Can. J. Math., Vol. XL, No. 6, 1988, pp. 1281–1300

RADIAL ENTIRE SOLUTIONS OF EVEN ORDER SEMILINEAR ELLIPTIC EQUATIONS

TAKAŜI KUSANO, MANABU NAITO AND CHARLES A. SWANSON

1. Introduction. Semilinear elliptic partial differential equations of the type

$$(1) \qquad \Delta^m u = f(|x|, u), \quad x \in \mathbf{R}^N, \quad N \geqq 3$$

will be considered throughout real Euclidean N-space, where $m \geqq 2$ is a positive integer, Δ denotes the N-dimensional Laplacian, and f is a real-valued continuous function in $[0, \infty) \times (0, \infty)$. Detailed hypotheses on the structure of f are listed in Section 3.

Our objective is to prove the existence of radially symmetric positive entire solutions $u(x)$ of (1) which are asymptotic to positive constant multiples of $|x|^{2m-2i}$ as $|x| \to \infty$ for every $i = 1, \dots, m, N \geqq 2i+1$. An *entire solution* of (1) is defined to be a function $u \in C^{2m}(\mathbf{R}^N)$ satisfying (1) pointwise in $\mathbf{R}^N$. Theorem 1 establishes, in particular, sufficient conditions for the existence of infinitely many positive radial entire solutions of (1) which are bounded above and below by constant multiples of $1 + |x|^{2m-2}$ in $\mathbf{R}^N, N \geqq 3$. This theorem also implies the existence of bounded positive entire solutions of (1) in $\mathbf{R}^N, N \geqq 2m+1$.

The sharpness of our existence criteria is indicated by Theorem 2: If $f(t, u)$ in (1) has constant sign in $[0, \infty) \times (0, \infty)$, these criteria are in fact necessary and sufficient conditions for positive entire solutions of (1) to exist which are asymptotic to constant multiples of $|x|^{2m-2i}, i = 1, \dots, m$, as $|x| \to \infty$.

Theorem 3 shows that equation (1) can have infinitely many radial positive entire solutions which grow more rapidly than any of the above solutions as $|x| \to \infty$.

The problem of existence of entire solutions which decay to zero as $|x| \to \infty$ has proved to be very difficult even for the second order case, i.e., $m = 1$ in (1). Indeed, only special second order results are known to date. A surprising result, in Theorem 4 below, is that there is a class of equations of the form (1) which possess positive entire solutions decaying uniformly to zero as $|x| \to \infty$. Theorem 5 contains another result of this type for a mixed sublinear-superlinear equation.

Considerable attention has been given to (1) in the case $m = 1$; recent bibliographies appear in [**2**, **3**, **5**]. Entire solutions of (1) for $m \geqq 2$ were first investigated by Walter [**8**, **9**] and Walter and Rhee [**10**]. A systematic study of the existence of entire solutions in the plane of $\Delta^m u = p(|x|)f(u)$ is contained in [**4**]. The case $m = 2$ is considered in [**6**]. As far as we are aware,

Received February 6, 1986 and in revised form April 2, 1987. The first author was supported by JSPS Bilateral Exchange Program with NSERC (Canada), 1985, while the third was supported by NSERC (Canada) under Grant A-3105.

there are no known results guaranteeing the existence of entire solutions of (1) for $N \geqq 3, m \geqq 2$, with specific information about the asymptotic behavior at infinity of these solutions.

Our method requires the existence of solutions of singular integral equations via the Schauder-Tychonov fixed point theorem. The integral equations are formed from iterates of integral operators Φ and Ψ. The estimates in Section 2 for these iterated operators are critical in the proofs of the theorems. The main results are proved in Sections 3 and 4.

2. Estimates for iterated integral operators. Let $L^1_\lambda(0,\infty), \lambda \geqq 0$, denote the set of all real-valued measurable functions g in $(0,\infty)$ such that

$$\int_0^\infty t^\lambda |g(t)| dt < \infty.$$

Let $\Phi : C[0,\infty) \longrightarrow C^2[0,\infty)$ and $\Psi : C[0,\infty) \cap L^1_1(0,\infty) \longrightarrow C^2[0,\infty)$ be the integral operators defined by

$$(2) \qquad (\Phi h)(t) = \frac{1}{N-2} \int_0^t \left[1 - \left(\frac{s}{t}\right)^{N-2}\right] sh(s) ds, \quad t \geqq 0, \quad N \geqq 3,$$

$$(3) \qquad (\Psi h)(t) = \frac{1}{N-2} \left[\int_0^t \left(\frac{s}{t}\right)^{N-2} sh(s) ds + \int_t^\infty sh(s) ds\right], \quad t \geqq 0, \quad N \geqq 3.$$

It is sometimes useful to notice that Φ and Ψ can be rewritten as

$$(\Phi h)(t) = t^{2-N} \int_0^t s^{N-3} \int_0^s rh(r) dr ds, \quad t > 0,$$
$$(\Psi h)(t) = t^{2-N} \int_0^t s^{N-3} \int_s^\infty rh(r) dr ds, \quad t > 0.$$

The operator Φ has been used by Kawano [**3**], Kusano and Oharu [**5**], and Ψ has been used by Fukagai [**2**] in existence theory of entire solutions of second order semilinear elliptic equations.

LEMMA 1. *Φ and Ψ have the following properties:*

(*A*) $(\Delta \Phi h)(|x|) = h(|x|), \quad x \in \mathbf{R}^N$, *for all* $h \in C[0,\infty)$;

(*B*) $(\Delta \Psi h)(|x|) = -h(|x|), \quad x \in \mathbf{R}^N$, *for all* $h \in C[0,\infty) \cap L^1_1(0,\infty)$;

(*C*) $\lim_{t\to\infty} (\Psi h)(t) = 0 \quad$ *if* $h \in C[0,\infty) \cap L^1_1(0,\infty)$.

These properties are easily verified from (2) and (3) and the polar form of Δ:

$$\Delta = t^{1-N} \frac{d}{dt} t^{N-1} \frac{d}{dt} = t^{-1} \frac{d}{dt} t^{3-N} \frac{d}{dt} t^{N-2}, \quad t = |x|.$$

LEMMA 2. *If $h \in C[0,\infty)$ and $h(t) \geqq 0$ for $t \geqq 0$, then*

$$(4) \qquad 0 \leqq (\Phi^i h)(t) \leqq \frac{t^{2i-2}}{(i-1)!2^{i-1}(N-2)^i} \int_0^t sh(s)ds$$

for all $t \geqq 0$ and $i = 1, 2, \ldots,$ and the limit

$$(5) \qquad \lim_{t\to\infty} \frac{(\Phi^i h)(t)}{t^{2i-2}} = A(h)$$

exists with $A(h) \in (0,\infty)$ if and only if $h \in L_1^1(0,\infty)$.

Proof. Use of (2) gives

$$0 \leqq (\Phi h)(t) \leqq \frac{1}{N-2} \int_0^t sh(s)ds, \quad t \geqq 0,$$

and hence (4) is true if $i = 1$. The truth of (4) for an integer $i (i \geqq 1)$ implies by (2) that

$$\begin{aligned} 0 \leqq (\Phi^{i+1} h)(t) &\leqq \frac{1}{N-2} \int_0^t s(\Phi^i h)(s)ds \\ &\leqq \frac{1}{(i-1)!2^{i-1}(N-2)^{i+1}} \int_0^t s^{2i-1} \int_0^s rh(r)drds \\ &= \frac{1}{(i-1)!2^{i-1}(N-2)^{i+1}} \int_0^t \left(\int_r^t s^{2i-1} ds \right) rh(r)dr \\ &\leqq \frac{t^{2i}}{2i(i-1)!2^{i-1}(N-2)^{i+1}} \int_0^t rh(r)dr, \quad t \geqq 0, \end{aligned}$$

proving (4) by induction. Clearly (5) follows from (4) and L'Hospital's rule.

LEMMA 3. *If $\lambda \geqq 0$ and $N > \lambda + 3$, then Ψ maps $C[0,\infty) \cap L_{\lambda+2}^1(0,\infty)$ into $C^2[0,\infty) \cap L_\lambda^1(0,\infty)$, and*

$$0 \leqq \int_0^\infty s^\lambda \Psi h(s)ds \leqq \frac{1}{(\lambda+1)(N-\lambda-3)} \int_0^\infty s^{\lambda+2} h(s)ds$$

for all nonnegative $h \in C[0,\infty) \cap L_{\lambda+2}^1(0,\infty)$.

Proof. Let h be any nonnegative function in $C[0,\infty) \cap L_{\lambda+2}^1(0,\infty)$. Since

$$L_{\lambda+2}^1(0,\infty) \subset L_1^1(0,\infty),$$

$\Psi h(t)$ is well defined, and for any $T > 0$ we have

$$\int_0^T s^\lambda \Psi h(s)ds = \frac{1}{N-2} \int_0^T s^\lambda \left(\int_s^\infty rh(r)dr + s^{2-N} \int_0^s r^{N-1} h(r)dr \right) ds$$

$$(6)\qquad = \frac{1}{N-2}\left\{\left[\frac{s^{\lambda+1}}{\lambda+1}\int_s^\infty rh(r)dr\right]_0^T + \frac{1}{\lambda+1}\int_0^T s^{\lambda+2}h(s)ds\right.$$

$$\left. + \left[\frac{s^{\lambda+3-N}}{\lambda+3-N}\int_0^s r^{N-1}h(r)dr\right]_0^T - \frac{1}{\lambda+3-N}\int_0^T s^{\lambda+2}h(s)ds\right\}.$$

Noting that $\lambda+3-N<0$,

$$T^{\lambda+1}\int_T^\infty rh(r)dr \leqq \int_T^\infty r^{\lambda+2}h(r)dr \longrightarrow 0 \quad \text{as} \quad T\longrightarrow\infty,$$

and

$$s^{\lambda+3-N}\int_0^s r^{N-1}h(r)dr \leqq \int_0^s r^{\lambda+2}h(r)dr \longrightarrow 0 \quad \text{as} \quad s\longrightarrow 0+,$$

we see from (6) that

$$\int_0^\infty s^\lambda \Psi h(s)ds$$

$$\leqq \frac{1}{N-2}\left\{\frac{1}{\lambda+1}\int_0^\infty s^{\lambda+2}h(s)ds - \frac{1}{\lambda+3-N}\int_0^\infty s^{\lambda+2}h(s)ds\right\}$$

$$= \frac{1}{(\lambda+1)(N-\lambda-3)}\int_0^\infty s^{\lambda+2}h(s)ds.$$

LEMMA 4. *Let j be a positive integer. If $N \geqq 2j+1$, then Ψ^j is well defined on $C[0,\infty)\cap L^1_{2j-1}(0,\infty)$.*

Proof. The proof is by induction on j. Clearly, Lemma 4 is true if $j=1$. Assume that Lemma 4 holds for some $j \geqq 1$. Let $N \geqq 2j+3$ and take an

$$h \in C[0,\infty)\cap L^1_{2j+1}(0,\infty).$$

Lemma 3 with $\lambda = 2j-1$ then implies that

$$\Psi h \in L^1_{2j-1}(0,\infty),$$

so that the operator Ψ^j can be applied to Ψh. It follows that Ψ^{j+1} is defined for all $h \in C[0,\infty)\cap L^1_{2j+1}(0,\infty)$.

LEMMA 5. *Let $N \geqq 2j+3$, where j is a positive integer. If $h \in C[0,\infty)\cap L^1_{2j+1}(0,\infty)$ and $h(t) \geqq 0$ for $t \geqq 0$, then*

$$(7)\qquad 0 \leqq (\Phi^i\Psi^j h)(t) \leqq c(i,j,N)t^{2i-2}\int_0^\infty s^{2j+1}h(s)ds$$

for all $t \geqq 0$ *and* $i = 1, 2, \ldots,$ *where*

$$(8) \qquad c(i,j,N) = \frac{1}{(i-1)!2^{i-1}(N-2)^{i-1} \cdot j!2^j(N-2)(N-4)\cdots(N-2j-2)}.$$

Furthermore, the limit

$$(9) \qquad \lim_{t\to\infty} \frac{(\Phi^i\Psi^j h)(t)}{t^{2i-2}} = B(h)$$

exists with $B(h) \in (0, \infty)$.

Proof. Note that $\Psi^j h$ is well defined by Lemma 4. We first prove (7) for $i = 1$ by induction on j. That (7) holds for $i = j = 1$ has been proved by Kusano and Swanson [**6**, Lemma 3.3]. Assume that (7) is true for $i = 1$ and some $j \geqq 1$. Using this assumption, we see that if $N \geqq 2j+5$ and h is a nonnegative function in $C[0,\infty) \cap L^1_{2j+3}(0,\infty)$, then

$$(10) \qquad \Phi\Psi^{j+1}h(t) = \Phi\Psi^j(\Psi h)(t) \leqq c(1,j,N)\int_0^\infty s^{2j+1}\Psi h(s)ds, \quad t \geqq 0.$$

From Lemma 3 with $\lambda = 2j+1$ it follows that

$$\int_0^\infty s^{2j+1}\Psi h(s)ds \leqq \frac{1}{2(j+1)(N-2j-4)}\int_0^\infty s^{2j+3}h(s)ds.$$

Combining this with (10), we obtain

$$\Phi\Psi^{j+1}h(t) \leqq \frac{c(1,j,N)}{2(j+1)(N-2j-4)}\int_0^\infty s^{2j+3}h(s)ds = c(1,j+1,N)\int_0^\infty s^{2j+3}h(s)ds, \quad t \geqq 0,$$

proving (7) for $i = 1$ and j replaced by $j+1$.

Next we show that (7) holds for any $i \geqq 2$ and $j \geqq 1$ by induction on i. Let $j \geqq 1$ be fixed. Assume the truth of (7) for some $i \geqq 1$ and j. Let $N \geqq 2j+3$ and $h \in C[0,\infty) \cap L^1_{2j+1}(0,\infty)$, $h(t) \geqq 0$ for $t \geqq 0$. We then have in view of (2)

$$\begin{aligned}\Phi^{i+1}\Psi^j h(t) &= \Phi(\Phi^i\Psi^j h)(t) \\ &\leqq \frac{1}{N-2}\int_0^t s\Phi^i\Psi^j h(s)ds \\ &\leqq \frac{1}{N-2}\int_0^t s\left(c(i,j,N)s^{2i-2}\int_0^\infty r^{2j+1}h(r)dr\right)ds \\ &= \frac{c(i,j,N)}{(N-2)2i}t^{2i}\int_0^\infty r^{2j+1}h(r)dr \\ &= c(i+1,j,N)t^{2i}\int_0^\infty r^{2j+1}h(r)dr, \quad t \geqq 0,\end{aligned}$$

https://doi.org/10.4153/CJM-1988-056-3 Published online by Cambridge University Press

implying that (7) with i replaced by $i+1$ is true. The verification of (9) is immediate. Thus the proof of Lemma 5 is complete.

LEMMA 6. *Let $N \geqq 2j+1$, where j is a positive integer. If h is a nonnegative function in $C[0,\infty)\cap L^1_{N-1}(0,\infty)$, then*

$$(11) \qquad I_1(j,N;h)\min\{1,t^{2j-N}\} \leqq \Psi^j h(t) \leqq I_2(j,N;h)\min\{1,t^{2j-N}\}$$

for $t \geqq 0$, where

$$(12) \qquad I_1(j,N;h) = \frac{1}{(N-2)^j(N-4)\cdots(N-2j)}\int_0^\infty \min\{s,s^{N-1}\}h(s)ds,$$

$$(13) \qquad I_2(j,N;h) = \frac{1}{2^{j-1}(N-2)(N-4)\cdots(N-2j)}\int_0^\infty \max\{s,s^{N-1}\}h(s)ds.$$

Proof. The proof is by induction on j. Let $j=1$. Then, since

$$\frac{d}{dt}\Psi h(t) = -t^{1-N}\int_0^t s^{N-1}h(s)ds \leqq 0, \quad t>0,$$

and

$$\frac{d}{dt}[t^{N-2}\Psi h(t)] = t^{N-3}\int_t^\infty sh(s)ds \geqq 0, \quad t>0,$$

we have in particular

$$\Psi h(1) \leqq \Psi h(t) \leqq \Psi h(0) \quad \text{for } 0 \leqq t \leqq 1$$

and

$$\Psi h(1) \leqq t^{N-2}\Psi h(t) \leqq \lim_{t\to\infty} t^{N-2}\Psi h(t) \quad \text{for } t \geqq 1.$$

It is easy to see that

$$\Psi h(1) = \frac{1}{N-2}\int_0^1 s^{N-1}h(s)ds + \frac{1}{N-2}\int_1^\infty sh(s)ds$$
$$= \frac{1}{N-2}\int_0^\infty \min\{s,s^{N-1}\}h(s)ds,$$
$$\Psi h(0) = \frac{1}{N-2}\int_0^\infty sh(s)ds,$$

and

$$\lim_{t\to\infty} t^{N-2}\Psi h(t)$$
$$= \lim_{t\to\infty}\left(\frac{1}{N-2}\int_0^t s^{N-1}h(s)ds + \frac{t^{N-2}}{N-2}\int_t^\infty sh(s)ds\right)$$
$$\leqq \frac{1}{N-2}\int_0^\infty s^{N-1}h(s)ds.$$

That (11) holds for $j = 1$ follows from the above observations.

We now assume that (11) is true for some $j \geqq 1$. Suppose that $N \geqq 2j+3$ and let h be a nonnegative function in $C[0,\infty)\cap L^1_{N-1}(0,\infty)$. Then $\Psi^{j+1}h(t)$ is estimated from above and below as follows. Since $\Psi^{j+1}h(t)$ is nonincreasing, we have

$$\Psi^{j+1}h(1) \leqq \Psi^{j+1}h(t) \leqq \Psi^{j+1}h(0) \quad \text{for } 0 \leqq t \leqq 1,$$

where

$$\begin{aligned}\Psi^{j+1}h(0) &= \frac{1}{N-2}\int_0^\infty s\Psi^j h(s)ds\\ &\leqq \frac{1}{N-2}\int_0^1 s[I_2L(j,N;h)]ds\\ &+\frac{1}{N-2}\int_1^\infty s[I_2(j,N;h)s^{2j-N}]ds\\ &=\frac{1}{N-2}\left(\frac{1}{2}+\frac{1}{N-2j-2}\right)I_2(j,N;h)\\ &\leqq \frac{I_2(j,N;h)}{2(N-2j-2)} = I_2(j+1,N;h),\end{aligned}$$

and

$$\begin{aligned}\Psi^{j+1}h(1) &= \frac{1}{N-2}\int_0^1 s^{N-1}\Psi^j h(s)ds + \frac{1}{N-2}\int_1^\infty s\Psi^j h(s)ds\\ &\geqq \frac{1}{N-2}\int_1^\infty s[I_1(j,N;h)s^{2j-N}]ds\\ &= \frac{I_1(j,N;h)}{(N-2)(N-2j-2)} = I_1(j+1,N;h).\end{aligned}$$

On the other hand, if $t \geqq 1$, then we find

$$\begin{aligned}t^{N-2j-2}\Psi^{j+1}h(t) &\geqq \frac{t^{N-2j-2}}{N-2}\int_t^\infty s\Psi^j h(s)ds\\ &\geqq \frac{t^{N-2j-2}}{N-2}\int_t^\infty s[I_1(j,N;h)s^{2j-N}]ds\\ &= \frac{I_1(j,N;h)}{(N-2)(N-2j-2)} = I_1(j+1,N;h)\end{aligned}$$

and

$$
\begin{aligned}
&t^{N-2j-2}\Psi^{j+1}h(t)\\
&\quad = \frac{t^{-2j}}{N-2}\int_0^t s^{N-1}\Psi^j h(s)ds + \frac{t^{N-2j-2}}{N-2}\int_t^\infty s\Psi^j h(s)ds\\
&\quad \leqq \frac{t^{-2j}}{N-2}\left(\int_0^1 s^{N-1}[I_2(j,N;h)]ds\right.\\
&\quad \left. + \int_1^t s^{N-1}[I_2(j,N;h)s^{2j-N}]ds\right)\\
&\quad + \frac{t^{N-2j-2}}{N-2}\int_t^\infty s[I_2(j,N;h)s^{2j-N}]ds\\
&\quad = \frac{t^{-2j}}{N-2}\left(\left(\frac{1}{N}-\frac{1}{2j}\right)I_2(j,N;h) + \frac{t^{2j}}{2j}I_2(j,N;h)\right)\\
&\quad + \frac{I_2(j,N;h)}{(N-2)(N-2j-2)}\\
&\quad \leqq \frac{1}{N-2}\left(\frac{1}{2j}+\frac{1}{N-2j-2}\right)I_2(j,N;h)\\
&\quad = \frac{I_2(j,N;h)}{2j(N-2j-2)} \leqq I_2(j+1,N;h).
\end{aligned}
$$

This proves the truth of (11) with j replaced by $j+1$, and the proof is complete.

3. Existence of positive entire solutions. Existence theorems for equation (1) will be obtained under the standing hypothesis that f is continuous in $Q = [0,\infty)\times(0,\infty)$, denoted by $f \in C(Q)$, and under one of the following structure hypotheses:

(L_∞) $|f(t,u)| \leqq F(t,u)$ in Q, where $F \in C(Q)$ and $u^{-1}F(t,u)$ is nonincreasing in $u \in (0,\infty)$ for each $t \geqq 0$ and satisfies

$$\lim_{u\to\infty} u^{-1}F(t,u) = 0, \quad t \geqq 0;$$

(L_0) $|f(t,u)| \leqq F(t,u)$ in Q, where $F \in C(Q)$ and $u^{-1}F(t,u)$ is nondecreasing in $u \in (0,\infty)$ for each $t \geqq 0$ and satisfies

$$\lim_{u\to 0+} u^{-1}F(t,u) = 0, \quad t \geqq 0.$$

Note that a function $F(t,u)$ in (L_0) is nondecreasing in $u \in (0,\infty)$ for each $t \geqq 0$, and that a function $F(t,u)$ which is nonincreasing in $u \in (0,\infty)$ for each $t \geqq 0$ has the property of $F(t,u)$ stated in (L_∞).

Our first main theorem is stated below.

https://doi.org/10.4153/CJM-1988-056-3 Published online by Cambridge University Press

THEOREM 1. *Suppose that $f \in C(Q)$ satisfies either* (L_∞) *or* (L_0). *Let i be an integer with $1 \leqq i \leqq m$. Suppose that $N \geqq 2i+1$ and that there exists a constant $c > 0$ such that*

$$\int_0^\infty t^{2i-1} F(t, c(1+t^{2m-2i}))dt < \infty. \tag{14}$$

Then equation (1) *has infinitely many radial positive entire solutions which are bounded above and below by positive constant multiples of $1+|x|^{2m-2i}$ in $\mathbf{R}^N$.*

Proof. We first consider the case that (L_∞) is satisfied. Let $k \geqq 2c$. Then

$$c(1+t^{2m-2i}) < k + \frac{k}{2}t^{2m-2i}, \quad t \geqq 0,$$

and so (L_∞) implies that

$$\frac{F(t, k+\frac{k}{2}t^{2m-2i})}{k+\frac{k}{2}t^{2m-2i}} \leqq \frac{F(t, c(1+t^{2m-2i}))}{c(1+t^{2m-2i})}, \quad t \geqq 0,$$

whence it follows that

$$k^{-1}t^{2i-1}F\left(t, k+\frac{k}{2}t^{2m-2i}\right) \leqq c^{-1}t^{2i-1}F(t, c(1+t^{2m-2i})), \quad t \geqq 0.$$

Since

$$\lim_{k\to\infty} k^{-1}t^{2i-1}F\left(t, k+\frac{k}{2}t^{2m-2i}\right) = 0$$

for each $t \geqq 0$ by (L_∞), condition (14) and the Lebesgue dominated convergence theorem show that

$$\lim_{k\to\infty}\int_0^\infty k^{-1}t^{2i-1}F\left(t, k+\frac{k}{2}t^{2m-2i}\right)dt = 0.$$

Therefore there is a $k_0 > 0$ such that

$$\int_0^\infty t^{2i-1}F\left(t, k+\frac{k}{2}t^{2m-2i}\right)dt \leqq \frac{k}{8c(m-i+1, i-1, N)} \tag{15}$$

for $k \geqq k_0$, where $c(m-i+1, i-1, N)$ is defined by (8). For any such k, we define

$$Y = \left\{y \in C[0,\infty) : k+\frac{k}{2}t^{2m-2i} \leqq y(t) \leqq k+2kt^{2m-2i}, t \geqq 0\right\} \tag{16}$$

https://doi.org/10.4153/CJM-1988-056-3 Published online by Cambridge University Press

and

(17) $$\mathrm{M}y(t) = k(1 + t^{2m-2i}) + (-1)^{i-1}\Phi^{m-i+1}\Psi^{i-1}f(t, y(t)), \quad t \geqq 0.$$

Clearly, Y is a closed convex subset of the Fréchet space $C[0, \infty)$ of all continuous functions in $[0, \infty)$ with the usual metric topology. If $y \in Y$, then by (L_∞)

(18) $$\begin{aligned} |f(t, y(t))| &\leqq \frac{F(t, y(t))}{y(t)} y(t) \\ &\leqq \frac{F\left(t, k + \frac{k}{2}t^{2m-2i}\right)}{k + \frac{k}{2}t^{2m-2i}}(k + 2kt^{2m-2i}) \\ &\leqq 4F\left(t, k + \frac{k}{2}t^{2m-2i}\right), \quad t \geqq 0, \end{aligned}$$

so that $f(t, y(t)) \in L^1_{2i-1}(0, \infty)$ by (14) and M is defined on Y by Lemma 4. From Lemma 2 (in case $i = 1$) or Lemma 5 (in case $2 \leqq i \leqq m$) it follows in view of (15) and (18) that if $y \in Y$, then

$$\begin{aligned} &|\Phi^{m-i+1}\Psi^{i-1}f(t, y(t))| \\ &\leqq \Phi^{m-i+1}\Psi^{i-1}\left[4F\left(t, k + \frac{k}{2}t^{2m-2i}\right)\right] \\ &\leqq 4c(m - i + 1, i - 1, N)t^{2m-2i}\int_0^\infty s^{2i-1}F\left(s, k + \frac{k}{2}s^{2m-2i}\right) ds \\ &\leqq \frac{k}{2}t^{2m-2i}, \quad t \geqq 0, \end{aligned}$$

and hence

$$\begin{aligned} \mathrm{M}y(t) &\leqq k(1 + t^{2m-2i}) + \frac{k}{2}t^{2m-2i} \\ &\leqq k + 2kt^{2m-2i}, \quad t \geqq 0, \\ \mathrm{M}y(t) &\geqq k(1 + t^{2m-2i}) - \frac{k}{2}t^{2m-2i} \\ &= k + \frac{k}{2}t^{2m-2i}, \quad t \geqq 0. \end{aligned}$$

Thus M maps Y into itself. Furthermore, it can be shown without difficulty that M is a continuous operator and that $\mathrm{M}(Y)$ is relatively compact in the topology of $C[0, \infty)$. The well- known Schauder-Tychonov fixed point theorem then implies that there exists a function $y(t)$ in Y such that $y(t) = \mathrm{M}y(t), t \geqq 0$. The function $u(x) = y(|x|), x \in \mathbf{R}^N$, gives a radial entire solution of (1), since repeated application of Lemma 1 shows that

$$\Delta^m((-1)^{i-1}\Phi^{m-i+1}\Psi^{i-1}h)(|x|) = h(|x|), \quad x \in \mathbf{R}^N,$$

for $h \in C[0,\infty) \cap L^1_{2i-1}(0,\infty)$ and $N \geqq 2i+1, i = 1,2,\dots,m$. Since $k \geqq k_0$ is arbitrary, there is an infinitude of such entire solutions of (1).

Next, suppose that (L_0) is satisfied. In this case one can easily show that

$$\lim_{k \to 0+} \int_0^\infty k^{-1} t^{2i-1} F(t, k + 2kt^{2m-2i}) dt = 0.$$

Choose $k_1 > 0$ so that

$$\int_0^\infty t^{2i-1} F(t, k + 2kt^{2m-2i}) dt \leqq \frac{k}{2c(m-i+1, i-1, N)}$$

for $0 < k \leqq k_1$, and define for such a k, Y and M by (16) and (17), respectively. Then, as in the case of (L_∞), M is shown to map Y continuously into a compact subset of Y, so that M has a fixed point $y \in Y$, which gives rise to a radial entire solution $u(x) = y(|x|)$ of (1) with the desired asymptotic behavior as $|x| \to \infty$. Since k is arbitrary in $(0, k_1]$, (1) has infinitely many such entire solutions. This completes the proof.

Remark 1. In Theorem 1 hypotheses (L_∞) and (L_0) can be replaced by the following more general hypotheses:

$(\tilde{\mathrm{L}}_\infty)$ $f(t,u) = o(u)$ as $u \to \infty$ for each fixed $t \geqq 0$;
$(\tilde{\mathrm{L}}_0)$ $f(t,u) = o(u)$ as $u \to 0+$ for each fixed $t \geqq 0$.

Then the roles of F in (L_∞) and (L_0) are played by the functions $\tilde{F}$ defined by

$$\tilde{F}(t,u) = u \sup\{v^{-1}|f(t,v)| : u \leqq v < \infty\}, \quad (t,u) \in Q$$

in the case of $(\tilde{\mathrm{L}}_\infty)$ and

$$\tilde{F}(t,u) = u \sup\{v^{-1}|f(t,v)| : 0 < v \leqq u\}, \quad (t,u) \in Q$$

in the case of $(\tilde{\mathrm{L}}_0)$. Although, in each case, $\tilde{F}(t,u)$ is not necessarily continuous in (t,u), it is not difficult to verify that the conclusion (as well as the proof) of Theorem 1 remains valid provided $\tilde{F}$ is required to satisfy condition (14).

Similar remarks also apply to the subsequent theorems and corollaries. Note that the actual computation of $\tilde{F}$ is not easy.

THEOREM 2. *Suppose that* $f \in C(Q)$ *is of constant sign in* Q *and satisfies either* (L_∞) *or* (L_0), *where* $F(t,u)$ *is taken to be* $|f(t,u)|$. *Let* i *be an integer with* $1 \leqq i \leqq m$ *and* $N \geqq 2i+1$. *Then* (14) *is a necessary and sufficient condition for equation* (1) *to have a radial positive entire solution* $u(x)$ *in* $\mathbf{R}^N$ *such that the limit*

$$\text{(19)} \qquad \lim_{|x| \to \infty} \frac{u(x)}{|x|^{2m-2i}} = A(u)$$

https://doi.org/10.4153/CJM-1988-056-3 Published online by Cambridge University Press

exists with $A(u) \in (0, \infty)$.

Proof. The sufficiency part of this theorem follows from Theorem 1 combined with the observation that when $f(t, u)$ is of constant sign, the term

$$\Phi^{m-i+1}\Psi^{i-1}f(t, y(t))$$

has that sign and has a finite (nonzero) limit

$$\lim_{t\to\infty} \frac{\Phi^{m-i+1}\Psi^{i-1}f(t, y(t))}{t^{2m-2i}}$$

in view of the second part of Lemma 2 or Lemma 5.

To prove the necessity part it suffices to apply a result of Fink and Kusano [**1**], noting that each t^{2m-2i}, $1 \leqq i \leqq m$, is a solution of the unperturbed differential equation $L^m y = 0$ in $(0, \infty)$, where

$$L = t^{1-N}\frac{d}{dt}t^{N-1}\frac{d}{dt}.$$

We denote by S the set of all radial positive entire solutions of equation (1). For $j \in \{0, 1, \dots, m-1\}$ let S_j denote the set of all $u \in S$ which are bounded above and below by positive constant multiples of $1 + |x|^{2j}$ in $\mathbf{R}^N$. If f is one-signed in $[0, \infty) \times (0, \infty)$, then Theorem 2 shows that S_j is the set of all $u \in S$ such that the positive limit

$$\lim_{|x|\to\infty} u(x)/|x|^{2j}$$

exists and is finite.

Suppose that either (L_∞) or (L_0) is satisfied. If (L_∞) holds, then for $c > 0$ and for t sufficiently large

$$\begin{aligned} F(t, c(1 + t^{2m-2i})) &\leqq \frac{c(1 + t^{2m-2i})}{c(1 + t^{2m-2i-2})}F(t, c(1 + t^{2m-2i-2})) \\ &\leqq 2t^2 F(t, c(1 + t^{2m-2i-2})), \end{aligned}$$

and hence

$$\int_0^\infty t^{2i+1}F(t, c(1 + t^{2m-2i-2}))dt < \infty \tag{20}$$

implies

$$\int_0^\infty t^{2i-1}F(t, c(1 + t^{2m-2i}))dt < \infty. \tag{21}$$

From Theorem 1 it then follows that our condition ensuring $S_j \neq \emptyset$ implies $S_{j+1} \neq \emptyset$; in particular, condition (14) (with $i = m$) implies that $S_j \neq \emptyset$ for

$j = 0, 1, \ldots, m-1$. On the other hand, if (L_0) holds, then we see that (21) implies (20), since

$$F(t, c(1+t^{2m-2i})) \geqq \frac{c(1+t^{2m-2i})}{c(1+t^{2m-2i-2})} F(t, c(1+t^{2m-2i-2}))$$
$$\geqq \frac{1}{2} t^2 F(t, c(1+t^{2m-2i-2}))$$

for $c > 0$ provided t is sufficiently large. In this case, by Theorem 1 again, we conclude that our condition ensuring $S_j \neq \emptyset$ implies $S_{j-1} \neq \emptyset$; in particular, (14) (with $i = 1$) implies $S_j \neq \emptyset$ for $j = 0, 1, \ldots, m-1$. If in addition f is one-signed, then the above observation combined with Theorem 2 shows that if (L_∞) (with $F = |f|$) holds, then $S_{m-1} = \emptyset$ implies $S_j = \emptyset$ for $j = 0, 1, \ldots, m-2$, and if (L_0) (with $F = |f|$) holds, $S_0 = \emptyset$ implies $S_j = \emptyset$ for $j = 1, 2, \ldots, m-1$. The foregoing results are summarized in the following corollaries.

COROLLARY 1. *Suppose that $f \in C(Q)$ satisfies* (L_∞). *Let $N \geqq 2m+1$.*

(i) *If there exists a constant $c > 0$ such that*

$$\int_0^\infty t^{2m-1} F(t, c)dt < \infty,$$

then $S_j \neq \emptyset$ for $j = 0, 1, \ldots, m-1$.

(ii) *Suppose in addition that f is one-signed. If for every constant $c > 0$*

$$\int_0^\infty t|f(t, c(1+t^{2m-2}))|dt = \infty,$$

then $S_j = \phi$ for $j = 0, 1, \ldots, m-1$.

COROLLARY 2. *Suppose that $f \in C(Q)$ satisfies (L_0). Let $N \geqq 2m+1$.*

(i) *If there exists a constant $c > 0$ such that*

$$\int_0^\infty tF(t, c(1+t^{2m-2}))dt < \infty,$$

then $S_j \neq \phi$ for $j = 0, 1, \ldots, m-1$.

(ii) *Suppose in addition that f is one-signed. If for every constant $c > 0$.*

$$\int_0^\infty t^{2m-1}|f(t, c)|dt = \infty,$$

then $S_j = \emptyset$ for $j = 0, 1, \ldots, m-1$.

It is possible that equation (1) possesses entire solutions which do not belong to $\bigcup_{j=0}^{m-1} S_j$ as the following example shows.

Example 1. Consider the generalized Emden-Fowler equation

$$\Delta^m u = p(|x|)u^\gamma, \quad x \in \mathbf{R}^N, \tag{22}$$

where γ is a constant and $p : [0, \infty) \longrightarrow \mathbf{R}$ is continuous. Let $\phi(t)$ be positive and smooth on $[0, \infty)$, but otherwise arbitrary. Then, $u(x) = \phi(|x|^2)$ is an entire solution of (22) if $p(t)$ is given by

$$p(t) = [\phi(t^2)]^{-\gamma} \left(t^{1-N} \frac{d}{dt} t^{N-1} \frac{d}{dt} \right)^m \phi(t^2).$$

This solution belongs to none of $S_j, j = 0, 1, \ldots, m-1$, if the asymptotic behavior of $\phi(t^2)$ as $t \longrightarrow \infty$ is different from any constant multiple of $t^{2j}, j = 0, 1, \ldots, m-1$.

It will then be natural to ask if one can actually detect or construct entire solutions, not belonging to $\bigcup_{j=0}^{m-1} S_j$, of equation (1) for general f, or equation (22) for a given p, in particular. For instance, a question will arise as to how to find conditions on f (resp. p) which guarantee the existence of a positive entire solution of equation (1) (resp. (22)) having one of the properties:

$$\text{(I)} \qquad \lim_{|x|\to\infty} u(x)/|x|^{2j} = 0 \quad \text{and}$$

$$\lim_{|x|\to\infty} u(x)/|x|^{2j-2} = \infty \quad \text{for some } j = 1, 2, \ldots, m-1;$$

$$\text{(II)} \qquad \lim_{|x|\to\infty} u(x)/|x|^{2m-2} = \infty, \quad \text{or } |x| \longrightarrow \infty,$$

$$\text{(III)} \qquad \lim_{|x|\to\infty} u(x) = 0.$$

A partial answer to this question will be given in Theorem 3 below and in Section 4.

THEOREM 3. *Suppose that $f(t, u)$ is nonnegative and continuous in $[0, \infty) \times (0, \infty)$ and is nonincreasing in u for each fixed $t \geqq 0$. If*

$$\int_0^\infty tf(t, c(1 + t^{2m-2}))dt = \infty \tag{23}$$

for every constant $c > 0$, then equation (1), $N \geqq 3$, possesses infinitely many radial positive entire solutions $u(x)$ such that

$$\lim_{|x|\to\infty} \frac{u(x)}{|x|^{2m-2}} = \infty. \tag{24}$$

Proof. Let $c > 0$ be any fixed number, and consider the set $Y \subset C[0, \infty)$ and the mapping

$$\mathrm{F} : Y \longrightarrow C^{2m}[0, \infty)$$

defined by

$$Y = \{y \in C[0,\infty) : c(1+t^{2m-2}) \leqq y(t) \\ \leqq c(1+t^{2m-2}) + \Phi^m f(t, c(1+t^{2m-2})), t \geqq 0\}$$

and

$$\mathrm{F}y(t) = c(1+t^{2m-2}) + \Phi^m f(t, y(t)), \quad t \geqq 0.$$

If $y \in Y$, then, by the nonincreasing nature of f with respect to u,

$$0 \leqq \Phi^m f(t, y(t)) \leqq \Phi^m f(t, c(1+t^{2m-2})), \quad t \geqq 0,$$

which ensures that F maps Y into itself. The continuity of F and the relative compactness of F(Y) in the topology of $C[0,\infty)$ can be proved without difficulty. Therefore, there exists a fixed point $y \in Y$ of F, which gives an entire solution $u(x) = y(|x|)$ of (1).

It remains to study the asymptotic behavior of $u(x)$ as $|x| \to \infty$. As is easily verified,

$$\text{(25)} \quad \lim_{t\to\infty} \frac{y(t)}{t^{2m-2}} = c + \lim_{t\to\infty} \frac{\Phi^m f(t, y(t))}{t^{2m-2}} \\ = c + \frac{\lim_{t\to\infty} \Phi f(t, y(t))}{2^{m-1}(m-1)!N(N+2)\cdots(N+2m-4)},$$

which is finite or infinite according as $f(t, y(t)) \in L_1^1(0,\infty)$ or $f(t, y(t)) \notin L_1^1(0,\infty)$. Suppose that the above limit is finite. Then,

$$f(t, y(t)) \in L_1^1(0,\infty),$$

i.e.,

$$\text{(26)} \quad \int_0^\infty tf(t, y(t))dt < \infty$$

and there exists a constant $k > 0$ such that

$$y(t) \leqq k(1+t^{2m-2}) \quad \text{for } t \geqq 0.$$

Combining the last inequality with (26) and noting that f is nonincreasing in u, we obtain

$$\int_0^\infty tf(t, k(1+t^{2m-2}))dt < \infty,$$

which contradicts (23). It follows therefore that the limit (25) must be infinite, that is, the obtained entire solution $u(x) = y(|x|)$ necessarily has the asymptotic behavior (24). Since $c > 0$ is arbitrary, there is an infinitude of such growing entire solutions $u(x)$. This completes the proof.

Remark 2. A nonnegative continuous function $f(t, u)$ in $[0, \infty) \times (0, \infty)$ which is nonincreasing in u clearly satisfies hypothesis (L_∞), and hence (ii) of Corollary 1 applies to equation (1) with this nonlinearity. In view of Theorem 3 and (ii) of Corollary 1 we conjecture that under condition (23) all radial entire solutions of such an equation (1) have the same asymptotic behavior (24) as $|x| \to \infty$.

Example 2. Consider equation (22) as in Example 1. In this case $f(t, u) = p(t)u^\gamma$ and the function $F(t, u)$ can be taken to be

$$F(t, u) = |p(t)|u^\gamma;$$

(L_∞) or (L_0) is satisfied according as $\gamma < 1$ or $\gamma > 1$. Condition (14) reduces to

$$\int^\infty t^{2i-1+2\gamma(m-i)}|p(t)|dt < \infty,$$

which, in case $\gamma \neq 1$, guarantees the existence of infinitely many members of S_{m-i} (see Theorem 1). Corollaries 1 and 2 show that $S_j \neq \emptyset, j = 0, 1, \ldots, m-1$, for (22) if

$$\int^\infty t^{2m-1}|p(t)|dt < \infty, \quad \gamma < 1, \quad N \geqq 2m+1,$$

or if

$$\int^\infty t^{1+2\gamma(m-1)}|p(t)|dt < \infty, \quad \gamma > 1, \quad N \geqq 2m+1,$$

and that $S_j = \emptyset, j = 0, 1, \ldots, m-1$, for (22) with one-signed $p(t)$ if

$$\int^\infty t^{1+2\gamma(m-1)}|p(t)|dt = \infty, \quad \gamma < 1, \quad N \geqq 2m+1,$$

or if

$$\int^\infty t^{2m-1}|p(t)|dt = \infty, \quad \gamma > 1, \quad N \geqq 2m+1.$$

Theorem 3 implies that equation (22) with $\gamma \leqq 0$ and $p(t) \geqq 0$ satisfying

$$\int^\infty t^{1+2\gamma(m-1)}p(t)dt = \infty, \quad N \geqq 3,$$

has radial positive entire solutions which grow more rapidly than $|x|^{2m-2}$ as $|x| \longrightarrow \infty$.

4. Existence of decaying positive entire solutions. Additional examples of entire solutions of (1) not belonging to $\bigcup_{j=0}^{m-1} S_j$ are radial positive entire solutions which decay to zero as $|x| \longrightarrow \infty$. The problem of existence of decaying entire solutions is difficult, and not completely resolved to date, even for the second order case of (1), i.e., for $m = 1$, and accordingly the same generality as in Section 3 cannot be expected in this direction for higher order elliptic equations. Below it is shown that there are two classes of equations of the form (1) which possess positive decaying entire solutions.

THEOREM 4. *Suppose that* $N \geqq 2m+1, -1 < \gamma < 1$, *and* $p : [0, \infty) \longrightarrow (0, \infty)$ *is continuous. Then the equation*

$$\Delta^m u = (-1)^m p(|x|)u^\gamma, \quad x \in \mathbf{R}^N, \tag{27}$$

has a radial decaying positive entire solution $u(x)$ *such that the limit*

$$\lim_{|x| \to \infty} |x|^{N-2m} u(x) = A(u) \in (0, \infty) \tag{28}$$

exists if and only if

$$\int^\infty t^{N-1-\gamma(N-2m)} p(t) dt < \infty. \tag{29}$$

Proof. Suppose first that $0 \leqq \gamma < 1$. Let

$$\rho(t) = \min\{1, t^{2m-N}\},$$

and define

$$Y = \{y \in C[0, \infty) : k_1\rho(t) \leqq y(t) \leqq k_2\rho(t), \quad t \geqq 0\},$$

where k_1 and k_2 are positive constants satisfying the inequalities

$$k_1 \leqq [I_1(m, N; p\rho^\gamma)]^{1/(1-\gamma)} \leqq [I_2(m, N; p\rho^\gamma)]^{1/(1-\gamma)} \leqq k_2,$$

and $I_i(m, N; p\rho^\gamma), i = 1, 2$, denote the functionals in (12) and (13) with $j = m$. Condition (29) ensures that $I_i(m, N; p\rho^\gamma), i = 1, 2$, are finite. Let

$$\mathrm{F} : Y \longrightarrow C^{2m}[0, \infty)$$

be the mapping defined by

$$\mathrm{F}y(t) = \Psi^m(py^\gamma)(t), \quad t \geqq 0.$$

If $y \in Y$, then

$$t^{N-1}p(t)y^{\gamma}(t) \leqq k_2^{\gamma}t^{N-1-\gamma(N-2m)}p(t),$$

so that $py^{\gamma} \in L^1_{N-1}(0,\infty)$ and Lemma 6 ($j = m$) is applicable to $h = py^{\gamma}$. It then follows that for $y \in Y$

$$I_1(m,N;py^{\gamma})\rho(t) \leqq \mathrm{F}y(t) \leqq I_2(m,N;py^{\gamma})\rho(t), \quad t \geqq 0,$$

and hence

$$k_1^{\gamma}I_1(m,N;p\rho^{r})\rho(t) \leqq \mathrm{F}y(t) \leqq k_2^{\gamma}I_2(m,N;p\rho^{\gamma})\rho(t), \quad t \geqq 0.$$

Consequently, in view of the choice of k_1, k_2,

$$\mathrm{F}y(t) \leqq k_2^{\gamma}k_2^{1-\gamma}\rho(t) = k_2\rho(t)$$

and

$$\mathrm{F}y(t) \geqq k_1^{\gamma}k_1^{1-\gamma}\rho(t) = k_1\rho(t),$$

from which $\mathrm{F}y \in Y$. Thus, F maps Y into itself, and it can be verified that F is continuous and $\mathrm{F}(Y)$ is relatively compact. By the Schauder-Tychonov fixed point theorem, there is a function $y \in Y$ such that $y(t) = \mathrm{F}y(t), t > 0$. The function $u(x) = y(|x|)$ is an entire solution of (27). It is easy to show that $u(x)$ satisfies (28).

The proof in the singular case $-1 < \gamma < 0$ is virtually the same, except that the constants k_1 and k_2 in the above definition of Y are replaced by

$$k_1 = [I_1(m,N;p\rho^{\gamma})I_2^{\gamma}(m,N;p\rho^{\gamma})]^{1/(1-\gamma^2)}$$

and

$$k_2 = [I_1^{\gamma}(m,N;p\rho^{\gamma})I_2(m,N;p\rho^{\gamma})]^{1/(1-\gamma^2)},$$

respectively. Again the desired decaying entire solution is obtained as a fixed point of F in Y. Thus the proof of the "if" part of the theorem is complete.

The "only if" part can be proved with the aid of a theory of Fink and Kusano [**1**] on the asymptotic behavior of perturbed general disconjugate ordinary differential equations.

We have been unable to solve the problem of existence of decaying entire solutions for the superlinear case of (27), i.e., $\gamma > 1$. However, a mixed sublinear-superlinear equation of the form

$$(30) \qquad \Delta^m u = (-1)^m[p(|x|)u^{\gamma} + q(|x|)u^{\delta}], \quad x \in \mathbf{R}^N,$$

where $0 < \gamma < 1$ and $\delta > 1$, may have a decaying positive entire solution, as the following theorem shows.

THEOREM 5. *Suppose that* $N \geqq 2m+1, 0 < \gamma < 1$ *and* $\delta > 1$, *and that* $p, q : [0, \infty) \longrightarrow (0, \infty)$ *are continuous and satisfy*

$$(31) \qquad \int^{\infty} t^{N-1-\gamma(N-2m)} p(t) dt < \infty,$$

$$(32) \qquad \int^{\infty} t^{N-1-\delta(N-2m)} q(t) dt < \infty.$$

Let $\rho(t) = \min\{1, t^{2m-N}\}$ *and denote by* $I_1(h)$ *and* $I_2(h)$, *respectively, the functionals* $I_1(m, N; h)$ *and* $I_2(m, N; h)$ *defined by* (12) *and* (13). *Suppose that*

$$(33) \qquad [I_2(p\rho^{\gamma})]^{(\delta-1)/(\delta-\gamma)} [I_2(q\rho^{\delta})]^{(1-\gamma)/(\delta-\gamma)} \left[\left(\frac{\delta-1}{1-\gamma} \right)^{(1-\gamma)/(\delta-\gamma)} + \left(\frac{1-\gamma}{\delta-1} \right)^{(\delta-1)/(\delta-\gamma)} \right] \leqq 1.$$

Then, equation (30) *has a radial positive entire solution* $u(x)$ *satisfying* (28).

Proof. We adapt the method used by Kusano and Trench [**7**] for the second order case of (30). Consider the mapping

$$\mathrm{F}y(t) = \Psi^m(py^{\gamma} + qy^{\delta}), \quad t \geqq 0,$$

on the set

$$Y = \{y \in C[0, \infty) : k_1\rho(t) \leqq y(t) \leqq k_2\rho(t), \quad t \geqq 0\},$$

where k_1 and k_2 are positive constants. Because of (31) and (32), F is well-defined on Y for any positive k_1 and k_2. It suffices to show that under (33) k_1 and k_2 can be chosen in such a way that F maps Y into itself, since the continuity of F and the relative compactness of F(Y) can be proved easily by standard arguments. If $y \in Y$, Lemma 6 shows that

$$\begin{aligned} \mathrm{F}y(t) &\leqq k_2^{\gamma}\Psi^m(p\rho^{\gamma}) + k_2^{\delta}\Psi^m(q\rho^{\delta}) \\ &\leqq [k_2^{\gamma}I_2(p\rho^{\gamma}) + k_2^{\delta}I_2(q\rho^{\delta})]\rho(t), \quad t \geqq 0, \end{aligned}$$

and

$$\begin{aligned} \mathrm{F}y(t) &\geqq k_1^{\gamma}\Psi^m(p\rho^{\gamma}) + k_1^{\delta}\Psi^m(q\rho^{\delta}) \\ &\geqq [k_1^{\gamma}I_1(p\rho^{\gamma}) + k_1^{\delta}I_1(q\rho^{\delta})]\rho(t), \quad t \geqq 0. \end{aligned}$$

It is then sufficient to show that $k_1 > 0$ and $k_2 > 0$ can be chosen so that

$$k_2^{\gamma-1} I_2(p\rho^\gamma) + k_2^{\delta-1} I_2(q\rho^\delta) \leqq 1 \tag{34}$$

and

$$k_1^{\gamma-1} I_1(p\rho^\gamma) + k_1^{\delta-1} I_1(q\rho^\delta) \geqq 1. \tag{35}$$

It is a matter of elementary computation to see that the left-hand side of (34) considered as a function of $k_2 > 0$ attains its least value

$$\begin{aligned}&\left[I_2(p\rho^\gamma)\right]^{(\delta-1)/(\delta-\gamma)} \left[I_2(q\rho^\delta)\right]^{(1-\gamma)/(\delta-\gamma)} \\ &\quad \times \left[\left(\frac{\delta-1}{1-\gamma}\right)^{(1-\gamma)/(\delta-\gamma)} + \left(\frac{1-\gamma}{\delta-1}\right)^{(\delta-1)/(\delta-\gamma)}\right]\end{aligned}$$

at

$$k_2 = [(1-\gamma)I_2(p\rho^\gamma)/(\delta-1)I_2(q\rho^\delta)]^{1/(\delta-\gamma)}.$$

Consequently, the existence of a $k_2 > 0$ for which (34) holds is assured if (33) is satisfied. Now, since $\gamma - 1 < 0$ and $I_1(p\rho^\gamma) > 0$, we can choose $k_1 > 0$ so small that $k_1 < k_2$ and (35) holds.

The Schauder-Tychonov theorem then guarantees that F has a fixed point y in Y, giving rise to the desired decaying entire solution $u(x) = y(|x|)$ of (30).

We are grateful to the Referee for his useful suggestions.

References

1. A. M. Fink and T. Kusano, *Nonoscillation theorems for differential equations with general deviating arguments*, Lecture Notes in Mathematics *1032* (Springer, 1983), 224–239.
2. N. Fukagai, *On decaying entire solutions of second order sublinear elliptic equations*, Hiroshima Math. J. *14* (1984), 551–562.
3. N. Kawano, *On bounded entire solutions of semilinear elliptic equations*, Hiroshima Math. J. *14* (1984), 125–158.
4. T. Kusano, M. Naito and C. A. Swanson, *Radial entire solutions to even order semilinear elliptic equations in the plane*, Proc. Roy. Soc. Edinburgh Sect. A *105* (1987), 275–287.
5. T. Kusano and S. Oharu, *Bounded entire solutions of second order semilinear elliptic equations with application to a parabolic initial value problem*, Indiana Univ. Math J. *34* (1985) 85–95.
6. T. Kusano and C. A. Swanson, *Positive entire solutions of semilinear biharmonic equations*, Hiroshima Math. J. *17* (1987), 13–28.
7. T. Kusano and W. F. Trench, *Global existence of solutions of mixed sublinear-superlinear differential equations*, Hiroshima Math. J. *16* (1986), 597–606.
8. W. Walter, *Ganze Lösungen der Differentialgleichung $\Delta^p u = f(u)$*, Math. Z. *67* (1957), 32–37.
9. ——— *Zur Existenz ganzer Lösungen der Differentialgleichung $\Delta^p u = e^u$*, Arch. Math. *9* (1958), 308–312.
10. W. Walter and H. Rhee, *Entire solutions of $\Delta^p u = f(r, u)$*, Proc. Roy. Soc Edinburgh Sect. A *82* (1979), 189–192.

Hiroshima University
Hiroshima, Japan;
The University of British Columbia,
Vancouver, British Columbia