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PATHS BETWEEN BANACH SPACES

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Abstract. The *Kadets path distance* between Banach spaces X and Y is defined to be the infimum of the lengths with respect to the Kadets distance of all curves joining X and Y. If there is no curve joining X and Y, the *Kadets path distance* between X and Y is defined to be ∞.

Some approaches to estimates of the Kadets path distance from above and from below are developed. In particular, the Kadets path distances between the spaces l_p^n, $p \in [1, +\infty]$, $n \in \mathbf{N}$ are estimated.

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1. Introduction. Path metrics (or inner metrics, or geodesic distances) are very important objects in both classical geometry and metric geometry. See the books A. D. Aleksandrov [**1**], A. D. Aleksandrov and V. A. Zalgaller [**2**], L. M. Blumenthal [**4**], M. R. Bridson and A. Haefliger [**5**], H. Busemann [**6**], M. Gromov [**11**], W. Rinow [**21**], J. J. Schäffer [**23**], and A. C. Thompson [**25**], where such metrics are introduced under different names and in different contexts.

The general construction is as follows. Let M be a metric space with metric μ. Let $x, y \in M$ and let $f : [0, 1] \to M$ be a continuous mapping satisfying $f(0) = x$ and $f(1) = y$. We call such a mapping a *curve* joining x and y. The *length* of the curve f is defined as

$$\sup \sum_{i=1}^{n} \mu(f(\alpha_{i-1}), f(\alpha_i)),$$

where the supremum is taken over all finite subsets

$$0 \leq \alpha_0 < \alpha_1 < \ldots < \alpha_n \leq 1.$$

If M is such that each two points of M are joined by at least one curve of finite length, we introduce the corresponding *path metric* on M, defined by $\mu_P(x, y) =$ infimum of the lengths of curves joining x and y.

The main purpose of this paper is to study the distances that we get if we apply this general construction to distances studied in Geometric Functional Analysis. It is easy to verify that if we apply this construction to the Banach-Mazur distance we do not get a new distance (this assertion follows, for example, from Lemma 1 below). In this paper we are going to consider the Kadets distance. This distance was introduced by M. I. Kadets in [**15**] and was studied in several papers; see [**16**] and [**18**].

The Kadets distance may be considered as a Banach-space analogue of the Gromov-Hausdorff distance, well known in geometry. See [**11**, Chapter 3], where this distance is called the *Hausdorff distance*. See also [**5**] and [**20**].

We shall use some standard notation of Banach Space Theory. See for example [**14**]. We recall the necessary definitions. Let X be a Banach space and let Y and Z be closed subspaces of X. The *opening* (sometimes called also *gap*) between Y and Z is defined to be the Hausdorff distance between their unit balls; that is

$$\Lambda(Y, Z) = \max\{ \sup_{y\in B(Y)} \operatorname{dist}(y, B(Z)), \sup_{z\in B(Z)} \operatorname{dist}(z, B(Y))\},$$

where by $B(Y)$ and $B(Z)$ we denote the unit balls of Y and Z respectively.

If X and Y are arbitrary Banach spaces we define the *Kadets distance*

$$d_K(X, Y) = \inf_{Z,U,V} \Lambda(UX, VY),$$

where the infimum is taken over all Banach spaces Z and all linear isometric embeddings $U : X \to Z$ and $V : Y \to Z$.

For known properties of Kadets distance we refer to [**16**] and the references therein.

Adapting the general definition discussed above to the Kadets distance we get the following definition.

Definition 1. Let X and Y be Banach spaces. Suppose that there exists a mapping $Z : [0, 1] \to$ {Banach spaces}, continuous with respect to d_K, such that $Z(0) = X$ and $Z(1) = Y$. We call such a mapping a *curve* joining X and Y. The *Kadets path distance* $d_{KP}(X, Y)$ between X and Y is defined to be the infimum of the lengths with respect to d_K of all curves joining X and Y.

If there is no curve joining X and Y, then $d_{KP}(X, Y)$ is defined to be ∞.

In addition to being a special case of a well-known construction, the Kadets path distance attracted my attention for the following reasons.

(1) The values of the Kadets distance between Banach spaces do not reflect adequately how different are the spaces. We illustrate this statement with the following example.

In [**18**] (see, also [**19**, p. 303]) the author proved that there exist a separable Banach space X and a non-separable Banach space Y such that $d_K(X, Y) \leq 2\sqrt{2} - 2$. On the other hand, it is easy to show that $d_K(l_1^n, l_2^n) \geq 1 - \frac{1}{\sqrt{n}}$. Hence, $d_K(l_1^{34}, l_2^{34})$ exceeds the Kadets distance between the set of separable and nonseparable spaces.

It seems that the values of Kadets path distance reflect the difference between Banach spaces more properly than the values of Kadets distance itself.

(2) The study of the behavior of Kadets path distance is one of the natural approaches to the following problem from [**16**]: whether each separable infinite dimensional super-reflexive space belongs to the same connected component (with respect to d_K) as l_2?

(3) S. Semmes [**24**] suggested the following problem. What is an optimal way of connecting two convex bodies in $\mathbf{R}^n$ by a curve of convex bodies in $\mathbf{R}^n$? The study of the Kadets path distance is one of the approaches to this problem restricted to symmetric convex bodies.

The main result of this paper is an analogue of the well-known result of V. I. Gurarii, M. I. Kadets and V. I. Matsaev [**13**] on Banach-Mazur distances between spaces l_p^n, $p \in [1, +\infty]$, $n \in \mathbf{N}$. (See Theorem 1 below.) In Section 5 we use the construction due to A. Douady [**8**] to show that for any Banach space X and for any closed subspace $Y \subset X$

$$d_{KP}(Y \oplus_\infty X/Y, X) \leq 2.$$

This estimate shows that Banach spaces that can be joined by "short" curves can be quite different in many Banach-space-theoretical senses.

It turns out that results on Kadets path distance can be used in the study of finite-dimensional versions of super-reflexivity. I am going to present my results in this direction in a separate paper.

We use the following notation. Let $f, g : \mathbf{N} \to \mathbf{R}^+$ be defined for all sufficiently large $n \in \mathbf{N}$. We write $f \approx g$ if $\exists c > 0,\ \exists C < +\infty$ such that

$$c \leq \frac{f(n)}{g(n)} \leq C \text{ for all large enough } n.$$

THEOREM 1. (a) *If* $2 \leq p, q < \infty$ *then* $d_{KP}(l_p^n, l_q^n) \leq \frac{\pi}{2}|\ln\left(\frac{p}{q}\right)|$. *If* $1 < p, q \leq 2$ *then* $d_{KP}(l_p^n, l_q^n) \leq \frac{\pi}{2}|\ln\left(\frac{1-1/p}{1-1/q}\right)|$.

(b) *Let* $1 < p < \infty$. *Then* $d_{KP}(l_p^n, l_1^n) \approx \ln\ln n$.

(c) *Let* $1 < p < \infty$. *Then* $d_{KP}(l_p^n, l_\infty^n) \approx \ln\ln n$.

(d) $d_{KP}(l_1^n, l_\infty^n) \approx \ln\ln n$.

We describe where the proofs of different parts of Theorem 1 can be found.

Part (a) is proved in Remark 2 after Corollary 1.

All other estimates from above are proved at the end of Section 2.

Parts (b) and (c) are equivalent by Corollary 3.

Estimates from below in (b) and (d) are proved in Section 4.

2. Estimates from above. In this section we use some results on the interpolation spaces. See [**7**] or Chapter 4 of [**3**].

We start with a very simple estimate. This is useful mainly for small values of $d(X, Y)$. Proposition 2 and Theorem 3 below give better asymptotic estimates.

PROPOSITION 1. $d_{KP}(X, Y) \leq \ln d(X, Y)$.

We need the following well-known result.

LEMMA 1. *Let X and Y be isomorphic Banach spaces and* $D > d(X, Y)$. *Then there exists a collection* $\{X(\tau)\}_{\tau\in[0,1]}$ *of Banach spaces such that* $X(0) = X$, $X(1) = Y$ *and* $d(X(\theta), X(\eta)) \leq D^{|\theta-\eta|}$.

Sketch of proof. Let $T : X \to Y$ be an isomorphism satisfying $||T|| \leq 1$ and $||T^{-1}|| \leq D$.

Now, let X and Y be complex Banach spaces. We identify $x \in X$ with $Tx \in Y$. With such identification the pair (X, Y) is a *compatible couple* (in the sense of [**3**, p. 24]) or *interpolation pair* (in the sense of [**7**, p. 114]). It is well known that the complex interpolation spaces corresponding to this compatible couple (interpolation pair) (see [**7** Paragraph **3**] or [**3**, p. 88]) have the required property.

To consider the real case we need some results on complexifications. See [**22**, Section 1.3] for more detailed discussion of complexifications.

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Let X be a real Banach space. By $X^{\mathbf{C}}$ we denote the complex Banach space that we obtain if we endow the direct sum $X \oplus X$ with the multiplication

$$(\alpha + i\beta)(x, y) := (\alpha x - \beta y, \beta x + \alpha y)(\alpha \in \mathbf{R}, \beta \in \mathbf{R}, x \in X, y \in X),$$

and the norm

$$||(x, y)|| := \max_{-\pi \leq \gamma \leq \pi} ||(\cos \gamma)x + (\sin \gamma)y||_X.$$

The space X will be identified with the subspace $\{(x, 0) : \ x \in X\} \subset X^{\mathbf{C}}$.

Let $T : X \to Y$ be a linear operator between real Banach spaces. The mapping $T^{\mathbf{C}} : X^{\mathbf{C}} \to Y^{\mathbf{C}}$ defined by

$$T(x, y) = (Tx, Ty)$$

is called a *complexification* of T.

LEMMA 2. *The mapping $T^{\mathbf{C}}$ is a linear operator between the complex Banach spaces $X^{\mathbf{C}}$ and $Y^{\mathbf{C}}$, and $||T^{\mathbf{C}}|| = ||T||$.*

The proof is straightforward.

REMARK. It is worthwhile to mention that Lemma 2 is not valid for some other natural norms on complexifications of real normed spaces.

We continue the proof of Lemma 1. If X and Y are real Banach spaces, we consider the operator $T^{\mathbf{C}}$. We have $||T^{\mathbf{C}}|| \leq 1$ and $||(T^{\mathbf{C}})^{-1}|| \leq D$. We identify $x \in X^{\mathbf{C}}$ with $T^{\mathbf{C}}x$. With such identification $(X^{\mathbf{C}}, Y^{\mathbf{C}})$ becomes a compatible couple (interpolation pair). Let $X(\tau)$, $\tau \in [0, 1]$ be the intersections of the complex interpolation spaces corresponding to the couple with X (or Y, observe that X and Y coincide as real linear subspaces of the couple). It is easy to check that the real Banach spaces $X(\tau)$ have the required property. □

Proof of Proposition 1. If $d(X, Y) = \infty$, there is nothing to prove, and so we suppose that $d(X, Y) < \infty$. Let $D > d(X, Y)$ and $\{X(\tau)\}_{\tau \in [0,1]}$ satisfy the conditions of Lemma 1. To get an estimate for $d_{KP}(X, Y)$ it is sufficient to estimate the length of this curve from above.

We have $d(X(\theta), X(\eta)) \leq D^{|\theta - \eta|}$. Now we apply the estimate $d_K \leq d - 1$ obtained in **[18]**. (See Proposition 6.2 in **[19]** for a proof. The argument in **[18]** contains some unpleasant misprints.) We get

$$d_K(X(\theta), X(\eta)) \leq D^{|\theta - \eta|} - 1.$$

Recalling the definition of the length of a curve (and using the continuity of our curve) we get $d_{KP}(X, Y) \leq \lim_{n \to \infty} n(D^{1/n} - 1)$. However

$$\lim_{n \to \infty} n(D^{1/n} - 1) = \lim_{x \to 0} \frac{e^{x \ln D} - 1}{x} = \ln D. \qquad \blacksquare$$

We shall use the result of **[16]** on complex interpolation spaces. Working with complex interpolation spaces we are forced to consider Banach spaces over $\mathbf{C}$ only.

Let $\bar{X} = (X_0, X_1)$ be a compatible couple of Banach spaces (interpolation pair) and let $\{\bar{X}_{[\theta]}\}_{\theta \in (0,1)}$ be the standard complex interpolation spaces (as they are defined in [**7**, Paragraph **3**] or [**3**, p. 88]). For $\theta, \eta \in (0, 1)$ we let

$$h(\theta, \eta) = \frac{\sin(\pi(\eta - \theta)/2)}{\sin(\pi(\eta + \theta)/2)}.$$

THEOREM 2. [**16**, Corollary 4.6]. If $0 < \theta < \eta < 1$ then

$$d_K(\bar{X}_{[\theta]}, \bar{X}_{[\eta]}) \leq 2h(\theta, \eta).$$

We immediately get the next result.

COROLLARY 1. If $0 < \theta < \eta < 1$ then

$$d_{KP}(\bar{X}_{[\theta]}, \bar{X}_{[\eta]}) \leq \pi \int_{\theta}^{\eta} \frac{d\tau}{\sin(\pi\tau)}.$$

REMARK 1. We can find the integral from Corollary 1 explicitly and get

$$d_{KP}(\bar{X}_{[\theta]}, \bar{X}_{[\eta]}) \leq \ln\left(\frac{\csc(\pi\eta) - \cot(\pi\eta)}{\csc(\pi\theta) - \cot(\pi\theta)}\right).$$

REMARK 2. Part (a) of Theorem 1 for complex spaces can be derived from Corollary 1 in the following way. We apply Corollary 1 to the standard interpolation between $X_0 = l_1^n$ and $X_1 = l_\infty^n$. Let $\theta, \eta \in [0, 1]$ be such that

$$\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{\infty} \quad \text{and} \quad \frac{1}{q} = \frac{\eta}{1} + \frac{1-\eta}{\infty}.$$

Hence $\theta = 1/p$ and $\eta = 1/q$. First, we consider the case $1/p < 1/q \leq 1/2$. Corollary 1 gives us the estimates

$$d_{KP}(l_p^n, l_q^n) \leq \pi \int_{1/p}^{1/q} \frac{d\tau}{\sin(\pi\tau)} \leq \pi \int_{1/p}^{1/q} \frac{d\tau}{2\tau} = \frac{\pi}{2} \ln\left(\frac{p}{q}\right).$$

In the case $1/q > 1/p \geq 1/2$ we have

$$d_{KP}(l_p^n, l_q^n) \leq \pi \int_{1/p}^{1/q} \frac{d\tau}{\sin(\pi\tau)} \leq \pi \int_{1/p}^{1/q} \frac{d\tau}{2 - 2\tau} = \frac{\pi}{2} \ln\left(\frac{1 - 1/p}{1 - 1/q}\right).$$

The same argument works in the real case also since, as it was observed in [**16**] (see Remark after Corollary 4.7), the estimate

$$d_K(l_p^n, l_q^n) \leq 2h(1/p, 1/q), \quad p, q \in (1, +\infty)$$

is valid in the real case also.

PROPOSITION 2. *Let X and Y be complex Banach spaces. If* $\ln d(X, Y) \geq \pi$, *then*

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$$d_{KP}(X, Y) \le \pi \ln\ln d(X, Y).$$

Proof. We may suppose that $d(X, Y) < \infty$. Let $D > d(X, Y)$ and let $J : X \to Y$ be an operator satisfying $||J||||J^{-1}|| \le D$. We identify $x \in X$ with $Jx \in Y$. With this identification (X, Y) becomes a compatible couple (interpolation pair). We denote it by $\bar{X}$. Then for arbitrary $0 < \theta < \eta < 0$ we have

$$d_{KP}(X, Y) \le d_{KP}(X, \bar{X}_{[\theta]}) + d_{KP}(\bar{X}_{[\theta]}, \bar{X}_{[\eta]}) + d_{KP}(\bar{X}_{[\eta]}, Y)$$

$$\le \ln d(X, \bar{X}_{[\theta]}) + \pi \int_\theta^\eta \frac{d\tau}{\sin(\pi\tau)} + \ln d(\bar{X}_{[\eta]}, Y),$$

by Proposition 1 and Corollary 1. We choose $\eta = 1 - \theta$ and suppose $\theta \le 1/2$. Observe that $d(X, \bar{X}_{[\theta]}) \le D^\theta$ and $d(Y, \bar{X}_{[1-\theta]}) \le D^\theta$. Since we may choose $D > d(X, Y)$ arbitrarily, we get

$$d_{KP}(X, Y) \le 2\theta \ln d(X, Y) + \pi \int_\theta^{1-\theta} \frac{d\tau}{\sin(\pi\tau)}. \tag{1}$$

We have

$$\pi \int_\theta^{1-\theta} \frac{d\tau}{\sin(\pi\tau)} = 2\pi \int_\theta^{1/2} \frac{d\tau}{\sin(\pi\tau)} \le 2\pi \int_\theta^{1/2} \frac{d\tau}{2\tau} \le \pi \ln(\frac{1}{2\theta}).$$

Hence

$$d_{KP}(X, Y) \le 2\theta \ln d(X, Y) - \pi \ln\theta - \pi \ln 2.$$

The right-hand side attains its minimum when

$$\theta = \frac{\pi}{2} \frac{1}{\ln d(X, Y)}.$$

In addition we have to restrict ourselves to the case in which $\theta \le \frac{1}{2}$. Thus, if $\ln d(X, Y) \ge \pi$ we get

$$d_{KP}(X, Y) \le \pi - \pi \ln \pi + \pi \ln\ln d(X, Y).$$

(In the case in which $\ln d(X, Y) < \pi$ the argument above doesn't give a new estimate.) □

In the present context it is important to find an analogue of Proposition 2 for real spaces. In the appendix we shall prove such an analogue. Its proof is based on a modification of the complex interpolation method developed in **[7]** and on a modification of the argument of **[16]**. This is why some of the details of the proof are omitted; we refer to **[7]** and **[16]** instead.

THEOREM 3. *Let X_0 and X_1 be real Banach spaces. If* $\ln d(X_0, X_1) \ge \pi$, *then*

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$$d_{KP}(X_0, X_1) \leq \pi \ln \ln d(X_0, X_1).$$

Proof. See Section 6. □

Corollary 2. *The diameter of the nth Minkowski compactum with respect to d_{KP} is* $\approx \ln \ln n$.

Proof of estimates from above in Parts (b), (c) and (d) of Theorem 1. The complex case immediately follows from Proposition 2. The real case follows from Theorem 3. Here is an alternative approach to the proof in the real case.

We use the same construction as in Proposition 2 with $X = l_1^n$, $Y = l_\infty^n$ and $J : l_1^n \to l_\infty^n$ the canonical embedding, so that $||J|| = 1$, $||J^{-1}|| = n$. Let $\bar{X} = (X, Y)$.

Consider the curve $f : [0, 1] \to \{\text{Banach spaces}\}$ defined by $f(\theta) = l_{p(\theta)}^n$, where $p(\theta) = \frac{1}{1-\theta}$. It is well known that $f(\theta)$ is isometric to $\bar{X}_{[\theta]}$.

By the remark about the real spaces mentioned before Proposition 2 we can use the formula (1) for real spaces and any $\theta \leq 1/2$:

$$d_{KP}(l_1^n, l_\infty^n) \leq \ln d(l_1^n, l_{p(\theta)}^n) + \pi \int_\theta^{1-\theta} \frac{d\tau}{\sin \pi\tau} + \ln d(l_{p(1-\theta)}^n, l_\infty^n)$$
$$\leq 2\theta \ln n - \pi \ln \theta - \pi \ln 2 \leq \pi - \pi \ln \pi + \pi \ln \ln n \ (\text{if } \ln n \geq \pi).$$

Because the considered curve contains the spaces l_p^n ($1 < p < \infty$), this computation proves the estimates from above in Parts (b) and (c) of Theorem 1 also. □

3. Duality. We need the following result from [**16**, Theorem 4.3].

Theorem 4. [**16**]. *Suppose that X and Y are Banach spaces. Then*

$$d_K(X^*, Y^*) \leq 2d_K(X, Y).$$

We immediately get the following result.

Corollary 3. *Suppose that X and Y are Banach spaces. Then*

$$d_{KP}(X^*, Y^*) \leq 2d_{KP}(X, Y).$$

If X and Y are reflexive, then $d_{KP}(X, Y) \leq 2d_{KP}(X^*, Y^*)$.

4. Estimates from below. Our approach to estimates from below is based on repeated use of the following results.

Lemma 3. (D. P. Giesy [**10**, Lemmas I.4 and I.6]; see also [**17**, p. 62].) *If a Banach space X contains an n^2-dimensional subspace X_1 satisfying $d(X_1, l_1^{n^2}) \leq \beta$, then X contains an n-dimensional subspace X_2 satisfying $d(X_2, l_1^n) \leq \sqrt{\beta}$.*

The following lemma is well known. To the best of my knowledge the first result of this type is due to V. I. Gurarii [**12**].

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LEMMA 4. *Let X and Y be finite dimensional Banach spaces such that $d_K(X, Y) \leq \alpha$ and X contains a k-dimensional subspace X_0 satisfying $d(X_0, l_1^k) \leq \gamma$. If $\alpha < 1/\gamma$, then Y contains a subspace Y_0 satisfying $d(Y_0, l_1^k) \leq 1/((1/\gamma) - \alpha)$.*

Proof of Lemma 4. (We give it for the convenience of the reader.) Let $\varepsilon > 0$. We may assume that X and Y are subspaces of some Banach space and $\Lambda(X, Y) < \alpha + \varepsilon$. The condition on X imples that there exist $x_1, \ldots, x_k \in X$ such that

$$\frac{1}{\gamma}\sum_{i=1}^{k}|a_i| \leq ||\sum_{i=1}^{k}a_ix_i|| \leq \sum_{i=1}^{k}|a_i| \quad \forall\{a_i\} \in \mathbf{R}^k;$$

in particular, $x_i \in B(X)$. Since $\Lambda(X, Y) < \alpha + \varepsilon$, there exist $y_1, \ldots, y_k \in B(Y)$ such that $||x_i - y_i|| < \alpha + \varepsilon$. Since $y_i \in B(Y)$,

$$||\sum_{i=1}^{k}a_iy_i|| \leq \sum_{i=1}^{k}|a_i| \quad \forall\{a_i\} \in \mathbf{R}^k.$$

On the other hand

$$||\sum_{i=1}^{k}a_iy_i|| \geq ||\sum_{i=1}^{k}a_ix_i|| - \sum_{i=1}^{k}|a_i|||x_i - y_i|| \geq$$

$$\frac{1}{\gamma}\sum_{i=1}^{k}|a_i| - (\alpha+\varepsilon)\sum_{i=1}^{k}|a_i| = (\frac{1}{\gamma} - \alpha - \varepsilon)\sum_{i=1}^{k}|a_i|.$$

Hence the subspace $Y_\varepsilon \subset Y$ spanned by $y_1, \ldots, y_n$ satisfies $d(Y_\varepsilon, l_1^k) \leq 1/(\frac{1}{\gamma} - \alpha - \varepsilon)$. Since $\varepsilon > 0$ is arbitrary and Y is finite dimensional, the lemma follows. □

Proof of estimates from below in Parts (b) and (d) of Theorem 1. To estimate $d_{KP}(l_1^n, l_p^n)$ from below we let $m = m(p, n)$ be the smallest positive integer satisfying the condition: the space l_p^n does not contain an m-dimensional subspace Y with $d(Y, l_1^m) \leq 4$. It is well known (and it follows, e.g., from the uniform convexity of l_p $(1 < p < \infty)$) that for $1 < p < \infty$ there exists a constant $C(p)$ such that $m(p, n) \leq C(p)$. It is also known that $m(\infty, n) \leq C \ln n$ for some absolute constant C. (This assertion follows, e.g., from the estimates on the dimensions of almost euclidean sections in l_1^m and l_∞^n; see **[9]**).

First we shall use Lemmas 3 and 4 to show that

$$d_{KP}(l_1^n, l_p^n) \geq \frac{3}{4} + \frac{k}{4}, \tag{2}$$

where k is the maximal integer satisfying $m^{2^k} \leq n$.

Let $f: [0, 1] \to$ {Banach spaces} be a continuous curve joining l_1^n and l_p^n. Let $p(0) = m^{2^k}, p(1) = m^{2^{k-1}}, \ldots, p(k) = m$. Let $X_0 = l_1^n$. Since $n \geq m^{2^k}$, the space X_0 contains a subspace Y_0 satisfying $d(l_1^{p(0)}, Y_0) = 1$. (The first step of our argument is somewhat artificial, for the uniformity of the construction.) Observe that $d_K(X_0, l_p^n) > \frac{3}{4}$ (otherwise, by Lemma 4 the space l_p^n would contain a subspace W_0 of

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dimension $p(0) \geq m$ satisfying $d(l_1^{p(0)}, W_0) \leq 4$, contrary to the definition of m). Because the curve is continuous, there exists a point X_1 on the curve corresponding to $0 < t_1 < 1$ satisfying $d_K(X_0, X_1) = \frac{3}{4}$. By Lemma 4 (we apply it with $\gamma = 1$ and $\alpha = \frac{3}{4}$) the space X_1 contains a subspace Y_1 of dimension $p(0)$ satisfying $d(l_1^{p(0)}, Y_1) \leq 4$. By Lemma 3 (we apply it with $\beta = 4$ and $\sqrt{\beta} = 2$) X_1 contains a subspace Z_1 of dimension $p(1)$ satisfying $d(l_1^{p(1)}, Z_1) \leq 2$. Observe that $d_K(X_1, l_p^n) > \frac{1}{4}$ (otherwise, by Lemma 4 the space l_p^n would contain a subspace W_1 of dimension $p(1) \geq m$ satisfying $d(l_1^{p(1)}, W_1) \leq 4$, contrary to the definition of m). Because the curve f is continuous with respect to d_K, there exists a point X_2 on the curve satisfying $d_K(X_2, X_1) = \frac{1}{4}$ and corresponding to a t_2 from the interval $(t_1, 1)$. We continue in an obvious way (all steps, except the first, are the same). In this way we find a sequence $X_1, \ldots, X_k$ of points on the curve corresponding to $0 < t_1 < \ldots < t_k < 1$ such that

$$d_K(X_0, X_1) = \frac{3}{4}, \; d_K(X_1, X_2) = \ldots = d_K(X_{k-1}, X_k) = \frac{1}{4}, \; d_K(X_k, l_p^n) > \frac{1}{4}.$$

Hence the length of the curve is at least $\frac{3}{4} + \frac{k}{4}$. Since the curve was arbitrary, it proves (2).

Remark. This argument can be generalized in terms of l_1-properties discussed in [**19**, p. 293]. I am going to present such generalizations in a separate paper.

To finish the proof it remains to show that $\frac{3}{4} + \frac{k}{4} \geq c \ln \ln n$ when n is large enough.

The definition of k implies that $m^{2^{k+1}} > n$. This inequality can be rewritten as

$$k + 1 > \frac{\ln \ln n - \ln \ln m}{\ln 2}.$$

If $1 < p < \infty$, then the sequence $\{m(p, n)\}_{n=1}^{\infty}$ is bounded by a constant that depends on p only, and the estimate is immediate.

If $p = \infty$, then $m(p, n) \leq C \ln n$ for some absolute constant C, and we get

$$k + 1 > \frac{\ln \ln n - \ln \ln C - \ln \ln \ln n}{\ln 2},$$

and the required inequality follows. □

5 One more estimate. The following estimate for d_{KP} is of a quite different nature from those discussed above.

Proposition 3. *Let X be a Banach space and Y be a closed linear subspace of X. Then*

$$d_{KP}(Y \oplus_\infty X/Y, X) \leq 2.$$

Proof. We recall a construction from [**8**, pp. 15–16]; see also [**19**, Lemma 5.9]. Let us denote by Φ the quotient mapping $X \to X/Y$. In the space $X \oplus_\infty (X/Y)$ we introduce the subspaces $G_0 = Y \oplus_\infty (X/Y)$ and $G_\alpha = \{(\alpha x, \Phi x) : \; x \in X\}, \; 1 \geq \alpha > 0$. We need the following variant of Lemma 5.9 of [**19**].

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LEMMA 5. $\forall \alpha, \beta \in [0, 1] \quad \Lambda(G_\alpha, G_\beta) \le 2|\alpha - \beta|$.

Proof. We may suppose without loss of generality that $\alpha > 0$. Let $u = (\alpha x, \Phi x) \in B(G_\alpha)$. Hence, $\max\{\alpha||x||, ||\Phi x||\} \le 1$. Let $\delta > 0$ and let $y \in Y$ be such that $||x - y|| \le (1 + \delta)||\Phi x||$. Then $(\beta x + (\alpha - \beta)y, \Phi x) \in G_\beta$ and

$$||(\alpha x, \Phi x) - (\beta x + (\alpha - \beta)y, \Phi x)|| \le |\alpha - \beta|(1 + \delta)||\Phi x|| \le |\alpha - \beta|(1 + \delta).$$

Taking the infimum over $\delta > 0$ we get $\operatorname{dist}(u, G_\beta) \le |\alpha - \beta|$. It follows that

$$\operatorname{dist}(u, B(G_\beta)) \le 2|\alpha - \beta|.$$

If $\beta > 0$ the same argument works with the roles of α and β interchanged.

Now let $\beta = 0$. Let $u = (y, z) \in B(G_0)$, so that $\max\{||y||, ||z||\} \le 1$. Let $\delta > 0$ and let $x \in X$ be such that $\Phi x = z$ and $||x|| \le (1 + \delta)||z||$. It is clear that $(\alpha x + y, z) \in G_\alpha$ and

$$||u - (\alpha x + y, z)|| \le \alpha||x|| \le \alpha(1 + \delta).$$

Taking the infimum over $\delta > 0$ we get $\operatorname{dist}(u, G_\alpha) \le \alpha$. Hence

$$\operatorname{dist}(u, B(G_\alpha)) \le 2\alpha.$$

This lemma has the following immediate consequence.

COROLLARY 4. $\forall \alpha, \beta \in [0, 1] \quad d_{KP}(G_\alpha, G_\beta) \le 2|\alpha - \beta|$.

To finish the proof of Proposition 3 it remains to observe that G_1 is isometric to X. □

6. Appendix. Real case estimate.

Proof of Theorem 3. We are going to follow the proof of Proposition 2. Let D be any number satisfying $D > d(X_0, X_1)$. It is easy to see that in order to use the same argument as in Proposition 2 it is enough to find a collection of real Banach spaces X_θ such that

$$d(X_\theta, X_0) \le D^\theta, \tag{3}$$

$$d(X_\theta, X_1) \le D^{1-\theta}, \tag{4}$$

$$d_K(X_\theta, X_\eta) \le 2h(\theta, \eta). \tag{5}$$

Let $J : X_0 \to X_1$ be a linear operator satisfying $||J|| \le 1$ and $||J^{-1}|| < D$. Let $X_0^{\mathbf{C}}$ and $X_1^{\mathbf{C}}$ be the complexifications of X_0 and X_1 and $J^{\mathbf{C}}$ be the complexification of J (see the definitions preceding Lemma 2). By Lemma 2 we have $||J^{\mathbf{C}}|| \le 1$ and $||(J^{\mathbf{C}})^{-1}|| < D$.

We identify $x \in X_0^{\mathbf{C}}$ with $T^{\mathbf{C}}x \in X_1^{\mathbf{C}}$. With this identification $(X_0^{\mathbf{C}}, X_1^{\mathbf{C}})$ becomes a compatible couple (interpolation pair) of complex Banach spaces. We denote this couple by $\bar{X}$. Observe that

$$x \in X_0 \iff T^{\mathbf{C}}x \in X_1.$$

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Hence X_0 and X_1 are identical as real linear subspaces of $\bar{X}$. We denote the corresponding linear subspace of $\bar{X}$ by X and call it *the real part of* $\bar{X}$.

For couples of the described form we introduce the following modification of the complex interpolation method. Consider Banach couples $\bar{A} = (A_0, A_1)$ of complex spaces satisfying the following conditions:

(i) the spaces A_0 and A_1 are the complexifications of real Banach spaces P_0 and P_1 respectively,

(ii) there exists an isomorphism $T : P_0 \to P_1$ such that the couple $\bar{A}$ is obtained by identification of elements of A_0 and their images under $T^{\mathbf{C}}$.

Hence A_0 and A_1 are identical as linear spaces. We denote this linear space by A. An A-valued function will be called *bounded (continuous)* if it is bounded (continuous) as a function into A_0 or A_1 (these notions coincide because $T^{\mathbf{C}}$ is an isomorphism). A function from an open region in $\mathbf{C}$ into A will be called *analytic*, if it is analytic as a function with values in A_0 (or A_1, these notions coincide). The real part of $\bar{A}$ will be denoted by P.

Let $\mathcal{R}(\bar{A})$ be the **real** Banach space consisting of all A-valued bounded continuous functions on $\Omega := \{z : \ 0 \le \Re z \le 1\}$ satisfying the conditions:

(a) f is analytic in $\{z : \ 0 < \Re z < 1\}$,

(b) $f(\theta) \in P$ for every $0 \le \theta \le 1$,

(c) $||f||_{\mathcal{R}(\bar{A})} = \max\{\sup_{\tau\in\mathbf{R}} ||f(i\tau)||_{A_0}, \sup_{\tau\in\mathbf{R}} ||f(1+i\tau)||_{A_1}\}$,

(c) $\lim_{|z|\to\infty,\ z\in\Omega} f(z) = 0$.

Comment. The conditions (a), (c) and (d) are the same as in Calderón's construction. The condition (b) is needed in order to prove the inequality (5).

Now we introduce the real Banach space P_θ, $0 < \theta < 1$, as the linear space P endowed with the norm

$$||x||_\theta = \inf\{||f||_{\mathcal{R}(\bar{A})} : \ f(\theta) = x\}.$$

We need the following analogue of a result of A. P. Calderón.

Let $\bar{A}$ and $\bar{B}$ be two pairs of the described type with the real parts P and R respectively. Let $T : P \to R$ be such that

$$||T||_{P_0 \to R_0} = M_0$$

and

$$||T||_{P_1 \to R_1} = M_1.$$

Then

$$||T||_{P_\theta \to R_\theta} \le M_0^{1-\theta} M_1^{\theta}. \tag{6}$$

To prove (6) we observe that the function $M_0^{z-1} M_1^{-z}$ is real-valued on the real line, and $T^{\mathbf{C}}(P) = R$. Hence

$$g(z) := M_0^{z-1} M_1^{-z} T^{\mathbf{C}}(f(z)) \in \mathcal{R}(\bar{B})$$

provided that $f \in \mathcal{R}(\bar{A})$. By Lemma 2 $||T^{\mathbf{C}}||_{A_0 \to B_0} = ||T||_{P_0 \to R_0}$ and $||T^{\mathbf{C}}||_{A_1 \to B_1} = ||T||_{P_1 \to R_1}$. Therefore the inequality (6) can be proved using the well-known argument. (See [**7**, p. 129] and [**3**, p. 88].)

The following fact is obvious: if $T : P_0 \to P_1$ is an isometry, then the norms of all spaces P_θ are the same.

Using this fact and the estimate (6) we get (3) and (4).

To derive (5) from the argument of [**16**] (see pp. 35–36) it is enough to observe that the space $\mathcal{R}(\bar{A})$ is invariant under multiplication of functions of the form

$$\psi_\theta(z) = \frac{\sin(\pi(z-\theta)/2)}{\sin(\pi(z+\theta)/2)},$$

and $||\psi_\theta f||_{\mathcal{R}(\bar{A})} = ||f||_{\mathcal{R}(\bar{A})}$.

This observation immediately follows from the fact that ψ_θ is a conformal mapping of the strip onto the unit disk in **C** and because ψ_θ is real on [0, 1]. □

REFERENCES

1. A. D. Aleksandrov, *Intrinsic geometry of convex surfaces* (Russian) (OGIZ, Moscow-Leningrad, 1948).

2. A. D. Aleksandrov and V. A. Zalgaller, *Intrinsic geometry of surfaces*, Translations of Mathematical Monographs **15** (AMS, Providence, Rhode Island, 1967).

3. J. Bergh and J. Löfström, *Interpolation spaces. An introduction*, Grundlehren der mathematischen Wissenschaften **223** (Springer-Verlag, 1976).

4. L. M. Blumenthal, *Theory and applications of distance geometry* (Oxford University Press, 1953; reprinted by Chelsea Publishing Company, New York, 1970).

5. M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der mathematischen Wissenschaften **319** (Springer-Verlag, 1999).

6. H. Busemann, *The geometry of geodesics* (Academic Press, 1955).

7. A. P. Calderón, Intermediate spaces and interpolation, the complex method, *Studia Math.* **24** (1964), 113–190.

8. A. Douady, Le probléme des modules pour les sous-espaces d'un espace analytique donné, *Ann. Inst. Fourier* **15** (1966), no. 1, 1–94.

9. T. Figiel, J. Lindenstrauss and V. D. Milman, The dimension of almost spherical sections of convex bodies, *Acta Math.* **139** (1977), 53–94.

10. D. P. Giesy, On a convexity condition in normed linear spaces, *Trans. Amer. Math. Soc.* **125** (1966), 114–146. Additions and corrections to "On a convexity condition in normed linear spaces", *Trans. Amer. Math. Soc.* **140** (1969), 511–512.

11. M. Gromov, *Metric structures for Riemannian and non-Riemannian spaces*, Based on the 1981 French original. With appendices by M. Katz, P. Pansu and S. Semmes. Translated from the French by S. M. Bates. Progress in Mathematics **152** (Birkhüser, 1999).

12. V. I. Gurarii, On openings and inclinations of subspaces of a Banach space (Russian), *Teor. Funktsii, Funktsional. Anal. i Prilozhen.* **1** (1965), 194–204.

13. V. I. Gurarii, M. I. Kadets and V. I. Matsaev, Distances between finite-dimensional analogs of the L_p-spaces (Russian), *Mat. Sb.* (N.S.) **70** (1966), 481–489; English transl. in *Amer. Math. Soc. Transl.* (2) **76** (1968), 207–216.

14. G. J. O. Jameson, *Summing and nuclear norms in Banach space theory* (Cambridge University Press, 1987).

15. M. I. Kadets, Note on the gap between subspaces (Russian) *Funkts. Anal. Prilozhen.* **9** (1975), no. 2, 73–74; English transl. in *Funct. Anal. Appl.* **9** (1975), 156–157.

16. N. J. Kalton and M. I. Ostrovskii, Distances between Banach spaces, *Forum Math.* **11** (1999), 17–48.

17. J. Lindenstrauss and L. Tzafriri, *Classical Banach spaces*, II, Ergebnisse der Mathematik und ihrer Grenzgebiete **97** (Springer-Verlag, 1979).

18. M. I. Ostrovskii, On the properties of the opening and related closeness characterizations of Banach spaces (Russian), *Teor. Funktsii, Funktsional. Anal. i Prilozhen.* **42** (1984), 97–107; English transl. in *Amer. Math. Soc. Transl.* (2) **136** (1987), 109–119.

19. M. I. Ostrovskii, Topologies on the set of all subspaces of a Banach space and related questions of Banach space geometry, *Quaestiones Math.* **17** (1994), 259–319.

20. P. Petersen, Gromov-Hausdorff convergence of metric spaces, in *Differential geometry: Riemannian geometry*, ed. by R. Greene and S. T. Yau, AMS, Providence, 1993, pp. 489–504.

21. W. Rinow, *Die innere Geometrie der metrischen Räume*, Grundlehren der mathematischen Wissenschaften **105** (Springer-Verlag, 1961).

22. A. F. Ruston, *Fredholm theory in Banach spaces* (Cambridge University Press, 1986).

23. J. J. Schäffer, *Geometry of spheres in normed spaces*, Lecture Notes in Pure and Applied Math. **20** (Marcel Dekker, 1976).

24. S. Semmes, Interpolation of Banach spaces, differential geometry and differential equations, *Revista Mat. Iberoamericana* **4** (1988), 155–176.

25. A. C. Thompson, *Minkowski geometry*, Encyclopedia of mathematics and its applications **63** (Cambridge University Press, 1996).