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Homotopy Equivalence and Groups of Measure-Preserving Homeomorphisms

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Abstract. It is shown that the group of compactly supported, measure-preserving homeomorphisms of a connected, second countable manifold is locally contractible in the direct limit topology. Furthermore, this group is weakly homotopically equivalent to the more general group of compactly supported homeomorphisms.

1 Introduction

Let M be a connected, second countable, Hausdorff manifold and let μ_o be a ∂-good measure on M, that is, a rather "homogeneous" measure to be defined below. The main results to be proved hereafter are: First, the group of compactly supported, measure-preserving homeomorphisms of M, denoted by $\mathcal{H}_c(M,\mu_o)_{\underrightarrow{\lim}}$, is locally contractible in the direct limit topology. Second, the inclusion $\mathcal{H}_c(M,\mu_o)_{\underrightarrow{\lim}} \hookrightarrow \mathcal{H}_c(M)_{\underrightarrow{\lim}}$ is a weak homotopy equivalence, where $\mathcal{H}_c(M)$ stands for the group of compactly supported homeomorphisms.

In Fathi [7], the Cernavskii, Edwards-Kirby Theorem on the deformation of spaces of embeddings is adapted to the case of spaces of the so-called "μ_o-biregular embeddings". This, and the results in Berlanga [2] are the main tools to prove that $\mathcal{H}_c(M,\mu_o)_{\underrightarrow{\lim}}$ is locally contractible.

It is shown, in Berlanga [2], that the inclusion of $\mathcal{H}_c(M,\mu_o)_{\underrightarrow{\lim}}$ into the group of compactly supported μ_o-biregular embeddings $\mathcal{H}_c(M,\mu_o\text{-}\varepsilon\text{-reg})_{\underrightarrow{\lim}}$ is a homotopy equivalence. Therefore, in order to get the second result of the present work, it is sufficient to prove that the inclusion $\mathcal{H}_c(M,\mu_o\text{-}\varepsilon\text{-reg})_{\underrightarrow{\lim}} \hookrightarrow \mathcal{H}_c(M)_{\underrightarrow{\lim}}$ is a weak homotopy equivalence.

Although the spaces we are considering are not metrizable in general, the methods of Eilenberg and Wilder [6] apply to our situation. Analogous results hold for the groups of homeomorphisms fixing ∂M pointwise.

2 Definitions

Let M be a second countable n-dimensional, Hausdorff manifold, possibly with nonempty boundary ∂M. Denote by $\mathcal{H}(M)$ the group of homeomorphisms of M and by $\mathcal{H}^{\partial}(M)$ the subgroup of homeomorphisms fixing ∂M pointwise. Recall that the *compact-open* topology on $\mathcal{H}(M)$ is the topology having for a subbase the sets $[K,U] = \{h \in \mathcal{H}(M) \mid h(K) \subset U\}$ where K is compact, and U is open in M.

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Denote the resulting topological space by $\mathcal{H}(M)_\kappa$. In what follows, all manifolds are assumed to be T_2 and σ-compact, hence metrizable.

For h in $\mathcal{H}(M)$ define its *support* as the closure of $\{x \in M \mid h(x) \neq x\}$. Define $\mathcal{H}_c(M)$ to be the group of all homeomorphisms of M with compact support. If K is a compact subset in M, let $\mathcal{H}(K, M)$ be the group of all homeomorphisms with support contained in K. Closely related to $\mathcal{H}_c(M)$ and $\mathcal{H}(K, M)$ are the groups $\mathcal{H}_c^\partial(M) = \mathcal{H}_c(M) \cap \mathcal{H}^\partial(M)$ and $\mathcal{H}^\partial(K, M) = \mathcal{H}(K, M) \cap \mathcal{H}^\partial(M)$ of homeomorphisms fixing ∂M pointwise.

In order to topologize groups of compactly supported homeomorphisms, some elementary facts on *direct limit* topologies should be made precise: let X be a set and let $\{A_\lambda\}_{\lambda \in \Lambda}$ be a collection of topological spaces, each a subset of X. We say that $\{A_\lambda\}$ is a *coherent family* (of topological spaces) on X if and only if $X = \bigcup_\lambda A_\lambda$, and for every $\lambda, \rho \in \Lambda$, $A_\lambda \cap A_\rho$ is closed in A_λ and the topologies of A_λ and A_ρ agree on $A_\lambda \cap A_\rho$.

The *direct limit* topology defined on X by the family $\{A_\lambda\}$ is the largest or finest topology such that, for each $\lambda \in \Lambda$, the inclusion $\iota_\lambda \colon A_\lambda \hookrightarrow X$ is continuous. With this topology on X, usually denoted by $X_{\varinjlim}$, each of these inclusions is a closed embedding. Furthermore, a subset V in X is closed if and only if, for all $\lambda \in \Lambda$, $A_\lambda \cap V$ is closed (in A_λ).

If Y is a space and $f \colon X \to Y$ a function, then f is continuous if and only if the composition (restriction) $f \circ \iota_\lambda \colon A_\lambda \to Y$ is continuous for each $\lambda \in \Lambda$. In particular, suppose $\{B_\gamma\}_{\gamma \in \Gamma}$ is a coherent family of spaces on the set Y and suppose $f \colon X \to Y$ is *stratified* in the sense that for each $\lambda \in \Lambda$, $f(A_\lambda) \subseteq B_\gamma$ for some $\gamma \in \Gamma$ and $f \circ \iota_\lambda \colon A_\lambda \to B_\gamma$ is continuous. Then f is continuous when Y is given the direct limit topology given by the family $\{B_\gamma\}$.

For $K \subset M$ compact, $\mathcal{H}(K, M)$ is closed in $\mathcal{H}(M)_\kappa$, hence $\{\mathcal{H}(K, M)_\kappa \mid K \subset M \text{ compact}\}$ is a coherent family of spaces in $\mathcal{H}_c(M)$, so we can consider the resulting direct limit topology. The space $\mathcal{H}_c(M)_{\varinjlim}$ is one of the major objects of this paper.

Let X_τ be a Hausdorff space. Then its family of compact subsets is coherent. Denote by kX the resulting space with the topology induced by this family. If $X_\tau = kX$, then X_τ is called a *compactly generated space*. Observe that kX is always a compactly generated space. See Whitehead [11] for further details.

We also want to consider the Whitney topology on the group of homeomorphisms of M. The relationship between the Whitney topology and the direct limit topology is established by noting that both topologies have the same family of compact subsets. This observation is of some importance later on. The fine or Whitney topology on $\mathcal{H}(M)$, denoted by $\mathcal{H}(M)_m$, is the topology having for a basis the sets $\bigcap_{i \in \Lambda} [K_i, U_i]$ where $\{K_i\}_{i \in \Lambda}$ is a locally finite family of compact sets and $\{U_i\}_{i \in \Lambda}$ an (arbitrary) open family in M. Observe that $\mathcal{H}(K, M)_m = \mathcal{H}(K, M)_\kappa$ for $K \subseteq M$ compact. With this, $\mathcal{H}(M)_m$ becomes a topological group having $\mathcal{H}_c(M)$ as a closed, nowhere dense subset and such that the inclusions $\mathcal{H}_c(M)_{\varinjlim} \hookrightarrow \mathcal{H}(M)_m \hookrightarrow \mathcal{H}(M)_\kappa$ are continuous.

If A is a subset of a topological space X, denote its interior, closure and frontier by $\operatorname{Int} A$, $\operatorname{Cl} A$ and $\operatorname{Fr} A$, respectively.

Lemma 2.1 *Let $\mathcal{K}$ be a compact subset in $\mathcal{H}_c(M)_m$, where M is a given manifold. Then there is a compact subset $K \subset M$ such that every element in $\mathcal{K}$ has support in K.*

Proof We argue by contradiction: assume that a compact subset $\mathcal{K}$ in $\mathcal{H}_c(M)_m$ is given such that for every compact K in M there is an $h \in \mathcal{K}$ with $\operatorname{supp} h \not\subset K$. It is then easy to construct a sequence $\{K_i\}_{i=1}^{\infty}$ of compact subsets of M whose union is M with $K_i \subset \operatorname{Int} K_{i+1}$ for each $i \in \mathbb{N} \setminus \{0\}$, and a sequence $\{h_i\}_{i=1}^{\infty}$ of homeomorphisms in $\mathcal{K}$ such that, for each $i \in \mathbb{N} \setminus \{0\}$, $\operatorname{supp} h_i \not\subset K_i$ but $\operatorname{supp} h_i \subset K_{i+1}$.

We want to show that $\{h_i\}_{i=1}^{\infty}$ is a closed but not compact subset of $\mathcal{K}$. For this purpose, let $h_0 \in \mathcal{H}_c(M) \setminus \{h_i\}_{i=1}^{\infty}$. Using the fact that $\mathcal{H}_c(M)_m$ is Hausdorff and the "divergent nature" of the family $\{\operatorname{supp} h_i\}_{i=1}^{\infty}$, we can choose, for each $i \in \mathbb{N}$, open sets $\mathcal{N}_{1,i}$ and $\mathcal{N}_{2,i}$ in $\mathcal{H}_c(M)$ such that $h_i \in \mathcal{N}_{1,i} \cap \mathcal{N}_{2,i}$, $h_j \notin \mathcal{N}_{1,i}$ for $0 \le j < i$, and $h_j \notin \mathcal{N}_{2,i}$ for $i < j$. Therefore, $\mathcal{N}_{1,0} \cap \mathcal{N}_{2,0}$, is a neighbourhood of h_0 contained in $\mathcal{H}_c(M) \setminus \{h_i\}_{i=1}^{\infty}$, proving that $\{h_i\}_{i=1}^{\infty}$ is closed. Also,$\{\mathcal{N}_{1,i} \cap \mathcal{N}_{2,i}\}_{i=1}^{\infty}$ is an open cover of $\{h_i\}_{i=1}^{\infty}$ which has no finite subcover. ■

Assertion 2.2 *Let M be a manifold and let $\mathcal{U} \subseteq \mathcal{H}_c(M)_m$ be an open subspace. Let $\mathcal{U}_{\varinjlim}$ denote the space obtained from (the underlying set of) $\mathcal{U}$ with the direct limit topology given by the (coherent) family $\{\mathcal{H}(K,M)_\kappa \cap \mathcal{U} \mid K \subset M \text{ compact}\}$. Let $\mathcal{U}_S$ be the space obtained from $\mathcal{U}$ with the subspace topology from $\mathcal{H}_c(M)_{\varinjlim}$. Finally let $k\mathcal{U}$ be the associated compactly generated space of $\mathcal{U}$. Then $\mathcal{U}_{\varinjlim} = \mathcal{U}_S = k\mathcal{U}$ and $\mathcal{U}_{\varinjlim}$ is open in $\mathcal{H}_c(M)_{\varinjlim} = k(\mathcal{H}_c(M)_m)$.*

Proof Since the inclusion $\mathcal{H}_c(M)_{\varinjlim} \hookrightarrow \mathcal{H}_c(M)_m$ is continuous, $\mathcal{U}_S$ is open in $\mathcal{H}_c(M)_{\varinjlim}$. With this, it follows that $\mathcal{U}_{\varinjlim} = \mathcal{U}_S$ holds as a consequence of the definition of the relevant topologies.

By Lemma 2.1 $\mathcal{H}_c(M)_{\varinjlim}$ and $\mathcal{H}_c(M)_m$ have the same compact sets, therefore $k(\mathcal{H}_c(M)_{\varinjlim}) = k(\mathcal{H}_c(M)_m)$. But $\mathcal{H}_c(M)_{\varinjlim}$ is a compactly generated space for it is Hausdorff and each $\mathcal{H}(K,M)_\kappa$ is a compactly generated space (for it is first countable). Therefore $k(\mathcal{H}_c(M)_{\varinjlim}) = \mathcal{H}_c(M)_{\varinjlim}$. Finally, $k\mathcal{U}$ is just $\mathcal{U}_S$ because $\mathcal{U}_S$ is an open subspace of a Hausdorff compactly generated space. ■

As a corollary we have that the inclusion $\mathcal{H}_c(M)_{\varinjlim} \hookrightarrow k(\mathcal{H}(M)_m)$ is a closed embedding. We are mostly interested in the implication "if $\mathcal{U} \subset \mathcal{H}_c(M)_m$ is open then $\mathcal{U}_{\varinjlim} \subset \mathcal{H}_c(M)_{\varinjlim}$ is open", but we think it is pertinent to set these topological considerations straight.

A *Radon measure* μ on M is a locally finite positive measure defined on the σ-algebra of all Borel subsets. The *support* of μ is the complement of the largest open set in M which has $\mu-$measure zero. We say that μ is a *good measure* if it has no atoms (*i.e.,* points of positive measure) and its support is the whole of M. Let $\mathcal{M}(M)$ be the set of Radon measures on M, $\mathcal{M}_g(M)$ the set of good Radon measures, $\mathcal{M}^{\partial}(M)$ the set of measures having ∂M as a null set and $\mathcal{M}_g^{\partial}(M) = \mathcal{M}_g(M) \cap \mathcal{M}^{\partial}(M)$. For $\mu \in \mathcal{M}_g^{\partial}(M)$ and $h \in \mathcal{H}(M)$, $h_*\mu$ given by $h_*\mu(B) = \mu(h^{-1}(B))\ \forall B \subset M$ *Borel* defines a measure in $\mathcal{M}_g^{\partial}(M)$.

Let $\mu_o \in \mathcal{M}_g^{\partial}(M)$. Define the group of μ_o-measure-preserving homeomorphisms $\mathcal{H}(M,\mu_o)$ as the set $\{h \in \mathcal{H}(M) \mid h_*\mu_o = \mu_o\}$. Let $\mathcal{H}(K,M,\mu_o) = \mathcal{H}(K,M) \cap \mathcal{H}(M,\mu_o)$ for each $K \subset M$ compact, and $\mathcal{H}_c(M,\mu_o) = \mathcal{H}_c(M) \cap \mathcal{H}(M,\mu_o)$. Since $\mathcal{H}(M,\mu_o)$ is closed in $\mathcal{H}(M)_\kappa$ (see Berlanga [2]), $\mathcal{H}_c(M,\mu_o)$ is closed in $\mathcal{H}_c(M)_{\varinjlim}$.

Also, the topology that $\mathcal{H}_c(M,\mu_o)$ gets as a subspace of $\mathcal{H}_c(M)_{\varinjlim}$ is the same as the direct limit topology given by the coherent family $\{\mathcal{H}(K,M,\mu_o)_\kappa \mid K \subset M \text{ compact}\}$.

For the analogously defined groups of measure-preserving homeomorphisms fixing ∂M pointwise, let us only state that $\mathcal{H}^\partial(M,\mu_o)$ is closed in $\mathcal{H}(M)_\kappa$ (hence in $\mathcal{H}(M,\mu_o)_\kappa$) and $\mathcal{H}_c^\partial(M,\mu_o)$ is closed in $\mathcal{H}_c(M)_{\varinjlim}$ (hence in $\mathcal{H}_c(M,\mu_o)_{\varinjlim}$).

Denote by $\mathcal{H}(M,\mu_o\text{-reg})$ the group of all homeomorphisms h in $\mathcal{H}(M)$ such that $h_*\mu_o$ and μ_o have the same sets of measure zero. Let $\mathcal{H}_c(M,\mu_o\text{-}\varepsilon\text{-reg}) = \mathcal{H}_c(M) \cap \mathcal{H}(M,\mu_o\text{-reg})$ and, for any compact subset K of M, define $\mathcal{H}(K,M,\mu_o\text{-}\varepsilon\text{-reg}) = \mathcal{H}(K,M) \cap \mathcal{H}(M,\mu_o\text{-reg})$. Denote by $\mathcal{H}_c(M,\mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim}$ the direct limit topological space obtained from the (closed) family $\{\mathcal{H}(K,M,\mu_o\text{-}\varepsilon\text{-reg})_\kappa \mid K \subset M \text{ compact}\}$.

It is straightforward to define the groups $\mathcal{H}^\partial(M,\mu_o\text{-reg})$, $\mathcal{H}_c^\partial(M,\mu_o\text{-}\varepsilon\text{-reg})$ and $\mathcal{H}^\partial(K,M,\mu_o\text{-}\varepsilon\text{-reg})$ of biregular homeomorphisms fixing ∂M pointwise. Let $\mathcal{H}_c^\partial(M,\mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim}$ be $\mathcal{H}_c^\partial(M,\mu_o\text{-}\varepsilon\text{-reg})$ endowed with the direct limit topology imposed by the (closed) family $\{\mathcal{H}^\partial(K,M,\mu_o\text{-}\varepsilon\text{-reg})_\kappa \mid K \subset M \text{ compact}\}$. Note that the letter ε is not indispensable in the above notations (consistent with those in Berlanga [2]). Its purpose is to reflect the fact that the pertinent groups preserve "the behaviour of μ_o at infinity", in a technical sense not relevant in the present work.

3 Deformations

Let M be a (second countable) manifold and let $\mu_o \in \mathcal{M}_g^\partial(M)$. Let A be a subset of M. By a *proper embedding* ι of A into M we mean an injective (continuous) map $\iota\colon A \hookrightarrow M$ such that ι is a homeomorphism of A onto $\iota(A)$ and $\iota^{-1}(\partial M) = A \cap \partial M$. Denote by $\mathcal{I}(A,M)$ the space of proper embeddings of A into M.

If $\iota \in \mathcal{I}(A,M)$ and A is a Borel subset of M, we can define a measure $\iota^*\mu_o$ on A such that $\iota^*\mu_o(B) = \mu_o(\iota(B))$ for each Borel subset $B \subset A$. We say that a proper embedding $\iota\colon A \hookrightarrow M$ is *biregular (with respect to μ_o)* if $\iota^*\mu_o$ and $\mu_o \mid A$ have the same sets of measure zero. Denote by $\mathcal{I}(A,M;\mu_o\text{-reg})$ the set of all proper biregular embeddings of A into M. Suppose B is a subset of M. We define

$$\mathcal{I}(A,B,M) = \left\{ \iota \in \mathcal{I}(A,M) \;\middle|\; \iota|_{B\cap A} = \mathrm{Id} \right\},$$
$$\mathcal{I}(A,B,M;\mu_o\text{-reg}) = \mathcal{I}(A,B,M) \cap \mathcal{I}(A,M;\mu_o\text{-reg}).$$

All spaces of proper embeddings will be endowed with the compact open topology.

Suppose M is a manifold with subsets Q and S. A *deformation of Q into S* is a continuous map $\phi\colon Q \times I \to M$ such that $\phi|_{Q\times\{0\}} = \mathrm{Id}_Q$ and $\phi(Q \times \{1\}) \subset S$. If $T \subset M$ and $\phi(Q \times I) \subset T$, we say that ϕ *takes place in* T.

Let $\mathcal{P}$ be a subset of $\mathcal{I}(A,M)$ and W a subset of A. A deformation $\phi\colon \mathcal{P} \times I \to \mathcal{I}(A,M)$ of $\mathcal{P}$ *is modulo* W if $\phi(\iota,t)|_W = \iota|_W$ for all $\iota \in \mathcal{P}$ and $t \in I$.

The following theorem, where no measures intervene, is due to Černavskii [4]. A much more readable and elegant approach, again without mention of measures, is due to Edwards–Kirby [5]. The version which follows appears in the paper by Fathi [7], who gives the credit for this result to M. Rogalski.

Theorem 3.1 *Let U be a neighbourhood of a compact C in a manifold M. Let μ_o be a measure in $\mathcal{M}_g^{\partial}(M)$. Given any neighbourhood $\mathcal{N}$ of the inclusion $\eta\colon U \hookrightarrow M$ in $\mathcal{I}(U, M; \mu_o\text{-reg})_\kappa$, there is a neighbourhood $\mathcal{P}$ of η in $\mathcal{I}(U, M; \mu_o\text{-reg})_\kappa$ and a deformation $\phi\colon \mathcal{P} \times I \to \mathcal{N}$ into $\mathcal{I}(U, C, M; \mu_o\text{-reg})$ such that*

(i) *ϕ is modulo the complement of a compact neighbourhood of C in Int U;*
(ii) *$\phi(\eta, t) = \eta$ for all $t \in I$;*
(iii) *$\phi \mid [\mathcal{P} \cap \mathcal{I}(U, \partial M, M; \mu_o\text{-reg})] \times I$ takes place in $\mathcal{I}(U, \partial M, M; \mu_o\text{-reg})$.*

Furthermore, suppose in addition to the above hypothesis that a closed set D in M (respectively, ∂M) and a neighbourhood V of D in M (respectively ∂M) are given. Then ϕ can be chosen so that the deformation $\phi\big| [\mathcal{P} \cap \mathcal{I}(U, V, M; \mu_o\text{-reg})] \times I$ takes place in $\mathcal{I}(U, D, M; \mu_o\text{-reg})$.

4 Local Contractibility

Recall that a space X is locally contractible if for each $x \in X$ there is a neighbourhood U of x such that U is deformable into $\{x\}$ by a deformation fixing x. This condition implies that every neighbourhood V of a point $x \in X$ contains a neighbourhood U of x deformable to $\{x\}$ in V.

Proposition 4.1 *Let M be a manifold and let $\mu_o \in \mathcal{M}_g^{\partial}(M)$. Then $\mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim}$ and $\mathcal{H}_c^{\partial}(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim}$ are locally contractible.*

Proof It suffices to prove (*cf.* [5, Corollary 6.2]) that the identity has a contractible neigbourhood. Fix a metric d on M. If $U \subset M$ is compact and $\delta > 0$, let $\mathcal{N}(\eta, \delta)$ denote the (basic) neighbourhood of the inclusion $\eta\colon U \hookrightarrow M$ in $\mathcal{I}(U, M; \mu_o\text{-reg})$ given by the set $\{\iota \in \mathcal{I}(U, M; \mu_o\text{-reg}) \mid d(\iota(x), x) < \delta \, \forall x \in U\}$.

Let $\{(U_i, C_i) \mid i \in \mathbb{N}\}$ be a countable collection of pairs of compact subsets of M such that for each $i \in \mathbb{N}$, U_i is a neighbourhood of C_i, $M = \bigcup_{i \in \mathbb{N}} \operatorname{Int} C_i$ and $U_i \cap U_j \neq \varnothing$ only if $|i - j| \leq 1$.

It follows from Theorem 3.1 (letting $U = U_{2i}, C = C_{2i}, V = C_{2i-1} \cup C_{2i+1}$ and $D = \operatorname{Cl}(U_{2i} \backslash C_{2i})$) that there is a sequence $\{\delta_{2i}\}$ of positive numbers such that if $\mathcal{P}_{2i}$ is defined to be the neighbourhood $\mathcal{N}(\eta, \delta_{2i})$ of $\eta\colon U_{2i} \hookrightarrow M$ in $\mathcal{I}(U_{2i}, M; \mu_o\text{-reg})_\kappa$ then there is a deformation $\phi_{2i}\colon \mathcal{P}_{2i} \times I \to \mathcal{I}(U_{2i}, M; \mu_o\text{-reg})_\kappa$ of $\mathcal{P}_{2i}$ into $\mathcal{I}(U_{2i}, C_{2i}, M; \mu_o\text{-reg})$ such that ϕ_{2i} deforms $\mathcal{P}_{2i} \cap \mathcal{I}(U_{2i}, C_{2i-1} \cup C_{2i+1}, M; \mu_o\text{-reg})$ into $\{\eta\}$ and ϕ_{2i} is modulo $\operatorname{Fr}_M U_{2i}$.

Likewise, there is a sequence $\{\delta_{2i-1}\}$ of positive numbers such that if $\mathcal{P}_{2i-1}$ is defined to be the neighbourhood $\mathcal{N}(\eta, \delta_{2i-1})$ of $\eta\colon U_{2i-1} \hookrightarrow M$ in $\mathcal{I}(U_{2i-1}, M; \mu_o\text{-reg})_\kappa$, then there is a deformation $\phi_{2i-1}\colon \mathcal{P}_{2i-1} \times I \to \mathcal{I}(U_{2i-1}, M; \mu_o\text{-reg})_\kappa$ of $\mathcal{P}_{2i-1}$ into $\mathcal{I}(U_{2i-1}, C_{2i-1}, M; \mu_o\text{-reg})$ such that ϕ_{2i-1} takes place in $\mathcal{N}(\eta, \min\{\delta_{2i-2}, \delta_{2i}\})$ and ϕ_{2i-1} is modulo $\operatorname{Fr}_M U_{2i-1}$.

Let $\delta\colon M \to \langle 0, \infty \rangle$ be continuous and such that $\delta(U_i) \subset \langle 0, \delta_i \rangle$ for each $i \in \mathbb{N}$ and let the set $\mathcal{U} = \{h \in \mathcal{H}(M, \mu_o\text{-reg}) \mid d(h(x), x) < \delta(x) \, \forall x \in M\}$ be endowed with the compact open topology. Define the continuous function $\phi\colon \mathcal{U} \times I \to$

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$\mathcal{H}(M, \mu_o\text{-reg})_\kappa$ by the formula

$$\phi(h,t) = \begin{cases} \phi_{2i-1}(h\mid_{U_{2i-1}}, 2t) & \text{on } U_{2i-1} \text{ for } t \in [0,1/2], \\ h & \text{on } M \setminus \bigcup_{i\in\mathbb{N}} U_{2i-1}; \end{cases}$$

$$\phi(h,t) = \begin{cases} \phi_{2i}(\phi(h,1/2)\mid_{U_{2i}}, 2t-1) & \text{on } U_{2i} \text{ for } t \in [1/2,1], \\ \phi(h,1/2) & \text{on } M \setminus \bigcup_{i\in\mathbb{N}} U_{2i}. \end{cases}$$

Furthermore, ϕ restricts to $\phi_c\colon \mathcal{U}_c \times I \to \mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})$ where $\mathcal{U}_c = \mathcal{U} \cap \mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})$ and it is stratified for the reason that

$$\phi(\Big[\mathcal{U} \cap \mathcal{H}\Big(\bigcup_{j=0}^{2i-1} C_j, M, \mu_o\text{-}\varepsilon\text{-reg}\Big)\Big] \times I)$$

is contained in $\mathcal{H}(\bigcup_{j=0}^{2i} C_j, M, \mu_o\text{-}\varepsilon\text{-reg})$. Hence

$$\phi_c\colon [\mathcal{U}_c \times I]_{\varinjlim} \to \mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim}$$

is continuous. Also $[\mathcal{U}_c \times I]_{\varinjlim} = [\mathcal{U}_c]_{\varinjlim} \times I$ (for I is locally compact) and $[\mathcal{U}_c]_{\varinjlim}$ is an open subspace in $\mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim}$ (by 2.2). This implies that $\phi_c\colon [\mathcal{U}_c]_{\varinjlim} \times I \to \mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim}$ is a deformation of $\mathcal{U}_c$ into $\{\mathrm{Id}\}$. By Theorem 3.1, $\phi(\{\mathrm{Id}\} \times I) = \{\mathrm{Id}\}$, and ϕ restricted to $[\mathcal{U}_c \cap \mathcal{H}_c^\partial(M, \mu_o\text{-}\varepsilon\text{-reg})] \times I$ takes place in $\mathcal{H}_c^\partial(M, \mu_o\text{-}\varepsilon\text{-reg})$. Hence the conclusion follows. ∎

Recall that a space X is *semi-locally simply connected* if for each $x \in X$ there is a neighbourhood U of x in X such that the homomorphism $\pi(U,x) \to \pi(X,x)$ between the fundamental groups of U and X (at x) induced by the inclusion $U \hookrightarrow X$ is trivial. Obviously, a locally contractible space X is semi-locally simply connected.

Remark 4.2 Part of the content of [2, Proposition 5.1; Corollary 5.2] is that, for a good measure μ_o in $\mathcal{M}_g^\partial(M)$ on a manifold M, there is a strong deformation retraction $\rho\colon \mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim} \to \mathcal{H}_c(M, \mu_o)_{\varinjlim}$ which restricts to a strong deformation retraction on the corresponding spaces of homeomorphisms fixing ∂M pointwise.

Theorem 4.3 *Let M be a manifold and μ_o a measure in $\mathcal{M}_g^\partial(M)$. Then $\mathcal{H}_c(M, \mu_o)_{\varinjlim}$ and $\mathcal{H}_c^\partial(M, \mu_o)_{\varinjlim}$ are (locally path connected and) locally contractible.*

Furthermore, there exists a (Whitney) neighbourhood $\mathcal{U}$ of the identity in $\mathcal{H}_c(M, \mu_o)_m$ such that $\mathcal{U}_{\varinjlim}$ is deformed into $\{\mathrm{Id}\}$ by a contraction ϕ fixing the identity and such that ϕ restricted to $[\mathcal{H}_c^\partial(M, \mu_o) \cap \mathcal{U}] \times I$ takes place in $\mathcal{H}_c^\partial(M, \mu_o)$. Hence $\mathcal{H}_c(M, \mu_o)_m$ and $\mathcal{H}_c^\partial(M, \mu_o)_m$ are locally path connected and semi-locally simply connected in the Whitney topology.

Proof Let x be a point in a space X. Then X is *connected im kleinen at x* if each open neighbourhood V of x contains an open neighbourhood U of x such that any pair points in U lie in some connected subset K of V. It is not difficult to show that if X is locally connected im kleinen at each point, then X is locally connected. Furthermore, X is locally path connected if the subsets K in the above definition can be taken to be continuous images of the open interval.

Let $\mathcal{U}_c$ be a Whitney neighbourhood of the identity in $\mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})$ contractible in the direct limit topology, such as in Proposition 4.1. If $\mathcal{U} = \mathcal{U}_c \cap \mathcal{H}_c(M, \mu_o)$, then (using the retraction ρ of Remark 4.2) there is a contraction $\phi\colon \mathcal{U}_{\varinjlim} \times I \to \mathcal{H}_c^{\partial}(M, \mu_o)_{\varinjlim}$ having the desired properties, thus proving that $\mathcal{H}_c(M, \mu_o)_{\varinjlim}$ and $\mathcal{H}_c^{\partial}(M, \mu_o)_{\varinjlim}$ are locally contractible, hence connected im kleinen at each point and also locally path connected. Since any function γ from a compact space P (*e.g.*, $P = I$ or $P = I \times I$) into $\mathcal{H}_c(M, \mu_o)$ is continuous in the direct limit topology if and only if it is continuous in the Whitney topology, the last assertion follows. ∎

5 Weak Homotopy

In [6], S. Eilenberg and R. L. Wilder investigate "the properties of uniformly locally connected subsets of a metric separable space with particular reference to the relation between the set and its closure".

If a subset A contained in the separable metric space X is uniformly locally j-connected for $j = 0, 1, \dots, q$, then the Eilenberg–Wilder theorem states, in particular, that A and $\mathrm{Cl}_X A$ have the same homotopy and homology groups for the dimensions $0, 1, \dots, q$.

The Černavskii, Edwards–Kirby theorem implies a "stratified" local contractibility for the space of compactly supported biregular homeomorphisms of a manifold. Hence, it gives a corresponding form of "stratified" uniform local j-connectedness for each $j \in \mathbb{N}$.

Also, by Fathi [7, Lemma 4.7], it is easily seen that $\mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim}$ is dense in $\mathcal{H}_c(M)_{\varinjlim}$.

Although $\mathcal{H}_c(M)_{\varinjlim}$ is not metrizable when M is non-compact, we can adapt the methods of Eilenberg and Wilder in order to prove the following result:

Theorem 5.1 *Let M be a connected manifold and let $\mu_0 \in \mathcal{M}_g^{\partial}(M)$. Then the inclusions $\mathcal{H}_c(M, \mu_o)_{\varinjlim} \hookrightarrow \mathcal{H}_c(M)_{\varinjlim}$ and $\mathcal{H}_c^{\partial}(M, \mu_o)_{\varinjlim} \hookrightarrow \mathcal{H}_c^{\partial}(M)_{\varinjlim}$ are weak homotopy equivalences.*

Proof The proof for both cases is the same. Therefore, we restrict ourselves to consider the inclusion $\mathcal{H}_c(M, \mu_o)_{\varinjlim} \hookrightarrow \mathcal{H}_c(M)_{\varinjlim}$.

By Remark 4.2, the inclusion $\mathcal{H}_c(M, \mu_o)_{\varinjlim} \hookrightarrow \mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim}$ is a homotopy equivalence. Hence, we need only prove that the inclusion $\mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim} \hookrightarrow \mathcal{H}_c(M)_{\varinjlim}$ is a weak homotopy equivalence. We divide the proof into three steps:

Step 1 Let d be a fixed metric on M. Define the right invariant metric d^* on $\mathcal{H}_c(M)$ by the formula $d^*(f, g) = \sup\{d(f(x), g(x)) \mid x \in M\}$ for each two homeomorphisms of M with compact support. Observe that the inclusion $\mathcal{H}_c(M)_{\varinjlim} \hookrightarrow \mathcal{H}_c(M)_{d^*}$ is continuous. By Berlanga and Epstein [3, Lemma 7], we can construct an increasing sequence $\varnothing \neq K_0 \subset \operatorname{Int} K_1 \subset K_1 \subset \operatorname{Int} K_2 \subset K_2 \subset \cdots \subset M$ of relative

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cells whose union is M and such that $\mu_0(\operatorname{Fr} K_i) = 0$ for each $i \in \mathbb{N}$. The following properties are satisfied:

(i) $\mathcal{H}(K_i, M, \mu_o\text{-}\varepsilon\text{-reg})$ is dense in $\mathcal{H}(K_i, M)$ for each $i \in \mathbb{N}$, so $\mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim}$ is dense in $\mathcal{H}_c(M)_{\varinjlim}$;

(ii) Let $i \in \mathbb{N}$ and let $\epsilon > 0$. Then there exists a $\delta > 0$ such that each continuous map $\gamma^j \colon \partial I^{j+1} \to \mathcal{H}(K_i, M, \mu_o\text{-}\varepsilon\text{-reg})$ with diameter$[\gamma^j(\partial I^{j+1})] < \delta$ is extended continuously to some $\overline{\gamma^j} \colon I^{j+1} \to \mathcal{H}(K_{i+1}, M, \mu_o\text{-}\varepsilon\text{-reg})$ with diameter$[\overline{\gamma^j}(I^{j+1})] < \epsilon$.

Remarks The first part of (i) is essentially in Fathi [7, Lemma 4.7]. Property (ii) is a consequence of Theorem 3.1 above and the right invariance of the metric d^*.

Step 2 We require the following definition:

Definition Let $\gamma \colon \partial I^{j+1} \to \mathcal{H}(K_i, M, \mu_o\text{-}\varepsilon\text{-reg})$ be a continuous map. Define a number $b(\gamma)$ as follows: $b(\gamma) = \infty$ if γ does not have a continuous extension $\overline{\gamma} \colon I^{j+1} \to \mathcal{H}(K_{i+1}, M, \mu_o\text{-}\varepsilon\text{-reg})$, and $b(\gamma) = \inf\{\text{diameter}[\overline{\gamma}(I^{j+1})] \mid \overline{\gamma} \colon I^{j+1} \to \mathcal{H}(K_{i+1}, M, \mu_o\text{-}\varepsilon\text{-reg})$ is a continuous extension of $\gamma\}$ otherwise.

Claim 1 (*cf.* [6, Theorem 1]) *Let B be a closed subset of a compact metric space Z such that the topological dimension of $Z \setminus B$ is finite. Let $f \colon B \to \mathcal{H}_c(M)_{\varinjlim}$ be a continuous map. Then there is an open subset $U \supset B$ and a continuous extension $f' \colon U \to \mathcal{H}_c(M)_{\varinjlim}$ of f such that $f'(U \setminus B) \subset \mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})$.*

Proof of Claim 1 Since B is compact, then $f(B) \subset \mathcal{H}(K_i, M)$ for some $i \in \mathbb{N}$. So assume without loss of generality that $i = 0$. Suppose that the dimension of $Z \setminus B$ is less than or equal to q. Then, according to Kuratowski [8, Theorem 2], $\mathcal{H}(K_0, M)_{d^*} = \mathcal{H}(K_0, M)_\kappa$ can be isometrically embedded in a metric separable space Y such that

(1) $\mathcal{H}(K_0, M)$ is a closed subset of Y.

(2) $Y \setminus \mathcal{H}(K_0, M) = P^q$ is an infinite polyhedron of dimension less than or equal to q.

(3) The map $f \colon B \to \mathcal{H}(K_0, M)$ has a continuous extension $\overline{f} \colon Z \to Y$ such that $\overline{f}(Z \setminus B) \subset P^q$.

In view of (3) it is therefore sufficient to prove that:

(4) There is an open set V such that $\mathcal{H}(K_0, M) \subset V \subset Y$ and a continuous function $\rho \colon V \to \mathcal{H}(K_q, M)$ such that $\rho(h) = h$ for each $h \in \mathcal{H}(K_0, M)$ and $\rho(V \setminus \mathcal{H}(K_0, M)) \subset \mathcal{H}(K_q, M, \mu_o\text{-}\varepsilon\text{-reg})$.

We proceed to construct the set V and the mapping ρ in exactly the same way as in Eilenberg and Wilder [6, Theorem 1]: let P^j denote the j-dimensional skeleton of P^q, that is, the subpolyhedron of P^q consisting only of the simplices of dimension less than or equal to j. To prove (4) it is sufficient to show that for $j = 0, 1, \dots, q$ it is true that:

(5) There is an open set V_j such that $\mathcal{H}(K_0, M) \subset V_j \subset Y$ and a continuous function $\rho_j \colon \mathcal{H}(K_0, M) \cup (V_j \cap P^j) \to \mathcal{H}(K_j, M)$ such that $\rho_j(h) = h$ for each $h \in \mathcal{H}(K_0, M)$ and $\rho_j(V_j \cap P^j) \subset \mathcal{H}(K_j, M, \mu_o\text{-}\varepsilon\text{-reg})$.

Hence we may proceed by induction. For $j = 0$, P^0 is the set of all vertices of P^q. Since $\mathcal{H}(K_0, M, \mu_o\text{-}\varepsilon\text{-reg})$ is dense in $\mathcal{H}(K_0, M)$, we can find for every vertex $x_\alpha \in P^0$ a point $h_\alpha \in \mathcal{H}(K_0, M, \mu_o\text{-}\varepsilon\text{-reg})$ such that $d^*(x_\alpha, h_\alpha) < 2d^*(x_\alpha, \mathcal{H}(K_0, M))$. Defining $V_0 = Y$ and $\rho_0(h) = h$ for $h \in \mathcal{H}(K_0, M)$, $\rho_0(x_\alpha) = h_\alpha$ for $x_\alpha \in P^0$, we verify (5) for $j = 0$.

Suppose that V_j and ρ_j of (5) are given for some $j < q$. Let $\triangle_1^{j+1}, \triangle_2^{j+1}, \dots,$ be the $(j+1)$-dimensional simplices contained $V_j \cap P^q$. Denote by S_α^j the boundary of the sphere of $\triangle_\alpha^{j+1}$. The mapping $\rho_j \colon S_\alpha^j \to \mathcal{H}(K_j, M, \mu_o\text{-}\varepsilon\text{-reg})$ is therefore defined. Denote this partial mapping by $\gamma_\alpha \colon S_\alpha^j \to \mathcal{H}(K_j, M, \mu_o\text{-}\varepsilon\text{-reg})$. If $b(\gamma_\alpha) < \infty$, then we can find a continuous extension $\overline{\gamma_\alpha} \colon \triangle_\alpha^{j+1} \to \mathcal{H}(K_{j+1}, M, \mu_o\text{-}\varepsilon\text{-reg})$ of γ_α such that diameter$[\overline{\gamma_\alpha}(\triangle_\alpha^{j+1})] < 2b(\gamma_\alpha)$. Now if the subsequence $\{\triangle_{\alpha_s}^{j+1}\}$ converges to a point $h \in \mathcal{H}(K_0, M)$ then diameter$[\rho_j(S_{\alpha_s}^j)] \to 0$ and diameter$[\gamma_{\alpha_s}(S_{\alpha_s}^j)] \to 0$. By Step 1(ii), we must have $b(\gamma_{\alpha_s}) \to 0$ and therefore $\overline{\gamma_{\alpha_s}}(\triangle_{\alpha_s}^{j+1})$ converges to h.

Consequently there is an open set V_{j+1} such that $\mathcal{H}(K_0, M) \subset V_{j+1} \subset Y$ and $b(\gamma_\alpha)$ is finite whenever $\triangle_\alpha^{j+1} \cap V_{j+1} \neq \varnothing$. Taking $\rho_{j+1}(x) = \rho_j(x)$ for $x \in \mathcal{H}(K_0, M) \cup (V_{j+1} \cap P^j)$; $\rho_{j+1}(x) = \overline{\gamma_\alpha}(x)$ for $x \in V_{j+1} \cap \triangle_\alpha^{j+1}$, we verify (5) for $j+1$. This concludes the proof of Claim 1. ■

Step 3 The following is taken, with minor modification, from Eilenberg and Wilder [6, Theorem 2]:

Claim 2 *Let B be a closed subset of a compact space such that* $\dim(Z \setminus B)$ *is finite. Let $f \colon Z \to \mathcal{H}_c(M)_{\varinjlim}$ be continuous. Then there is a continuous function $f^* \colon Z \to \mathcal{H}_c(M)_{\varinjlim}$ satisfying the following properties:*

(i) *f is homotopic to f^*;*
(ii) *$f^*(Z \setminus B) \subset \mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})$;*
(iii) *$f^*(z) = f(z)$ for $z \in B$.*

Proof of Claim 2 Consider the product space $Z^* = Z \times I$ and the closed subspace $B^* = Z \times \{0\} \cup B \times I$. Clearly $Z^* \setminus B^*$ is of finite dimension. Define the mapping $g \colon B^* \to \mathcal{H}_c(M)_{\varinjlim}$ as follows: $g(z, 0) = f(z)$ for $z \in Z$, and $g(z, t) = f(z)$ for $(z, t) \in B \times I$.

By Claim 1, we can find an open set U such that $B^* \subset U \subset Z^*$ and an extension $g^* \colon U \to \mathcal{H}_c(M)_{\varinjlim}$ of g^* such that $g^*(U \setminus B^*) \subset \mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})$. For sufficiently small $t_0 \in I$ we then have that $Z \times [0, t_0] \subset U$. Taking $f^*(z) = g^*(z, t_0)$ we complete the proof of Claim 2, since f^* is the desired function. ■

It now follows from Claim 2 that the inclusion $\mathcal{H}_c(M, \mu_o\text{-}\varepsilon\text{-reg})_{\varinjlim} \hookrightarrow \mathcal{H}_c(M)_{\varinjlim}$ is a weak homotopy equivalence. This concludes the proof of Theorem 5.1. ■

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