SOME APPROXIMATE RESULTS IN RENEWAL AND DAM THEORIES

R. M. PHATARFOD

(Received 14 August 1969)

Communicated by P. D. Finch

1. Introduction and preliminaries

It is well known that Wald's Fundamental Identity (F.I.) in sequential analysis can be used to derive approximate (and, sometimes exact) results in most situations wherein we have essentially a random walk phenomenon. Bartlett [2] used it for the gambler's ruin problem and also for a simple renewal problem. Phatarfod [18] used it for a problem in dam theory. It is the purpose of this paper to show how a generalization of the Fundamental Identity to Markovian variables, (Phatarfod [19]) can be used to derive approximate results in some problems in dam and renewal theories where the random variables involved have Markovian dependence. The reason for considering both the theories together is that the models usually proposed for both the theories – input distribution for dam theory, and life-distribution for renewal theory – are similar, and only a slight modification (to account for the 'release rules' in dam theory, plus the fact that we have two barriers) is necessary to derive results in dam theory from those of renewal theory.

The generalization of the F.I. is as follows: Let $X_0, X_1, X_2, \cdots$ be a Markov chain (discrete or continuous state space). Let S_N denote the cumulative sum $X_0+X_1+X_2+\cdots X_N$, and let n be the least positive integer such that S_N does not lie in the open interval (b, a), $(b \leqq 0, a > 0)$. If the m.g.f. (for real θ in an interval (θ_1, θ_2) around zero) of S_N can be written for large N as

$$M_N(\theta) \sim C(\theta)\lambda_1^N(\theta), \tag{1.1}$$

with

$$\lambda_1(0) = 1, \quad \lambda_1''(0) > \lambda_1'(0)^2,$$

then

$$E[\exp(\theta S_n)\lambda_1^{-n}(\theta)d(\theta|X_n)] = C(\theta) \tag{1.2}$$

for all real θ in (θ_1, θ_2) such that $\lambda_1(\theta) \geqq 1$. (The functions $C(\theta)$ and $d(\theta)$ have the relation $E[d(\theta|X_0)] = C(\theta)$).

From (1.2) one can derive results such as the probability of absorption at either barrier, probability distribution of 'n' etc. in a manner similar to the case when $\{X_i\}$ is a sequence of independent and identically distributed random variables.

A considerable amount of work has been done to establish conditions under which (1.1) holds. The problem is essentially of generalizing results of Perron and Frobenius for non-negative matrices. The earliest such generalization seems to be due to Jentzsch [9] for integral operators with a positive kernel. More recently results have been given by Krein and Rutman [11], Birkhoff [3], Karlin [10], Chung [4], Harris [8], and Vere-Jones [21]. For applications, the main problem is the determination of $\lambda_1(\theta)$, the 'largest eigen value', and it seems, to the present author, that this can be determined only by ad hoc methods.

We consider two important examples of Markov chains for which the quantity $\lambda_1(\theta)$ is easily determinable, and which serve as useful models for life-distribution in renewal theory and as input-distribution in dam theory. The cases considered a are:

(i) Gamma Markov Sequence. The conditional probability density of X_{r+1} given $X_r = y$ is given by

$$f(x|y) = \beta e^{-\alpha y-\beta x}\sum_{j=0}^{\infty}\frac{(\alpha y)^j}{j!}\frac{(\beta x)^{j+p-1}}{\Gamma(p+j)} \qquad (0 < x, y < \infty) \tag{1.3}$$

where $p > 0$, $0 < \alpha < \beta$.

(ii) Negative Binomial Markov chain. The transition probability

$$p_{ij} = \Pr\,[X_{r+1} = j|X_r = i]$$

is given by

$$p_{ij} = \frac{(p+i+j-1)!}{j!(p+i-1)!}\cdot\frac{\rho^j}{(1+\rho)^{p+i+j}} \qquad (0 \leqq i, j < \infty) \tag{1.4}$$

where $0 < \rho \leqq 1$, and p is a positive integer.

An extensive investigation of the processes i) and ii) has been carried out by Lampard [12] and Lee [13], who obtained them as output processes of a counter system. Among other results they show that the stationary distributions of the two processes are given by,

(i) the Gamma density:

$$p(x) = \frac{(\beta-\alpha)^p}{\Gamma(p)}x^{p-1}e^{-(\beta-\alpha)x} \qquad (x > 0) \tag{1.5}$$

and,

(ii) the negative binomial distribution:

$$p_i = \frac{(p+i-1)!}{i!(p-1)!}(1-\rho)^p\rho^i,\ (i = 0, 1, 2, 3, \cdots) \tag{1.6}$$

respectively.

In § 2 it is shown that for the cases considered above the m.g.f. of S_N is essentially of the form

$$M_N(\theta) = \frac{C}{A\mu_1^N(\theta)+B\mu_2^N(\theta)}$$

where $\mu_1(\theta)$, $\mu_2(\theta)$ are particular solutions of a homogeneous linear second-order difference equation. In an interval around zero, one of the solutions dominates the other, and hence as $N \to \infty$, the m.g.f. has the required asymptotic expression (1.1). §§ 3 and 4 deal with applications to renewal and dam theories respectively.

2. Derivation of moment generating functions

(i) Gamma Markov Sequence: Consider the Markov chain $X_0, X_1, X_2, \cdots$ where the conditional probability density of X_{r+1} given $X_r = y$ is given by (1.3). We have,

$$\begin{aligned} E[\exp(\theta X_{r+1})|x_r] &= e^{-\alpha x_r}\sum_{j=0}^{\infty}\frac{(\alpha x_r)^j}{j!}\int_0^\infty \frac{\beta^{j+p}x^{j+p-1}e^{-(\beta-\theta)x}}{\Gamma(j+p)}\,dx \\ &= \left(\frac{\beta}{\beta-\theta}\right)^p \exp\left(\frac{\alpha\theta x_r}{\beta-\theta}\right), \qquad (\theta<\beta). \end{aligned} \tag{2.1}$$

Assuming that X_0 has the stationary distribution given by (1.5), and writing $M_r(\theta) = E[\exp(\theta S_r)]$, we obtain,

$$M_0(\theta) = \left(\frac{\beta-\alpha}{\beta-\alpha-\theta}\right)^p, \qquad (\theta < \beta-\alpha), \tag{2.2}$$

$$\begin{aligned} M_1(\theta) &= \left(\frac{\beta}{\beta-\theta}\right)^p E\left\{\exp\left[\left(\theta+\frac{\alpha\theta}{\beta-\theta}\right)X_0\right]\right\}, \\ &= \left(\frac{\beta}{\beta-\theta}\right)^p\left(\frac{\beta-\alpha}{\beta-\alpha-K_1\theta}\right)^p, \qquad (K_1\theta<\beta-\alpha), \end{aligned} \tag{2.3}$$

where the operator K_1 is defined by $K_1\theta = \theta+\alpha\theta/(\beta-\theta)$. Now, because of the Markovian property,

$$E[\exp(\theta X_{r+1})|x_0, x_1, \cdots, x_r] = E[\exp(\theta X_{r+1})|x_r].$$

Hence, we have from (2.1), (2.2), and (2.3),

$$M_2(\theta) = \left(\frac{\beta}{\beta-\theta}\right)^p\left(\frac{\beta}{\beta-K_1\theta}\right)^p\left(\frac{\beta-\alpha}{\beta-\alpha-K_2\theta}\right)^p, \qquad (K_2\theta < \beta-\alpha),$$

where

$$K_2\theta = \theta+\frac{\alpha K_1\theta}{\beta-K_1\theta}.$$

https://doi.org/10.1017/S1446788700010284 Published online by Cambridge University Press

Continuing this way, we obtain the m.g.f. of S_N as

$$(2.4) \qquad M_N(\theta) = \left[\frac{\beta}{(\beta-\theta)}\cdot\frac{\beta}{(\beta-K_1\theta)}\cdots\frac{\beta}{(\beta-K_{N-1}\theta)}\cdot\frac{\beta-\alpha}{(\beta-\alpha-K_N\theta)}\right]^p$$

for all θ such that $\text{Max}_r\, K_r\theta < \beta-\alpha$, and where

$$(2.5) \qquad K_{r+1}\theta = \theta + \frac{\alpha K_r\theta}{\beta-K_r\theta}, \qquad (r = 0, 1, 2, \cdots, N-1);\ (K_0\theta = \theta).$$

Writing $\beta/(\beta-K_r\theta) = A_r/A_{r+1}$, $(r = 0, 1, 2, \cdots, N-1)$, reduces (2.5) to a second-order linear difference equation

$$(2.6) \qquad \beta A_{r+2} + (\theta-\alpha-\beta)A_{r+1} + \alpha A_r = 0.$$

Since, for $r = 0$, $A_0/A_1 = \beta/(\beta-\theta)$, we have the boundary conditions for the difference equation as $A_0 = \beta$, $A_1 = \beta-\theta$; from this we obtain

$$(2.7) \qquad A_r = \frac{\beta}{\mu_1-\mu_2}\{\mu_1(1-\mu_2)\mu_1^r - \mu_2(1-\mu_1)\mu_2^r\},$$

and hence, from (2.4),

$$(2.8) \qquad M_N(\theta) = \left[\frac{(\beta-\alpha)(\mu_1-\mu_2)}{\beta\{\mu_1^2(1-\mu_2)^2\mu_1^N - \mu_2^2(1-\mu_1)^2\mu_2^N\}}\right]^p,$$

where $\mu_1(\theta)$, $\mu_2(\theta)$ are the solution of $\beta x^2 + (\theta-\alpha-\beta)x + \alpha = 0$; we take,

$$\mu_1 = \frac{\alpha+\beta-\theta+\{(\alpha+\beta-\theta)^2-4\alpha\beta\}^{\frac{1}{2}}}{2\beta}, \quad \mu_2 = \frac{\alpha+\beta-\theta-\{(\alpha+\beta-\theta)^2-4\alpha\beta\}^{\frac{1}{2}}}{2\beta}.$$

It can be easily seen that for $\theta < \alpha+\beta-2\sqrt{\alpha\beta}$, $\mu_1(\theta) > \mu_2(\theta) > 0$, and $\mu_2(\theta) < 1$. We also note that for such values of θ, the condition $K_r\theta < \beta-\alpha$ is satisfied for $r = 0, 1, 2, \cdots, N$. Hence, as $N \to \infty$,

$$(2.9) \qquad M_N(\theta) \sim C(\theta)\lambda_1^N(\theta),$$

where

$$C(\theta) = \left\{\frac{(\beta-\alpha)(\mu_1-\mu_2)}{\beta\mu_1^2(1-\mu_2)^2}\right\}^p, \qquad \lambda_1(\theta) = \mu_1(\theta)^{-p}.$$

(ii) Negative Binomial Markov chain: Consider the Markov chain $X_0, X_1, X_2, \cdots$ where the conditional probability of $X_{r+1} = j$ given $X_r = i$ $(i = 0, 1, 2, 2, \cdots)$, is given by (1.4). It is convenient, in this case, to work with probability generating functions (p.g.f.). Writing $P_r(z) = E(z^{S_r})$, we have, in a manner similar to that in the previous case,

$$(2.10)\quad \begin{aligned} P_0(z) &= \left(\frac{1-\rho}{1-\rho z}\right)^p, \quad P_1(z) = \frac{(1-\rho)^p}{[1+\rho(1-z)]^p}\cdot\frac{1}{(1-\rho K_1 z)^p}, \\ P_N(z) &= \frac{(1-\rho)^p}{[\{1+\rho(1-z)\}\{1+\rho(1-K_1 z)\}\cdots\{1+\rho(1-K_{N-1} z)\}\{1-\rho K_N z\}]^p} \end{aligned}$$

where

$$(2.11)\quad K_{r+1} z = \frac{z}{1+\rho(1-K_r z)}, \ (r = 0, 1, 2, \cdots, N-1), \ (K_0 z = z).$$

Writing $\{1+\rho(1-K_r z)\}^{-1} = A_r/A_{r+1}$, we obtain the difference equation,

$$A_{r+2}-(1+\rho)A_{r+1}+z\rho A_r = 0, \quad \text{with } A_0 = 1, \ A_1 = 1+\rho(1-z).$$

We finally obtain,

$$(2.12)\quad M_N(\theta) = \left[\frac{(1-\rho)(\mu_1-\mu_2)}{\mu_1(1-\mu_2)^2\mu_1^N-\mu_2(1-\mu_1)^2\mu_2^N}\right]^p,$$

where

$$\mu_1(\theta),\ \mu_2(\theta) = \frac{(1+\rho)\pm\{(1+\rho)^2-4\rho e^\theta\}^{\frac{1}{2}}}{2}.$$

In this case, we have, that for $\theta < \log(1+\rho)^2/4\rho$, $\mu_1(\theta) > \mu_2(\theta) > 0$, and $\mu_2(\theta) < 1$, and hence as $N \to \infty$,

$$(2.13)\quad M_N(\theta) \sim C(\theta)\lambda_1^N(\theta),$$

where

$$C(\theta) = \left\{\frac{(1-\rho)(\mu_1-\mu_2)}{\mu_1(1-\mu_2)^2}\right\}^p, \quad \lambda_1(\theta) = \mu_1^{-p}(\theta).$$

Lee [13] has obtained results similar to (2.8) and (2.12) by using different methods.

3. Applications to renewal theory

One of the obvious generalizations of the standard renewal process is to assume that the lifetimes $\{X_r\}$ of articles to be renewed are not independent, but have a Markovian dependence. Wold, [22], [23] considered in some details the case where the conditional probability density of X_{r+1} given $X_r = y$ is given by $f(x|y) = \lambda(y)e^{-\lambda(y)x}$ where $\lambda(y) = \lambda y^{-\frac{1}{2}}$. Cox [5] and Cox and Lewis [6] considered the case where $\lambda(y)$ above, is replaced by $\lambda_0(1+\lambda_1 y)$. Here, we will consider the case where the dependence is given by (i) and (ii) of § 1. The Gamma Markov sequence is particularly useful because in the stationary case the first-order serial correlation is given by α/β, (Lampard [12]), and consequently can be controlled by varying both α and β in such a way that the stationary distribution given by (1.5) is not affected. By taking the particular case of $p = 1$, we obtain a sequence whose stationary distribution is a negative exponential.

We will be mainly interested in deriving the asymptotic c.f. $C_T(\phi)$ of the number N_T of renewals occurring in $(0, T)$. For simplicity, we consider an ordinary renewal process, i.e. at time $t = 0$, a new item is installed. The number N_T is then the least positive integer n such that $S_n \geqq T$, and its c.f. can be obtained by applying thr results in § 1.

Putting $\lambda_1(\theta) = e^{-i\phi}$ in (1.2), and neglecting the overshooting over the barrier T as well as the terms involving the initial and final states we obtain for the case (i)

$$\log C_T(\phi) \sim \frac{(\beta e^{i\phi/p}-\alpha)(e^{i\phi/p}-1)}{e^{i\phi/p}} T$$

from which, we obtain,

$$E(N_T) \sim \frac{(\beta-\alpha)}{p} T, \qquad V(N_T) \sim \frac{(\alpha+\beta)}{p^2} T.$$

For the case (ii), similarly.

$$C_T(\phi) \sim \left[\frac{\rho}{(1+\rho)e^{i\phi/p}-e^{2i\phi/p}}\right]^T,$$

giving

$$E(N_T) \sim \frac{(1-\rho)}{\rho p} T; \qquad V(N_T) \sim \frac{(1+\rho)}{\rho^2 p^2} T.$$

4. Applications to dam theory

A similar generalization to the theory of dams can be made, and in this case, the generalization is much more relevant. The theory of dams as formulated by Moran [17] and extended by Gani, Prabhu and others (for a review and references see Gani [7], Prabhu [20]) essentially dealt with the case of independent inputs to the dam. Recently, however, the case of Markov dependent inputs had received some attention. Lloyd [14], and Lloyd and Odoom [15], [16], considered the case when the sequence of inputs in a dam of finite capacity during consecutive time intervals form a Markov chain with a finite number of states. By considering the dam content Z_t at times $t = 0, 1, 2, \cdots$ just after release of the lesser of $M > 0$ or the total content, together with the inputs X_t in $(t, t+1)$ as a bivariate Markov chain, they derive the stationary probabilities of the dam content. Ali Kan and Gani [1] derive the time-dependent solutions of the same model for an infinite capacity dam. We will consider here the problem of emptiness of dam of a finite capacity dam fed by inputs forming a Markov chain as given in (i) and (ii) of § 1.

Let the initial content at time $t = 0$ be u $(0 < u < K)$, where K is the capacity of the dam. The net input in the dam during $(0, N)$, assuming it has neither overflowed nor become empty before, is $S'_N = \sum_{i=1}^{N} X_i - NM$, and we have a random walk problem with barriers 0 and K. For simplicity, we consider the case $M = 1$; this gives the m.g.f. of S'_N as

$$M_N'(\theta) \sim C(\theta)\lambda_1^N(\theta)e^{-N\theta}.$$

For the case (i), the unique non-zero solution of $\lambda_1(\theta)e^{-\theta} = 1$ is given by $\theta_0 = -p \log \delta$ where $\delta \neq 1$ is the unique solution of

$$(\beta\delta - \alpha)(\delta - 1) = p\delta \log \delta.$$

Putting $\theta = \theta_0$ in (1.2), $b = -u$, $a = K-u$, we obtain the probability P_u of the dam becoming empty before overlow as

$$P_u \sim \frac{\delta^{(K-u)p} - 1}{\delta^{Kp} - 1}.$$

For case (ii), the results are considerably simplified if we take $p = 1$, i.e. the stationary distribution is geometric. Equation $\lambda_1(\theta)e^{-\theta} = 1$ yields a cubic in $x = e^\theta$ which has only two positive roots, $x = 1$, and $x_0 = (-1+(1+4/\rho)^{\frac{1}{2}})/2$, from which we obtain

$$\theta_0 = \log \frac{-1 + \left(1 + \dfrac{4}{\rho}\right)^{\frac{1}{2}}}{2}.$$

This gives,

$$P_u \sim \frac{x_0^K - x_0^u}{x_0^K - 1}.$$

To find the p.g.f. of 'n' the time the dam becomes empty for the first time before overflow, we solve the equation $\lambda_1(\theta)e^{-\theta} = 1/s$ where $0 < s < 1$. This yields a cubic in $x = e^\theta$ whose positive roots are given by

$$x_1, x_2 = 2s^{\frac{1}{2}}\left(\frac{1+\rho}{3\rho}\right)^{\frac{1}{2}} \cos\left(\frac{\pi}{3} \pm \frac{\phi}{3}\right), \quad \phi = \cos^{-1}\left[\tfrac{3}{2}\left\{\frac{3\rho}{s(1+\rho)^3}\right\}^{\frac{1}{2}}\right].$$

Finally, the p.g.f. $G_u(s)$ is obtained by putting $\theta = \log x_1, \log x_2$ in (1.2) and solving the two resultant equations. This gives,

$$G_u(s) \sim (x_1 x_2)^u \frac{x_2^{K-u} - x_1^{K-u}}{x_2^K - x_1^K}.$$

Acknowledgement

I am grateful to Professor P. D. Finch for indicating to me the use of operators to derive the moment generating functions in § 2.

References

[1] M. S. Ali Khan and J. Gani, 'Infinite dams with inputs forming a Markov chain', *J. Appl. Prob.* 5 (1968), 72–83.

[2] M. S. Bartlett, *An Introduction to Stochastic Processes* (Cambridge Univ. Press., 1955).
[3] G. Birkhoff, 'Extensions of Jentzsch's theorem', *Trans. Amer. Math. Soc.* 85 (1957), 219–227.
[4] K. L. Chung, *Markov chains with stationary transition probabilities* (Springer-Verlag, Berlin, 1960).
[5] D. R. Cox, 'Some statistical methods connected with series of events', *J. R. Statist. Soc.* B17 (1955), 129–164.
[6] D. R. Cox and P. A. W. Lewis, *The statistical Analysis of Series of Events* (Methuen, London, 1966).
[7] J. Gani, 'Problems in the probability theory of storage systems', *J. R. Statist. Soc.* B19 (1957), 181–206.
[8] T. E. Harris, *The Theory of Branching Processes* (Springer-Verlag, Berlin, 1963).
[9] R. Jentzsch, 'Über Integralgleichungen mit positivem Kern', *J. Reine Angew. Math.* 141 (1912 235–244.
[10] S. Karlin, 'Positive Operators', *J. Math. Mech.* 8 (1957), 907–938.
[11] M. G. Krein and M. A. Rutman, 'Linear operators leaving invariant a cone in a Banach space' (*Amer. Math. Soc. Translation No.* 26., 1950).
[12] D. Lampard, 'A stochastic process whose successive intervals between events form a first order Markov chain — I', *J. Appl. Prob.* 5 (1968), 648–668.
[13] A. Lee, 'A stochastic process whose successive intervals between events form a first order Markov chain' — II, *Report No. MEE* 66-2 *Electrical Engineering Department, Monash University*, (1966).
[14] E. H. Lloyd, 'Reservoirs with correlated inflows', *Technometrics* 5 (1963), 85–93.
[15] E. H. Lloyd and S. Odoom, 'A note on the solution of dam equations', *J. R. Statist. Soc.* B 26 (1964), 338–344.
[16] E. H. Lloyd and S. Odoom, 'A note on the equilibrium distribution of levels in a semi-infinite reservoir subject to Markovian inputs and unit withdrawals', *J. Appl. Prob.*, 2 (1965), 215–222.
[17] P. A. P. Moran, 'A probability theory of dams and storage systems', *Aust. J. Appl. Sci.* 5 (1954), 116–124.
[18] R. M. Phatarfod, 'Application of methods in sequential analysis to dam theory', *Ann. Math. Statist.* 34 (1963), 1588–1592.
[19] R. M. Phatarfod, 'Sequential tests for normal Markov Sequence', (to be published).
[20] N. U. Prabhu, 'Time-dependent results in storage theory', *J. Appl. Prob.* 1 (1964), 1–46.
[21] D. Vere-Jones, 'Ergodic properties of non-negative matrices - I', *Pacific. J. Math.* 22 (1967), 361–386.
[22] H. Wold, 'On stationary point processes and Markov chains', *Skand. Aktuar* 31 (1948), 229–240.
[23] H. Wold, 'Sur les processus stationnaires ponctuels', *Colloques Internationaux du C.N.R.S.* 13 (1948), 75–86.

Monash University
Clayton, Victoria