}$, put $\hat{\epsilon} \in \{0, 1\}^{d+k}$ such that

$$\hat{\epsilon}(j) = \begin{cases} \epsilon(j) & 1 \le j \le d, \\ 0 & d + 1 \le j \le d + k. \end{cases}$$

Then $\hat{\epsilon} + \omega_i \in F$ for $1 \le i \le k$. Moreover, we have that

$$\rho(T^{\vec{n}\cdot\epsilon}x_i', x) = \rho(T^{\vec{n}\cdot\epsilon+\vec{m}\cdot\omega_i}x, x) = \rho(T^{\vec{m}\cdot(\hat{\epsilon}+\omega_i)}x, \mathbf{y}(\hat{\epsilon} + \omega_i)) < \delta,$$

by (6) and property (2) of Claim 3 which implies that $(x_1, \ldots, x_k)$ is k-regionally proximal of order d.

The proof is complete. $\square$

4. *Independence and minimal nilsystems*

The main aim of this section is to study $\mathrm{IN}^{[d]}$-pairs in minimal nilsystems. It turns out that for a minimal nilsystem, any regionally proximal of order d pair is an $\mathrm{IN}^{[d]}$-pair. We start by recalling some basic results in nilsystems. For more details and proofs, see [**2**, **21**].

If G is a nilpotent Lie group, let G^0 denote the connected component of its unit element 1_G. In the sequel, $s \ge 2$ is an integer and $(X = G/\Gamma, T)$ is a minimal s-step nilsystem. We let τ denote the element of G defining the transformation T. If (X, T) is minimal, let G' be the subgroup of G spanned by G^0 and τ and let $\Gamma' = \Gamma \cap G'$. Then we have that $G = G'\Gamma$. Thus the system (X, T) is conjugate to the system (X', T'), where $X' = G'/\Gamma'$ and T' is the translation by τ on X'. Therefore, without loss of generality, we can restrict to the case G is spanned by G^0 and τ.

We fix an enumeration $F_1, F_2, \ldots, F_{2^d}$ of all upper faces of $\{0, 1\}^d$, ordered such that codim (F_i) is non-decreasing with i. Then $F_1 = \{0, 1\}^d$, and the upper faces of codimension 1 are $F_2, \ldots, F_{d+1}$.

If F is a face of $\{0, 1\}^d$, for $g \in G$ we define $g^{(F)} \in G^{[d]}$ by

$$(g^{(F)})(\epsilon) = \begin{cases} g, & \epsilon \in F, \\ 1_G, & \epsilon \notin F. \end{cases}$$

Denote by $\mathcal{HK}^{[d]}$ the subgroup of $G^{[d]}$ spanned by

$$\{g^{(F_i)} : g \in G,\ 1 \le i \le d + 1\}.$$

LEMMA 4.1. [**12**, Ch. 12] *$\mathcal{HK}^{[d]}$ is a rational subgroup of $G^{[d]}$.*

Lemma 4.1 means that $\Gamma^{[d]} \cap \mathcal{HK}^{[d]}$ is cocompact in $\mathcal{HK}^{[d]}$, allowing us to define an s-step nilmanifold

$$\widetilde{X}^{[d]} = \frac{\mathcal{HK}^{[d]}}{\Gamma^{[d]} \cap \mathcal{HK}^{[d]}}.$$

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LEMMA 4.2. [12, Ch. 12] *The nilmanifold* $\widetilde{X}^{[d]}$ *is equal to* $\mathbf{Q}^{[d}(X)$.

By Lemma 4.2 we can view $\mathbf{Q}^{[d]}(X)$ as a nilsystem, and it is also $\mathcal{HK}^{[d]}$-invariant.

LEMMA 4.3. [12, Ch. 12] *Let F be a face of $\{0,1\}^d$ and let $g \in G_{\mathrm{codim}(F)}$. Then $g^{(F)} \in \mathcal{HK}^{[d]}$.*

The following corollary is an immediate consequence of Lemmas 4.2 and 4.3.

COROLLARY 4.4. *Let F be a face of $\{0,1\}^d$ and let $g \in G_{\mathrm{codim}(F)}$. Then $g^{(F)}\mathbf{x} \in \mathbf{Q}^{[d]}(X)$ for every $\mathbf{x} \in \mathbf{Q}^{[d]}(X)$.*

We are now in a position to show the main result of this section. In it proof we omit the nilpotency class as it is not important.

THEOREM 4.5. *Let $(X = G/\Gamma, T)$ be a minimal nilsystem. For $x \in X$ and $g \in G_{d+1}$, if $x \neq gx$, then (x, gx) is an $\mathrm{IN}^{[d]}$-pair.*

Proof. Let $x \in X$ and $g \in G_{d+1}$ with $x \neq gx$. Put $x_0 = x$ and $x_1 = gx$. For $i = 0, 1$, let U_i be a neighborhood of x_i and choose $\delta > 0$ with $B(x_i, \delta) \subset U_i$.

We fix an enumeration $\omega_1, \ldots, \omega_{2^d-1}$ of all elements of $\{0,1\}^d \backslash \{\vec{0}\}$.

Let $k \in \mathbb{N}$. For $1 \le i \le k$, $1 \le j \le 2^d - 1$, let

$$F_{ij} = \left\{ \epsilon \in \{0,1\}^{k(2^d+d)} : \begin{array}{c} \epsilon(k(i-1)+j) = 1, \text{ and} \\ \epsilon(k2^d + d(i-1) + s) = \omega_j(s), \ 1 \le s \le d \end{array} \right\}.$$

Notice that F_{ij} is a face of $\{0,1\}^{k(2^d+d)}$ of codimension $d+1$ for any i, j. It follows from Lemma 4.3 that $g^{(F_{ij})} \in \mathcal{HK}^{[k(2^d+d)]}$. Thus by Corollary 4.4 we have that

$$\mathbf{x} = \left(\prod_{i=1}^{k} \prod_{j=1}^{2^d-1} g^{(F_{ij})} \right) x^{[k(2^d+d)]} \in \mathbf{Q}^{[k(2^d+d)]}(X). \tag{7}$$

Recall that for any $\theta = \theta(t_1, \ldots, t_k, a, b) \in \Theta_{k,d}$, we have $\theta \in F_{ij}$ if and only if $a = i$, $b = j$ and $t_a(b) = 1$ (see §2.6). Thus by (7) we get that

$$\mathbf{x}(\theta) = \left(\prod_{\substack{1 \le i \le k, 1 \le j \le 2^d-1 \\ \theta \in F_{ij}}} g \right) x = x_{t_a(b)}.$$

By Lemma 2.12, we deduce that $(x, gx) = (x_0, x_1)$ is an $\mathrm{IN}^{[d]}$-pair. □

We refer to [4] for the following description of maximal factors of higher order of minimal nilsystems.

LEMMA 4.6. *For $1 \le r \le s$, if X_r is the maximal factor of order r of X, then X_r has the form $G/(G_{r+1}\Gamma)$, endowed with the translation by the projection of τ on G/G_{r+1}.*

The following corollary is an immediate consequence of Theorem 4.5 and Lemma 4.6.

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COROLLARY 4.7. *Let (X, T) be a minimal nilsystem and $d \in \mathbb{N}$. Then $(x_0, x_1) \in \mathrm{IN}^{[d]}(X)$ if and only if $(x_0, x_1) \in \mathbf{RP}^{[d]}(X)$.*

COROLLARY 4.8. *Let (X, T) be an inverse limit of minimal nilsystems and $d \in \mathbb{N}$. Then $(x_0, x_1) \in \mathrm{IN}^{[d]}(X)$ if and only if $(x_0, x_1) \in \mathbf{RP}^{[d]}(X)$.*

Proof. Let $(x_0, x_1) \in \mathbf{RP}^{[d]}(X)\backslash\Delta_X$. By the definition of $\mathrm{IN}^{[d]}$-pairs, it suffices to show that $(x_0, x_1) \in \mathrm{IN}^{[d]}(X)$.

Assume that X_i is a minimal nilsystem for every $i \in \mathbb{N}$, set $X = \lim_{\longleftarrow}\{X_i\}_{i\in\mathbb{N}}$ and assume that $\pi_i : X \to X_i$ and $\pi_{i,j} : X_j \to X_i$ are the factor maps.

Set $x_s^i = \pi_i(x_s)$, $i \in \mathbb{N}$, $s = 0, 1$. Then $\pi_{i,j}(x_s^j) = x_s^i$ and there is some $n \in \mathbb{N}$ such that $x_0^n \neq x_1^n$. For $j \geq n$ and $s = 0, 1$, we have $x_s^n = \pi_{n,j}(x_s^j)$ which implies $x_0^j \neq x_1^j$. Without loss of generality, we may assume $x_0^i \neq x_1^i$ for all $i \in \mathbb{N}$.

Let $k \in \mathbb{N}$. It follows from Theorem 2.5 that $(x_0^i, x_1^i) \in \mathbf{RP}^{[d]}(X_i)$ for all $i \in \mathbb{N}$. By Theorem 4.5 and Lemma 4.6, for every $i \in \mathbb{N}$ there exists some $\mathbf{x}_i \in \mathbf{Q}^{[k(2^d+d)]}(X_i)$ such that $\mathbf{x}_i(\theta) = x^i_{t_a(b)}$ for all $\theta \in \Theta_{k,d}$. Notice that $\mathbf{Q}^{[k(2^d+d)]}(X)$ is an inverse limit of the sequence $\{\mathbf{Q}^{[k(2^d+d)]}(X_i)\}_{i\in\mathbb{N}}$. Thus for every $i \in \mathbb{N}$ there exists some $\widetilde{\mathbf{x}}_i \in \mathbf{Q}^{[k(2^d+d)]}(X)$ such that

$$\pi_i^{[k(2^d+d)]}(\widetilde{\mathbf{x}}_i) = \mathbf{x}_i.$$

Without loss of generality, assume that $\widetilde{\mathbf{x}}_i \to \mathbf{x}$ as $i \to \infty$ for some $\mathbf{x} \in \mathbf{Q}^{[k(2^d+d)]}(X)$.

We claim that $\mathbf{x}(\theta) = x_{t_a(b)}$ for all $\theta \in \Theta_{k,d}$.

Actually, for any $\theta \in \Theta_{k,d}$ and $i \leq j$ we have

$$\pi_i(\widetilde{\mathbf{x}}_j(\theta)) = \pi_{i,j} \circ \pi_j(\widetilde{\mathbf{x}}_j(\theta)) = \pi_{i,j}(\mathbf{x}_j(\theta)) = \pi_{i,j}(x^j_{t_a(b)}) = x^i_{t_a(b)}.$$

By letting j go to infinity and the continuity of π_i, we get $\pi_i(\mathbf{x}(\theta)) = x^i_{t_a(b)}$ for all $i \in \mathbb{N}$. This shows the claim and thus $(x_0, x_1) \in \mathrm{IN}^{[d]}(X)$ by Lemma 2.12.

This completes the proof. □

5. *The structure of minimal systems without non-trivial* $\mathrm{IN}^{[d]}$*-pairs*

In this section we discuss the structure of minimal systems without non-trivial $\mathrm{IN}^{[d]}$-pairs. We will show that such systems are almost one-to-one extensions of their maximal factors of order d.

We start with the following useful lemma which can be found in the proof of [22, Theorem 3.1].

LEMMA 5.1. *Let (X, T) be a dynamical system and $d \in \mathbb{N}$. If $\mathbf{x} \in X^{[d]}$ is an $\mathrm{id} \times T_*^{[d]}$-minimal point, then it is an $\mathcal{F}^{[d]}$-minimal point.*

LEMMA 5.2. *Let (X, T) be a minimal system and $d \in \mathbb{N}$. For $\omega \in \{0, 1\}^d$ and $\mathbf{x} \in \mathbf{Q}^{[d]}(X)$, let $\mathbf{y} \in X^{[d]}$ such that $\mathbf{y}(\epsilon) = \mathbf{x}(\epsilon)$ if $\epsilon \in \{0, 1\}^d\backslash\{\omega\}$ and $(\mathbf{x}(\omega), \mathbf{y}(\omega)) \in \mathbf{RP}^{[\infty]}(X)$. If $\mathbf{y}$ is a $T^{[d]}$-minimal point, then $\mathbf{y} \in \mathbf{Q}^{[d]}(X)$.*

Proof. Let $\mathbf{x}(\omega) = x$, $\mathbf{y}(\omega) = y$ and $(x, y) \in \mathbf{RP}^{[\infty]}(X)$.

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Case 1: $\omega = \vec{0}$. As $\mathbf{x} \in \mathbf{Q}^{[d]}(X)$, there exists some sequence $\{S_i\}_{i\in\mathbb{N}} \subset \mathcal{F}^{[d]}$ such that $S_i\mathbf{x} \to x^{[d]}$ as $i \to \infty$ by Theorem 2.2. At the same time, we have that $S_i\mathbf{y} \to (y, x_*^{[d]}) \in \mathbf{Q}^{[d]}(X)$ as $i \to \infty$. By our hypothesis, $\mathbf{y}$ is a $T^{[d]}$-minimal point, so $\mathbf{y}$ is also an id $\times$ $T_*^{[d]}$-minimal point. Thus by Lemma 5.1, $\mathbf{y}$ is an $\mathcal{F}^{[d]}$-minimal point which implies that $\mathbf{y}$ also belongs to the orbit closure of $(y, x_*^{[d]})$ under the $\mathcal{F}^{[d]}$-action, and thus $\mathbf{y} \in \mathbf{Q}^{[d]}(X)$.

Case 2: $\omega \neq \vec{0}$. Recall that $\mathbf{Q}^{[d]}(X)$ is invariant under the Euclidean permutations of $X^{[d]}$. We can choose some Euclidean permutation f such that $f(\mathbf{y})(\vec{0}) = \mathbf{y}(\omega)$.

Now we have $f(\mathbf{x})(\epsilon) = f(\mathbf{y})(\epsilon)$ for any $\epsilon \in \{0, 1\}^d\backslash\{\vec{0}\}$ and $(f(\mathbf{x})(\vec{0}), f(\mathbf{y})(\vec{0})) \in \mathbf{RP}^{[\infty]}(X)$. Moreover, $f(\mathbf{y})$ is also a $T^{[d]}$-minimal point. By Case 1, we get that $f(\mathbf{y}) \in \mathbf{Q}^{[d]}(X)$ and thus $\mathbf{y} \in \mathbf{Q}^{[d]}(X)$. $\square$

Recall a characterization of Ind_{fip}-pairs in [4, Corollary 4.4].

LEMMA 5.3. *Let (X, T) be a minimal system and $(x_0, x_1) \in \mathbf{RP}^{[\infty]}(X)\backslash\Delta$. If (x_0, x_1) is a $T \times T$-minimal point, then (x_0, x_1) is an Ind_{fip}-pair.*

Analogously to Lemma 5.3, we provide a characterization of $\mathrm{IN}^{[d]}$-pairs.

LEMMA 5.4. *Let (X, T) be a minimal system and $d \in \mathbb{N}$, $(x_0, x_1) \in \mathbf{RP}^{[d]}(X)\backslash\Delta$. If (x_0, x_1) is a $T \times T$-minimal point, then (x_0, x_1) is an $\mathrm{IN}^{[d]}$-pair.*

Proof. Let $\pi : X \to X_\infty = X/\mathbf{RP}^{[\infty]}(X)$ be the factor map and let $u_j = \pi(x_j)$, $j = 0, 1$. If $u_0 = u_1$, then $(x_0, x_1) \in \mathbf{RP}^{[\infty]}(X)$ and thus $(x_0, x_1) \in \mathrm{Ind}_{fip}(X)$ by Lemma 5.3. In particular, we have $(x_0, x_1) \in \mathrm{IN}^{[d]}(X)$.

Now assume that $u_0 \neq u_1$. Then $(u_0, u_1) \in \mathbf{RP}^{[d]}(X_\infty)\backslash\Delta_{X_\infty}$ by Theorem 2.5 and (u_0, u_1) is a $T \times T$-minimal point as (x_0, x_1) is a $T \times T$-minimal point.

Fix $k \in \mathbb{N}$. By Lemma 2.12, it suffices to show that there is some $\mathbf{x} \in \mathbf{Q}^{[k(2^d+d)]}(X)$ such that $\mathbf{x}(\theta) = x_{t_a(b)}$ for all $\theta \in \Theta_{k,d}$.

Step 1: Reduction to the maximal factor of order ∞. It follows from Theorem 2.14 that X_∞ is an inverse limit of minimal nilsystems. By the argument in Corollary 4.8, there exists some $\mathbf{u} \in \mathbf{Q}^{[k(2^d+d)]}(X_\infty)$ such that $\mathbf{u}(\theta) = u_{t_a(b)}$ for all $\theta \in \Theta_{k,d}$.

Step 2: Lifting to X. Notice that $\pi^{[l]} : (\mathbf{Q}^{[l]}(X), \mathcal{G}^{[l]}) \to (\mathbf{Q}^{[l]}(X_\infty), \mathcal{G}^{[l]})$ is a factor map for every $l \in \mathbb{N}$, where $\pi^{[l]} : X^{[l]} \to X_\infty^{[l]}$ is defined from π coordinatewise.

We note that Theorem 2.5 also holds for general abelian group actions. Thus there is some $\mathbf{w} \in \mathbf{Q}^{[k(2^d+d)]}(X)$ such that

$$\pi^{[k(2^d+d)]}(\mathbf{w}) = \mathbf{u},$$

which implies that $\mathbf{w}(\theta) \in \mathbf{RP}^{[\infty]}[x_{t_a(b)}]$ for all $\theta \in \Theta_{k,d}$.

Step 3: Transformations.

Case 1: $(x_0, x_1, \mathbf{w})$ is a $T^{[k(2^d+d)]+2}$-minimal point. By Lemma 5.2, we can replace $\mathbf{w}(\theta)$ by $x_{t_a(b)}$ for all $\theta \in \Theta_{k,d}$ which implies that there is some $\mathbf{x} \in \mathbf{Q}^{[k(2^d+d)]}(X)$ such that $\mathbf{x}(\theta) = x_{t_a(b)}$ for all $\theta \in \Theta_{k,d}$.

Case 2: General cases. By property (3) of Proposition A.2, there is a minimal point

$$(x_0', x_1', \mathbf{w}') \in \overline{O((x_0, x_1, \mathbf{w}), T^{[k(2^d+d)]+2})} \tag{8}$$

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such that they are also proximal. Note that $\mathbf{Q}^{[k(2^d+d)]}(X)$ is $T^{[k(2^d+d)]}$-invariant, and we get $\mathbf{w}' \in \mathbf{Q}^{[k(2^d+d)]}(X)$.

Now by (8), $(x_i, x_i'), (\mathbf{w}(\epsilon), \mathbf{w}'(\epsilon)) \in \mathbf{P}(X)$ for all $i = 0, 1$ and $\epsilon \in \{0, 1\}^{[k(2^d+d)]}$. As $\mathbf{P}(X) \subset \mathbf{RP}^{[\infty]}(X)$ and $\mathbf{w}(\theta) \in \mathbf{RP}^{[\infty]}[x_{t_a(b)}]$ for all $\theta \in \Theta_{k,d}$, by equivalence of $\mathbf{RP}^{[\infty]}(X)$ we get that

$$\mathbf{w}'(\theta) \in \mathbf{RP}^{[\infty]}[x'_{t_a(b)}],$$

which implies $\mathbf{w}' \in \mathbf{Q}^{[k(2^d+d)]}(X)$ by Case 1.

Recall that (x_0, x_1) is a $T \times T$-minimal point and $(x_0', x_1') \in \overline{O((x_0, x_1), T \times T)}$. There exists some sequence $\{n_i\}_{i\in\mathbb{N}} \subset \mathbb{Z}$ such that

$$(T \times T)^{n_i}(x_0', x_1') \to (x_0, x_1)$$

as $i \to \infty$. Let $\mathbf{x}$ be some limit point of the sequence $\{(T^{n_i})^{[k(2^d+d)]}\mathbf{w}'\}_{i\in\mathbb{N}}$. Then we have $\mathbf{x} \in \mathbf{Q}^{[k(2^d+d)]}(X)$ and $\mathbf{x}(\theta) = x_{t_a(b)}$ for all $\theta \in \Theta_{k,d}$. □

As a consequence, we get the following corollary.

COROLLARY 5.5. *Let (X, T) be a minimal distal system and $d \in \mathbb{N}$. Then $(x_0, x_1) \in \mathrm{IN}^{[d]}(X)$ if and only if $(x_0, x_1) \in \mathbf{RP}^{[d]}(X)$.*

We are now able to show the main result of this section. We need the following theorem.

THEOREM 5.6. [4, Theorem 4.5] *Let (X, T) be a minimal system. If X does not contain any non-trivial* Ind_{fip}*-pair, then it is an almost one-to-one extension of its maximal factor of order ∞.*

THEOREM 5.7. *Let (X, T) be a minimal system and $d \in \mathbb{N}$. If X does not contain any non-trivial* $\mathrm{IN}^{[d]}$*-pair, then it is an almost one-to-one extension of its maximal factor of order d.*

Proof. Let (X, T) be a minimal system without non-trivial $\mathrm{IN}^{[d]}$-pairs, where $d \in \mathbb{N}$. Let $\pi : X \to X/\mathbf{RP}^{[d]}(X)$ be the factor map.

We first show that π is a proximal extension.

Remark that if $(x, y) \in R_\pi = \mathbf{RP}^{[d]}(X)$ is a $T \times T$-minimal point, then by Lemma 5.4 we have that (x, y) is an $\mathrm{IN}^{[d]}$-pair and thus we get that $x = y$. Now consider any $(x, y) \in R_\pi$ and $u \in E(X, T)$ a minimal idempotent. As (ux, uy) is a $T \times T$-minimal point, we have from previous observation that $ux = uy$, which implies that (x, y) is a proximal pair.

This shows that $\mathbf{P}(X) = \mathbf{RP}^{[\infty]}(X) = \mathbf{RP}^{[d]}(X)$, which implies that the maximal factor of order ∞ of X is $X/\mathbf{RP}^{[d]}(X)$.

As X does not contain any non-trivial $\mathrm{IN}^{[d]}$-pair, we get that $\mathrm{Ind}_{fip}(X)$ is trivial. By Theorem 5.6, X is an almost one-to-one extension of its maximal factor of order ∞. From this, we deduce that X is an almost one-to-one extension of its maximal factor of order d.

This completes the proof. □

To end this section, we pose a question which we cannot solve in this paper.

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Question 5.8. Let (X, T) be a minimal system without any non-trivial $\mathrm{IN}^{[d]}$-pair, where $d \in \mathbb{N}$. Is (X, T) uniquely ergodic?

6. *An example*

In the last part of this paper, we give the example which is mentioned in the introduction.

Example 6.1. For $d \in \mathbb{N}$, there is a minimal system (X, T) such that $\mathrm{IN}^{[d]}(X)$ is trivial but $\mathbf{RP}^{[d]}(X)$ is non-trivial.

We start with the following classical example. Let $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ be the one-dimensional torus. Consider an irrational rotation $(\mathbb{T}, R_\alpha)$. Choose $x_0 \in \mathbb{T}$ and split each point of the orbit $x_n = x_0 + n\alpha$ into two points x_n^+, x_n^-. This procedure results is a Sturmian symbolic dynamical system (X_0, σ) which is a minimal almost one-to-one extension of $(\mathbb{T}, R_\alpha)$, and we denote this extension by π. The minimal system (X_0, σ) is null, which implies $\mathrm{IN}^{[1]}(X_0) = \Delta_{X_0}$. For more details, see [9, Example 14.4].

For $d \geq 2$, let $X = X_0 \times \mathbb{T}^{d-1}$. Define a continuous map $T : X \to X$ by

$$(x, z_2, \ldots, z_d) \mapsto (\sigma x, z_2 + \pi(x), \ldots, z_d + z_{d-1}).$$

We next show that the system (X, T) meets our requirement.

Let $T_\alpha : \mathbb{T}^d \to \mathbb{T}^d$ such that

$$(z_1, z_2, \ldots, z_d) \mapsto (z_1 + \alpha, z_2 + z_1, \ldots, z_d + z_{d-1}).$$

Then $(\mathbb{T}^d, T_\alpha)$ is a minimal d-step nilsystem (see, for example, [23, Example 5.22]). There is a continuous map from X to $\mathbb{T}^d$ induced by π which will be denoted by $\widetilde{\pi}$:

$$\widetilde{\pi}(x, z_2, \ldots, z_d) = (\pi(x), z_2, \ldots, z_d).$$

Moreover, $\widetilde{\pi}$ induces a non-trivial extension $\widetilde{\pi} : (X, T) \to (\mathbb{T}^d, T_\alpha)$. Notice that the factor map π is almost one-to-one; a simple observation shows that the factor map $\widetilde{\pi}$ is also almost one-to-one. From this, we deduce that the system (X, T) is minimal.

CLAIM 1. $\mathbf{RP}^{[d]}(X) = R_{\widetilde{\pi}}$.

Proof of Claim 1. Write $z_1 = \pi(x_m^+) \in \mathbb{T}$ and $z_0 = \alpha$. A simple computation yields

$$T^n(x_m^*, z_2, \ldots, z_d) = \left(\sigma^n x_m^*, z_2 + nz_1 + \frac{n(n-1)}{2} z_0, \ldots, \sum_{i=0}^{d} \binom{n}{d-i} z_i\right),$$

where $* \in \{+, -\}$, $n \in \mathbb{Z}$ and $\binom{n}{0} = 1$, $\binom{n}{i} = n \cdot \cdots \cdot (n - i + 1)/i!$ for $i = 1, \ldots, d$.

From this, we get that for any $z_2, \ldots, z_d \in \mathbb{T}$ and $m \in \mathbb{Z}$, the points $(x_m^+, z_2, \ldots, z_d)$ and $(x_m^-, z_2, \ldots, z_d)$ can be close enough, which implies that two such points are proximal, and also are regionally proximal of order d by (2). It follows that $R_{\widetilde{\pi}} \subset \mathbf{RP}^{[d]}(X)$.

On the other hand, by Theorem 2.5, one has that

$$(\widetilde{\pi} \times \widetilde{\pi})\mathbf{RP}^{[d]}(X) = \mathbf{RP}^{[d]}(\mathbb{T}^d) = \Delta_{\mathbb{T}^d},$$

which implies $\mathbf{RP}^{[d]}(X) \subset R_{\widetilde{\pi}}$.

This shows Claim 1 and thus $\mathbf{RP}^{[d]}(X)$ is non-trivial. $\square$

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CLAIM 2. $\mathrm{IN}^{[d]}(X) = \Delta$.

To show this claim, we need the following proposition.

PROPOSITION 6.2. [17, Proposition 3.2] *Let $\pi : (Y, T) \longrightarrow (Z, T)$ be a factor map of dynamical systems. If $(y_1, y_2) \in \mathrm{IN}^{[1]}(Y)$, then $(\pi(y_1), \pi(y_2)) \in \mathrm{IN}^{[1]}(Z)$.*

Proof of Claim 2. Let $\mathbf{x} = (x_1, \ldots, x_d)$, $\mathbf{y} = (y_1, \ldots, y_d) \in X$ with $(\mathbf{x}, \mathbf{y}) \in \mathrm{IN}^{[d]}(X)$. By the definition of the $\mathrm{IN}^{[d]}$-pair, we get that $\mathrm{IN}^{[d]}(X) \subset \mathbf{RP}^{[d]}(X)$, which implies $x_2 = y_2, \ldots, x_d = y_d$ by Claim 1.

Let θ be the projection from X to X_0. It is easy to see θ induces a factor map $\theta : (X, T) \to (X_0, \sigma)$. Notice that $(\mathbf{x}, \mathbf{y}) \in \mathrm{IN}^{[d]}(X) \subset \mathrm{IN}^{[1]}(X)$. Thus by Proposition 6.2, one has $(x_1, y_1) = (\theta(\mathbf{x}), \theta(\mathbf{y})) \in \mathrm{IN}^{[d]}(X_0) = \Delta_{X_0}$, which implies $x_1 = y_1$.

This shows that $\mathrm{IN}^{[d]}(X) = \Delta$. □

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A. *Appendix. Basic facts about abstract topological dynamics*

We recall some basic definitions and results in abstract topological dynamical systems. For more details, see [1, 6].

A.1. *Topological transformation groups.* A *topological dynamical system* is a triple $\mathcal{X} = (X, \mathcal{T}, \Pi)$, where X is a compact metrizable space, $\mathcal{T}$ is a T_2 topological group and $\Pi : T \times X \to X$ is a continuous map such that $\Pi(e, x) = x$ and $\Pi(s, \Pi(t, x)) = \Pi(st, x)$. We shall fix $\mathcal{T}$ and suppress the action symbol. $\mathcal{X}$ is widely also called a *topological transformation group* or a *flow*. Usually we omit Π and denote a system by $(X, \mathcal{T})$.

Let $(X, \mathcal{T})$ be a system and $x \in X$. Then $\mathcal{O}(x, \mathcal{T})$ denotes the *orbit* of x, which is also denoted by $\mathcal{T}x$. A subset $A \subset X$ is called *invariant* if $ta \in A$ for all $a \in A$ and $t \in \mathcal{T}$. When $Y \subset X$ is a closed and $\mathcal{T}$-invariant subset of the system $(X, \mathcal{T})$ we say that the system $(Y, \mathcal{T})$ is a *subsystem* of $(X, \mathcal{T})$. If $(X, \mathcal{T})$ and $(Y, \mathcal{T})$ are two dynamical systems their *product system* is the system $(X \times Y, \mathcal{T})$, where $t(x, y) = (tx, ty)$. A system $(X, \mathcal{T})$ is called *minimal* if X contains no proper closed invariant subsets.

A.2. *Enveloping semigroups.* Given a system $(X, \mathcal{T})$, its *enveloping semigroup* or *Ellis semigroup* $E(X, \mathcal{T})$ is defined as the closure of the set $\{t : t \in \mathcal{T}\}$ in X^X (with its compact, usually non-metrizable, pointwise convergence topology). The maps $E \to E : p \mapsto pq$ and $p \mapsto tp$ are continuous for all $q \in E$ and $t \in \mathcal{T}$.

A.3. *Idempotents and ideals.* For a semigroup the element u with $u^2 = u$ is called an *idempotent*. The Ellis–Numakura theorem says that for any enveloping semigroup E the set $J(E)$ of idempotents of E is not empty [6]. A non-empty subset $I \subset E$ is a *left ideal*

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(respectively, *right ideal*) if it $EI \subset I$ (respectively, $IE \subset I$). A *minimal left ideal* is the left ideal that does not contain any proper left ideal of E. Obviously every left ideal is a semigroup and every left ideal contains some minimal left ideal.

An idempotent $u \in J(E)$ is *minimal* if $v \in J(E)$ and $vu = v$ implies $uv = u$. The following results are well known [5, 7]. Let L be a left ideal of enveloping semigroup E and $u \in J(E)$. Then there is some idempotent v in Lu such that $uv = v$ and $vu = v$; an idempotent is minimal if and only if it is contained in some minimal left ideal.

A useful result about minimal points is the following proposition.

PROPOSITION A.1. *Let I be a minimal left ideal. A point $x \in X$ is minimal if and only if $ux = x$ for some $u \in I$.*

A.4. *Proximality.* Two points x_1 and x_2 are called *proximal* if and only if

$$\overline{\mathcal{T}(x_1, x_2)} \cap \Delta_X \neq \emptyset.$$

Let $\mathcal{U}_X$ be the unique uniform structure of X. Then the following assertions hold.

$$\mathbf{P}(X) = \bigcap \{\mathcal{T}\alpha : \alpha \in \mathcal{U}_X\}$$

is the collection of proximal pairs in X, the *proximal relation*.

PROPOSITION A.2. *Let $(X, \mathcal{T})$ be a dynamical system. Then:*

(1) *The points x_1, x_2 are proximal in $(X, \mathcal{T})$ if and only if $px_1 = px_1$ for some $p \in E(X, \mathcal{T})$;*
(2) *If u is an idempotent in $E(X, \mathcal{T})$, then $(x, ux) \in \mathbf{P}(X)$ for every $x \in X$;*
(3) *There is a minimal point $x' \in \overline{O(x, \mathcal{T})}$ such that $(x, x') \in \mathbf{P}(X)$;*
(4) *If (X, T) is minimal, then $(x, y) \in \mathbf{P}(X)$ if and only if there is some minimal idempotent $u \in E(X, \mathcal{T})$ such that $y = ux$.*

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