ON METRIZABILITY OF TOPOLOGICAL SPACES

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1. Introduction. Our present work is divided into three sections. In §2 we study the metrizability of spaces with a G_δ-diagonal (see Definition 2.1). In §3 we study the metrization of topological spaces by means of collections of (not necessarily continuous) real-valued functions on a topological space. Our efforts, in §§2 and 3, are directed toward answering the following question: "Is every normal, metacompact (see Definition 2.4) Moore space a metrizable space?" which still remains unsolved. (However, Theorems 2.12 through 2.15 and Theorem 3.1 may be helpful in answering the preceding question.) In §4 we prove an apparently new necessary and sufficient condition for the metrizability of the Stone-Čech compactification of a metrizable space and hence for the compactness of a metric space.

Unless otherwise specified, we use the terminology of Kelley (**8**), except that *all our topological spaces are* T_1.

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2. Metrizability of spaces with a G_δ-diagonal. First we state some basic definitions and prove some relevant propositions.

Definition 2.1. A space X is said to have a G_δ-diagonal if $\{(x, x) | x \in X\}$ is a G_δ-subset of $X \times X$.

We shall frequently need the following result, concerning spaces with G_δ-diagonals, which is proved in Lemma 5.4 of Ceder (**3**):

LEMMA 2.2. *A space X has a G_δ-diagonal if and only if there exists a sequence of open covers $\{\mathfrak{U}_1, \mathfrak{U}_2, \ldots\}$ of X such that, for each $x, y \in X$, $x \neq y$ implies there exists m such that $y \notin \mathrm{st}(x, \mathfrak{U}_m)$.*

It turns out that a slight change (in view of Lemma 2.2) of Definition 2.1 will be quite useful:

Definition 2.3. A space X has a $\bar{G}_\delta$-diagonal provided that there exists a sequence of open covers $\{\mathfrak{U}_1, \mathfrak{U}_2, \ldots\}$ of X such that

(a) if $x, y \in X$ and $x \neq y$, then there exists m such that $y \notin \mathrm{st}(x, \mathfrak{U}_m)$;
(b) for each $x \in X$ and n there exists m such that $[\mathrm{st}(x, \mathfrak{U}_m)]^- \subset \mathrm{st}(x, \mathfrak{U}_n)$.

For the sake of completeness and clarity we state three more definitions before we prove some new results.

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Definition 2.4. Let X be a space. Then X is said to be metacompact if each open cover $\mathfrak{U}$ of X has an open refinement $\mathfrak{V}$ such that each $x \in X$ is an element of only finitely many elements of $\mathfrak{V}$; $\mathfrak{V}$ is said to be a point-finite refinement of $\mathfrak{U}$. X is said to be *para-Lindelöf* (*meta-Lindelöf*) if each open cover $\mathfrak{U}$ of X has an open refinement $\mathfrak{V}$ such that each $x \in X$ has a neighbourhood which intersects only countably many elements of $\mathfrak{V}$ (each $x \in X$ is an element of only countable many elements of $\mathfrak{V}$; $\mathfrak{V}$ is said to be a point-countable refinement of $\mathfrak{U}$).[1]

The term "metacompact" is not new; it appears on page 171 of Kelley's book (**8**) and elsewhere. Metacompact spaces are also called pointwise paracompact spaces by some authors.

Definition 2.5. Let X be a space. Then

(a) X is said to be a $w\Delta$-space if there exists a sequence $(\mathfrak{V}_1, \mathfrak{V}_2, \ldots\}$ of open covers of X such that for each $x_0 \in X$, if $x_n \in \mathrm{st}(x_0, \mathfrak{V}_n)$ for $n = 1, 2, \ldots$, then the sequence $\{x_1, x_2, \ldots\}$ has a cluster point. Equivalently, if $\{A_1, A_2, \ldots\}$ is a decreasing sequence of non-empty closed subsets of X and there exists $x_0 \in X$ for which $A_n \subset \mathrm{st}(x_0, \mathfrak{V}_n)$ for each n, then

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset.$$

(b) X is said to be a Δ-space if X is a $w\Delta$-space and the covers $\mathfrak{V}_n$ satisfying (a) can be chosen so that we also have, for each $x \in X$ and each $n = 1, 2, \ldots$, $[\mathrm{st}(x, \mathfrak{V}_{n+1})]^- \subset \mathrm{st}(x, \mathfrak{V}_n)$.

Definition 2.6 (Morita **9**). A space X is an M-space provided there exists a normal sequence[2] $\{\mathfrak{U}_1, \mathfrak{U}_2, \ldots\}$ of open covers of X satisfying the following: If $\{A_1, A_2, \ldots\}$ is a sequence of subsets of X, with the finite intersection property, and if there exists $x_0 \in X$ such that, for each $n = 1, 2, \ldots$, there exists some $A_k \subset \mathrm{st}(x_0, \mathfrak{U}_n)$, then

$$\bigcap_{n=1}^{\infty} A_n^- \neq \emptyset.$$

Clearly all *metrizable* or *countably compact* spaces are M-spaces.

LEMMA 2.7. *Every M-space X is a Δ-space.*

Proof (this proof is a modification of part 2 of the proof of Theorem 6.1 in (**9**)). Because of Theorem 6.1 in (**9**), there exists a closed continuous map $f: X \to T$ from X onto a metrizable space T such that $f^{-1}(t)$ is countably compact for each $t \in T$. It is easily shown that there exists a normal sequence

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[1]An open cover $\mathfrak{U}$ of a topological space X is σ-point finite if $\mathfrak{U} = \bigcup_{n=1}^{\infty}\mathfrak{U}_n$, where each $\mathfrak{U}_n$ is point finite. We similarly define σ-point-countable covers.

[2]A sequence $\{\mathfrak{U}_1, \mathfrak{U}_2, \ldots\}$ of open covers of a topological space X is said to be a normal sequence if, for each n, $\mathfrak{U}_{n+1}$ is a Δ-refinement of $\mathfrak{U}_n$ (i.e., $\mathrm{st}(x, \mathfrak{U}_{n+1}) = \bigcup\{U \in \mathfrak{U}_{n+1} | x \in U\}$ is a subset of some element of $\mathfrak{U}_n$ for each $x \in X$).

$\{\mathfrak{W}_1, \mathfrak{W}_2, \ldots\}$ of open covers of T such that $\{\mathrm{st}(x, \mathfrak{W}_i)\}_{i=1}^{\infty}$ is a neighbourhood base of t for each $t \in T$, and $\{w^- \mid w \in \mathfrak{W}_{n+1}\}$ is a locally finite (and thus closure-preserving) refinement of $\mathfrak{W}_n$. For each i, let $\mathfrak{U}_i = \{f^{-1}(w) \mid w \in \mathfrak{W}_i\}$. Then $\{\mathfrak{U}_1, \mathfrak{U}_2, \ldots\}$ is a normal sequence of open covers of X, and

$$[\mathrm{st}(x, \mathfrak{U}_{n+1})]^- \subset \mathrm{st}(x, \mathfrak{U}_n)$$

for each $x \in X$ and n (since f is continuous and clearly

$$[\mathrm{st}(t, \mathfrak{W}_{n+1})]^- \subset \mathrm{st}(t, \mathfrak{W}_n)$$

for each n). To complete the proof we only need show that whenever $\{A_1, A_2, \ldots\}$ is a decreasing sequence of non-empty closed subsets of X and there exists $x_0 \in X$ for which $A_n \subset \mathrm{st}(x_0, \mathfrak{U}_n)$ for each n, then

$$\bigcap_{n=1}^{\infty} A_n \neq \emptyset.$$

Let $t_0 = f(x_0)$. Then $f(A_i) \subset \mathrm{st}(t_0, \mathfrak{W}_i)$ since $A_i \subset \mathrm{st}(x_0, \mathfrak{U}_i)$, and clearly $\{f(A_1), f(A_2), \ldots\}$ is a decreasing sequence of closed subsets of X. Hence $t_0 \in f(A_i)$ for each i, since $\{\mathrm{st}(t_0, \mathfrak{W}_i)\}_{i=1}^{\infty}$ is a neighbourhood base of t_0. Hence

$$\bigcap_{i=1}^{\infty} (A_i \cap f^{-1}(t_0)) \neq \emptyset$$

since $f^{-1}(t_0)$ is countably compact.

PROPOSITION 2.8. *The following implications are valid for topological spaces:*
(a) *If X is a Moore space,*[3] *then X is a $w\Delta$-space and X has a $\bar{G}_\delta$-diagonal.*
(b) *If X is a Moore space, then X is semimetrizable.*[4]
(c) *If X is semimetrizable and Hausdorff, then X has a G_δ-diagonal.*

Proof. Part (a) follows easily from the definition of a Moore space. Part (b) is well known and easily proved. Part (c) is easily proved by use of Lemma 2.2 (simply let $\mathfrak{U}_n$ be the family of interiors of spheres in X of radius $1/n$).

PROPOSITION 2.9. *Let X be a metacompact, regular $w\Delta$-space with a G_δ-diagonal. Then X has a σ-point finite development* (see footnote 3). *Hence X is a Moore space with a uniform*[5] *base.*

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[3]A topological space X is a Moore space if it is regular and has a base $\mathfrak{B} = \bigcup_{n=1}^{\infty}\mathfrak{B}_n$ such that $\{\mathrm{st}(x, \mathfrak{B}_n)\}_{n=1}^{\infty}$ is a neighbourhood base for each $x \in X$ (see footnote 2). The sequence $\{\mathfrak{B}_1, \mathfrak{B}_2, \ldots\}$ is called a development.

[4]A topological space X is semimetrizable if there exists a non-negative real-valued function d on $X \times X$ such that $d(x, y) = d(y, x)$ for every $x, y \in X$, $d(x, y) = 0$ if and only if $x = y$, and d is compatible with the topology on X (i.e., $p \in A^-$ if and only if there exists a sequence $\{x_1, x_2, \ldots\} \subset A$ such that $\lim_n d(x_n, x) = 0$). The function d is called a semimetric.

[5]A base $\mathfrak{B}$ for a topological space X is said to be a uniform base if every infinite subfamily of $\mathfrak{B}$ having a common element $x \in X$ is a neighbourhood base for x.

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Proof. Since X has a G_δ-diagonal and is a $w\Delta$-space, there exists a sequence $\{\mathfrak{V}_1, \mathfrak{V}_2, \ldots\}$ of open covers of X such that if $x \neq y$, then $y \notin \mathrm{st}(x, \mathfrak{V}_n)$ for some n (see Lemma 2.2) and, for each $x_0 \in X$, if $x_n \in \mathrm{st}(x_0, \mathfrak{V}_n)$ for $n = 1, 2, \ldots$, then the sequence $(x_1, x_2, \ldots\}$ has a cluster point.

For each n, let $\mathfrak{V}_n$ be a point-finite open refinement of $\mathfrak{V}_n$ such that $\{B^-| B \in \mathfrak{V}_{n+1}\}$ is a refinement of $\mathfrak{V}_n$ (this can be done since X is regular). Letting

$$\mathfrak{V} = \bigcup_{n=1}^{\infty} \mathfrak{V}_n,$$

we show that $\mathfrak{V}$ is a σ-point-finite development for X (see footnotes 1 and 2): Clearly $\mathfrak{V}$ is σ-point-finite. Let $U \subset X$ be open and $x \in U$. Assume that for no n is $\mathrm{st}(x, \mathfrak{V}_n) \subset U$. Then there exists $x_n \in \mathrm{st}(x, \mathfrak{V}_n) - U$ for each n, and thus the sequence $\{x_1, x_2, \ldots\}$ has a cluster point $y \in X$ (clearly $y \neq x$). Since $[\mathrm{st}(x, \mathfrak{V}_{n+1})]^- \subset \mathrm{st}(x, \mathfrak{V}_n)$ for each n, we get that

$$y \in \mathrm{st}(x, \mathfrak{V}_n) \subset \mathrm{st}(x, \mathfrak{V}_n)$$

for every n, a contradiction. Then X is a Moore space, and, by Theorem 4 in (**7**), X has a uniform base.

PROPOSITION 2.10. *Let X be a regular, meta-Lindelöf, $w\Delta$-space with a $\bar{G}_\delta$-diagonal. Then X is a Moore space with a point-countable base.*[6]

Proof. Since X has a $\bar{G}_\delta$-diagonal, let $\{\mathfrak{U}_1, \mathfrak{U}_2, \ldots\}$ be a sequence of open covers of X satisfying Definition 2.3. If we let

$$\mathfrak{U}'_n = \left\{ \bigcap_{i=1}^{n} U_i \,\middle|\, U_i \in \mathfrak{U}_i \text{ for each } i \right\}$$

for each n, we easily see that the sequence $\{\mathfrak{U}'_1, \mathfrak{U}'_2, \ldots\}$ of open covers of X satisfies Definition 2.3 and, furthermore, $\mathfrak{U}'_{n+1}$ is a refinement of $\mathfrak{U}'_n$ for each n. Since X is a $w\Delta$-space, let $\{\mathfrak{W}_1, \mathfrak{W}_2, \ldots\}$ be a sequence of open covers of X satisfying Definition 2.5(a). Then, for each n, let $\mathfrak{V}_n = \{U \cap W| U \in \mathfrak{U}'_n$ and $W \in \mathfrak{W}_n\}$. It is easily seen that

(a) if $x \neq y$, then there exists n such that $y \notin \mathrm{st}(x, \mathfrak{V}_n)$;

(b) for each $x_0 \in X$, if $x_n \in \mathrm{st}(x_0, \mathfrak{V}_n)$ for $n = 1, 2, \ldots$, then the sequence $\{x_1, x_2, \ldots\}$ has a cluster point;

(c) for each $x \in X$ and n there exists m such that

$$[\mathrm{st}(x, \mathfrak{V}_{m+j})]^- \subset [\mathrm{st}(x, \mathfrak{U}'_m)]^- \subset \mathrm{st}(x, \mathfrak{U}'_n) \quad \text{for } j = 1, 2, \ldots$$

(because $\mathfrak{U}'_{m+j}$ is a refinement of $\mathfrak{U}'_m$).

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[6]A base $\mathfrak{V}$ for a topological space X is said to be a point-countable base if no element of X belongs to uncountably many elements of $\mathfrak{V}$.

For each n, let $\mathfrak{B}_n$ be a point-countable open refinement of $\mathfrak{V}_n$ and let

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n.$$

Clearly $\mathfrak{B}$ is point-countable. Using the argument in the proof of Proposition 2.9, one easily sees that $\mathfrak{B}$ is a development (see footnote 3) for X, which completes the proof.

We have not been able to settle the following natural question: Does Proposition 2.10 remain valid whenever we replace "$w\Delta$-space" by "Δ-space" and "$\bar{G}_\delta$-diagonal" by "G_δ-diagonal" in it? (We conjecture that the answer is "no".)

Because of Proposition 2.8 the following result improves Theorem 1 in Traylor (**14**).

THEOREM 2.11. *The following implications are valid*:

(a) *If X is a regular, locally separable, metacompact, $w\Delta$-space with a G_δ-diagonal, then X is metrizable.*

(b) *If X is a regular, locally separable, meta-Lindelöf, $w\Delta$-space with a $\bar{G}_\delta$-diagonal, then X is metrizable.*

Proof. (a) By Proposition 2.9, X has a point-countable base. Since a separable space with a point-countable base clearly has a countable base, X is the union of a point-countable family of open separable metrizable subspaces and is thus metrizable, by Theorem 5 in (**13**). Part (b) is similarly proved by the use of Proposition 2.10.

THEOREM 2.12. *If X is a regular, locally countably compact, metacompact space with a G_δ-diagonal (or a meta-Lindelöf space with a $\bar{G}_\delta$-diagonal), then X is metrizable and locally compact.*

Proof. By Proposition 2.9 (Proposition 2.10) X has locally a point-countable base (since a countably compact space is an M-space and thus a Δ-space, by Lemma 2.7). By Proposition 2.1 in (**4**), X is locally metrizable and countably compact (hence locally compact). Hence, by Theorem 2.11, X is metrizable and locally compact.

Because of Theorem 6.1 in (**9**), the following result substantially improves Theorem 8.2 in (**2**) or Theorem 2 in (**10**).

THEOREM 2.13. *A regular metacompact (meta-Lindelöf) M-space X is metrizable if and only if it has a G_δ-diagonal ($\bar{G}_\delta$-diagonal).*

Proof. First we prove the "if" part. By Theorem 6.1 in (**9**), there exists a metrizable space Y and a map f from X onto Y such that f is continuous, closed, and $f^{-1}(y)$ is countably compact for every $y \in Y$. Thus, by our Theorem 2.12, each $f^{-1}(y)$ is metrizable and countably compact. Hence each $f^{-1}(y)$ is compact, and f is a perfect map. Thus, by Theorem 2.2 in (**6**), X is paracompact (even though all spaces in (**6**) are assumed to be completely regular, the proof of Theorem 2.2 of (**6**) requires only that the spaces be regular.) By Theorem 8.2 in (**2**), X is metrizable. The "only if" part is obvious.

COROLLARY 2.14. *A regular, metacompact semimetrizable space is metrizable if and only if it is an M-space.*

Proof. Immediate, because of Proposition 2.8(c).

COROLLARY 2.15. *A regular, meta-Lindelöf Moore space is metrizable if and only if it is an M-space.*

Proof. Immediate, because of Proposition 2.8(a).

COROLLARY 2.16. *A regular space X with a uniform base is metrizable if and only if it is an M-space.*

Proof. Immediate from our Corollary 2.15 and Theorem 4 in (**7**).

Corollary 2.16 suggests the following question: "Is every normal space with a uniform base an M-space?" Certainly, a positive answer to this question would prove that every normal space with a uniform base is metrizable and would thus settle a rather old and famous question: "Is every normal space with a uniform base a metrizable space?"

The definitions of "Moore space" and "M-space" seem to be quite similar. Thus we formulate another question: "Is every normal Moore space an M-space?" A positive answer to this question would prove that a meta-Lindelöf, normal Moore space is metrizable (see Corollary 2.15), which would partially settle another very old question: "Is every normal Moore space a metrizable space?"

Another interesting consequence of Theorem 2.13 is the following:

COROLLARY 2.17. *Every regular, screenable,*[7] *semimetrizable M-space is metrizable.*

Proof. Immediate, because of Theorem 2 in (**7**) and Corollary 2.14.

We conclude this section with the following relevant results.

THEOREM 2.18. *A paracompact, locally connected, locally peripherally compact space X is metrizable if and only if it has a G_δ-diagonal.*

Proof. Immediate from Lemma 8.2 in (**4**) and Theorem 1 in (**11**).

Theorem 2.18 is a rather surprising result since the real line with the half-open interval topology is paracompact, locally peripherally compact, but non-metrizable (thus the hypothesis of local connectedness is crucial in Theorem 2.18).

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[7]A topological space X is said to be screenable if every open cover $\mathfrak{U}$ of X has an open refinement $\mathfrak{V} = \bigcup_{n=1}^{\infty} \mathfrak{V}_n$ such that each $\mathfrak{V}_n$ is pairwise disjoint.

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The following is an easy consequence of Theorem 2.18.

THEOREM 2.19. *Let $f\colon X \to Y$ be a monotone quotient map from the locally connected, locally peripherally compact, regular space X onto the metrizable space Y such that, for each $y \in Y$, $f^{-1}(y)$ is compact. If X has a G_δ-diagonal, then X is metrizable.*

Proof. By Lemma 3.4 in (**4**), f is a perfect map. By Theorem 2.2 in (**6**), X is paracompact (again we point out that Theorem 2.2 of (**6**) remains valid if all spaces are assumed to be only regular). By our Theorem 2.18, X is metrizable.

3. Collections of functions. Throughout we let I denote the closed unit interval, and $I^+ =]0, 1]$.

THEOREM 3.1. *Let X be a first countable space. Then X is metrizable if and only if there exists a base*

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$$

for X such that for each $B \in \mathfrak{B}$ there exists a (not necessarily continuous) function $f_B\colon X \to I$ satisfying

(a) $f_B{}^{-1}(0) = X - B$;

(b) *for each sequence* $(f_{B_1}, f_{B_2}, \ldots\}$, *with* $\{B_1, B_2, \ldots\} \subset \mathfrak{B}_j$ *for some* j,

$$\lim_n x_n = y \quad \textit{implies} \quad \lim_n |f_{B_n}(x_n) - f_{B_n}(y)| = 0.$$

Proof. First we prove the "if" part. For every $x, y \in X$ and positive integer n let

$$\rho_n(x, y) = \sup_{B\in\mathfrak{B}_n} |f_B(x) - f_B(y)|$$

and

$$\rho(x, y) = \textstyle\sum_n 2^{-n}\rho_n(x, y).$$

Clearly, ρ is a metric on X. We now show the following.

(a′) If $\lim_n x_n = y$, then $\lim_n \rho(x_n, y) = 0$. Let $\epsilon > 0$. Then there exists N such that

$$\sum_{k\geqslant N} 2^{-k}\rho_k(x_n, y) < \tfrac{1}{2}\epsilon$$

for every x_n. It suffices to show that, for each $n < N$, $\lim_j \rho_n(x_j, y) = 0$, for then one easily gets a positive integer M such that $\rho(x_m, y) < \epsilon$ for each $m > M$. Assume there exists $m < N$ such that it is false that $\lim_j \rho_m(x_j, y) = 0$. Then there exists a subsequence $\{w_1, w_2, \ldots\}$ of $\{x_1, x_2, \ldots\}$ and $\delta > 0$ such that $\rho_m(w_i, y) \geqslant \delta$ for each i. Then, for each i, there exists $B_i \in \mathfrak{B}_m$ such that $|f_{B_i}(w_i) - f_{B_i}(y)| \geqslant \delta/2$, contradicting assumption (b).

(b′) If $\lim_n \rho(x_n, y) = 0$, then $\lim_n x_n = y$. Assume this is not true. Then there exists an open neighbourhood B of y, with $B \in \mathfrak{B}_i$ for some i, and a subsequence $\{z_1, z_2, \ldots\}$ of $\{x_1, x_2, \ldots\}$ such that $B \cap \{z_1, z_2, \ldots\} = \emptyset$. Hence

$$f_B(y) \neq 0 \quad \text{and} \quad f_B(z_k) = 0 \quad \text{for each} \quad k.$$

Consequently, the sequence $\{\rho(z_k, y)\}_{k=1}^{\infty}$ does not converge to 0 (for each k, $\rho(z_k, y) \geqslant 2^{-i}|f_B(z_k) - f_B(y)| = 2^{-i}|f_B(y)| > 0$), a contradiction.

From (a′) and (b′) we get that the identity map $id: X \to (X, \rho)$ is a homeomorphism, completing the proof.

To prove the "only if" part, let d be a metric on X compatible with the topology of X, and let

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n$$

be a σ-locally finite base for X (see Theorem 18, page 127 in **12**). For each open B, define $f_B: X \to I$ by

$$f_B(x) = d(x, X - B) \quad \text{for each } x \in X.$$

It is easily seen that the functions $\{f_B |\, B \in \mathfrak{B}\}$ satisfy all requirements because the functions f_B are clearly continuous.

It is significant to observe that Theorem 3.1 remains valid even if the functions f_B are not continuous. Consequently, whenever X is a semimetrizable space and $\mathfrak{B}$ is a base for X, for each $B \in \mathfrak{B}$ there exists a function $f_B: X \to I$ satisfying condition (a) of Theorem 3.1. (Let d be a semimetric on X compatible with the topology of X. For each $B \in \mathfrak{B}$, let $f_B: X \to I$ be defined by $f_B(x) = d(x, X - B)$ for each $x \in X$. Note that, for each $y \in B$,

$$d(y, X - B) > 0$$

since y is not a limit point of $X - B$. Hence, for each $B \in \mathfrak{B}$, $f_B^{-1}(0) = X - B$, as required.)

Also, if each $\mathfrak{B}_n$ (in Theorem 3.1) is locally finite and the functions f_B are continuous, then condition (b) of Theorem 3.1 is satisfied automatically. Consequently, whenever X is a regular space with a σ-discrete (or σ-locally finite) base

$$\mathfrak{B} = \bigcup_{n=1}^{\infty} \mathfrak{B}_n,$$

then X is easily seen to be perfectly normal and hence there exist continuous functions (using Tietze's extension theorem) $f_B: X \to I$ satisfying conditions (a) and (b) of Theorem 3.1. Hence X is metrizable, and thus the easier halves of Theorem 3 of Bing (**1**) and Theorem 1 of Smirnov (**12**) are easy consequences of our Theorem 3.1.

COROLLARY 3.2. *A first countable space X is metrizable if and only if there exists a family $\mathfrak{F}$ of continuous functions from X to I such that*

(a) $\{f^{-1}(I^+) |\, f \in \mathfrak{F}\}$ *is a base for X;*

(b) $\mathfrak{F} = \bigcap_{n=1}^{\infty} \mathfrak{F}_n$, *where $\{f^{-1}(I^+) |\, f \in \mathfrak{F}_n\}$ is point-finite for each n;*

(c) *for each $x \in X$ and sequence $\{f_1, f_2, \ldots\} \subset \mathfrak{F}_j$ for some j, with $f_n(x) = 0$ for all n, there exists a continuous function $h: X \to I$ such that $h(x) = 0$ and $h \geqslant f_n$ for each n.*

Proof. To prove the "if" part we simply note that $\mathfrak{F}$ is easily seen to satisfy condition (b) of the hypothesis of Theorem 3.1. To prove the "only if" part we proceed, as in the proof of the "only if" part of Theorem 3.1, to obtain the functions f_B for each B in some σ-locally finite base for X. Then we simply observe that, given $x \in X$ and functions f_{B_n} such that $f_{B_n}(x) = 0$ for $n = 1, 2, \ldots$, then, letting

$$B = \bigcup_{n=1}^{\infty} B_n,$$

we have $f_B(x) = 0$ and $f_B \geqslant f_{B_n}$ for each n. This completes the proof.

4. A characterization of compact metric spaces. As an application of the results of §2, we prove a necessary and sufficient condition for the metrizability of the Wallman compactification (or the Stone–Cěch compactification) of a metrizable space, and thus we obtain an apparently new characterization of compact metric spaces. For terminology and necessary results, see Exercise R (page 167) in (**8**).

For the sake of clarity we point out that, for any space X, the Stone–Cěch compactification βX is topologically equivalent to the Wallman compactification $\omega(X)$ whenever $\omega(X)$ is Hausdorff. Furthermore, $\omega(X)$ is Hausdorff whenever X is normal.

LEMMA 4.1. *If X is any space, then the Wallman compactification $\omega(X)$ has a G_δ-diagonal if and only if there exists a sequence $\{\mathfrak{U}_1, \mathfrak{U}_2, \ldots\}$ of open covers of X such that*

(a) *for each n, $\mathfrak{U}^*_n = \{U^* \mid U \in \mathfrak{U}_n\}$ is a cover of $\omega(X)$;*

(b) *given distinct $\mathfrak{A}, \mathfrak{A}' \in \omega(X)$, there is an integer n such that if $A \in \mathfrak{A}$ and $U \in \mathfrak{U}_n$ are such that $A \subset U$, then $X - U \in \mathfrak{A}'$.*

Proof. Assume such a sequence $\{\mathfrak{U}_1, \mathfrak{U}_2, \ldots\}$ exists. Let $\mathfrak{A}, \mathfrak{A}'$ be distinct elements of $\omega(X)$ and pick n such that (b) is satisfied. Then $\mathfrak{A}' \notin \mathrm{st}(\mathfrak{A}, \mathfrak{U}^*_n)$. By Lemma 2.2, $\omega(X)$ has a G_δ-diagonal. The converse is easily seen, because of Lemma 2.2 and part (c) of Exercise R on page 167 of (**8**).

For the sake of convenience, we shall say that a sequence $\{\mathfrak{U}_1, \mathfrak{U}_2, \ldots\}$ of open covers of a space X is a *Wallman sequence* if it satisfies conditions (a) and (b) of Lemma 4.1. Then we can easily prove the following theorem.

THEOREM 4.2. *A normal space X is a compact metric space if and only if X has a Wallman sequence of open covers.*

Proof. Since the "only if" part is clear, we proceed with the proof of the "if" part. Assuming X is normal and has a Wallman sequence of open covers, we get that $\omega(X)$ is a compact Hausdorff space (and thus a paracompact M-space) with a G_δ-diagonal (by Lemma 4.1). Hence, by Theorem 2.13, $\omega(X)$ is metrizable. However, because of Corollary 9.6 of (**5**), $\omega(X) = X$. Consequently, X is a compact metrizable space, which completes the proof.

COROLLARY 4.3. *A metrizable space X is compact if and only if X has a Wallman sequence of open covers.*

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