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ON THE BLOCK STRUCTURE OF QUARTIC DESIGNS

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1. **Introduction.** Raghavarao and Chandrasekhararao [3] introduced a family of PBIB designs having three associate classes known as cubic designs. In this paper a detailed analysis of the case of PBIB designs having four associate classes, which are called quartic designs, is given. Results are obtained pertaining to construction and existence of quartic designs. Moreover, using methods similar to those used by Shah [5], [6], [7], the block structure of certain quartic designs is studied.

2. **Definition of the association scheme and parameters of quartic designs.** Let there be $v=s^4$ treatments numbered from 1 to s^4, with treatment number t assigned coordinates $(\alpha, \beta, \gamma, \delta)$ $(\alpha, \beta, \gamma, \delta=1, 2, \ldots, s)$ if

$$t = (\alpha-1)s^3+(\beta-1)s^2+(\gamma-1)s+\delta. \tag{2.1}$$

This may be interpreted geometrically by thinking of the s^4 treatments as lying in a four-dimensional hypercube of side s. Two treatments are then said to be ith associates if they differ in exactly i of their four coordinates.

Let n_i be the number of ith associates of each treatment and, if two treatments are kth associates, let p^k_{ij} be the number of treatments that are simultaneously ith associates of one and jth associates of the other. For the quartic association scheme, these parameters have the following values:

$$n_1 = 4(s-1), \quad n_2 = 6(s-1)^2, \quad n_3 = 4(s-1)^3, \quad n_4 = (s-1)^4 \tag{2.2}$$

and

$$P_1 = (p^1_{ij}) = \begin{bmatrix} s-2 & 3(s-1) & 0 & 0 \\ 3(s-1) & 3(s-1)(s-2) & 3(s-1)^2 & 0 \\ 0 & 3(s-1)^2 & 3(s-1)^2(s-2) & (s-1)^3 \\ 0 & 0 & (s-1)^3 & (s-1)^3(s-2) \end{bmatrix}$$

$$P_2 = (p^2_{ij}) = \begin{bmatrix} 2 & 2(s-2) & 2(s-1) & 0 \\ 2(s-2) & s^2 & 4(s-1)(s-2) & (s-1)^2 \\ 2(s-1) & 4(s-1)(s-2) & 2(s-1)(s^2-3s+3) & 2(s-1)^2(s-2) \\ 0 & (s-1)^2 & 2(s-1)^2(s-2) & (s-1)^2(s-2)^2 \end{bmatrix}$$

(2.3)

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(2.3)

$$P_3 = (p_{ij}^3) = \begin{bmatrix} 0 & 3 & 3(s-2) & s-1 \\ 3 & 6(s-2) & 3(s^2-3s+3) & 3(s-1)(s-2) \\ 3(s-2) & 3(s^2-3s+3) & (s-2)(s^2+2s-2) & 3(s-1)(s-2)^2 \\ s-1 & 3(s-1)(s-2) & 3(s-1)(s-2)^2 & (s-1)(s-2)^3 \end{bmatrix}$$

$$P_4 = (p_{ij}^4) = \begin{bmatrix} 0 & 0 & 4 & 4(s-2) \\ 0 & 6 & 12(s-2) & 6(s-2)^2 \\ 4 & 12(s-2) & 12(s-2)^2 & 4(s-2)^3 \\ 4(s-2) & 6(s-2)^2 & 4(s-2)^3 & (s-2)^4 \end{bmatrix}.$$

3. **Characterization of quartic designs.** Let $N=(n_{ij})$ be the $s^4 \times b$ incidence matrix of the quartic design, with $n_{ij}=1$ or 0 according as the ith treatment does or does not appear in the jth block. The matrix NN' is then symmetric with all diagonal elements equal to r and off-diagonal elements equal to Δ_1, Δ_2, Δ_3, or Δ_4, the (i,j)th element being Δ_k (which is the number of blocks in which two kth associates appear together) if treatments numbered i and j are kth associates. With the treatments numbered as in (2.1), NN' has the form

$$NN' = I_s \otimes (P-Q)+E_{ss} \otimes Q \tag{3.1}$$

in which I_s is the identity matrix of order s, E_{mn} is the $m \times n$ matrix with positive unit elements everywhere, $\otimes$ denotes the Kronecker product of matrices and, successively,

$$\begin{aligned} P &= I_s \otimes (X-Y)+E_{ss} \otimes Y, & Q &= I_s \otimes (Y-Z)+E_{ss} \otimes Z, \\ X &= I_s \otimes (A-B)+E_{ss} \otimes B, & Y &= I_s \otimes (B-C)+E_{ss} \otimes C, \end{aligned}$$
$$Z = I_s \otimes (C-D)+E_{ss} \otimes D,$$
$$\begin{aligned} A &= (r-\Delta_1)I_s+\Delta_1 E_{ss}, & B &= (\Delta_1-\Delta_2)I_s+\Delta_2 E_{ss}, \\ C &= (\Delta_2-\Delta_3)I_s+\Delta_3 E_{ss}, & D &= (\Delta_3-\Delta_4)I_s+\Delta_4 E_{ss}. \end{aligned}$$

Using this representation, the determinant of NN' can be found to be

$$|NN'| = \theta_0 \theta_1^{\alpha_1} \theta_2^{\alpha_2} \theta_3^{\alpha_3} \theta_4^{\alpha_4}, \tag{3.2}$$

where

$$\begin{aligned} \theta_0 &= r+4(s-1)\Delta_1+6(s-1)^2\Delta_2+4(s-1)^3\Delta_3+(s-1)^4\Delta_4 = rk \\ \theta_1 &= r+(3s-4)\Delta_1+3(s-1)(s-2)\Delta_2+(s-1)^2(s-4)\Delta_3-(s-1)^3\Delta_4 \\ \theta_2 &= r+2(s-2)\Delta_1+(s^2-6s+6)\Delta_2-2(s-1)(s-2)\Delta_3+(s-1)^2\Delta_4 \\ \theta_3 &= r+(s-4)\Delta_1-3(s-2)\Delta_2+(3s-4)\Delta_3-(s-1)\Delta_4 \\ \theta_4 &= r-4\Delta_1+6\Delta_2-4\Delta_3+\Delta_4 \end{aligned}$$

and

$$\alpha_1 = 4(s-1), \quad \alpha_2 = 6(s-1)^2, \quad \alpha_3 = 4(s-1)^3 \quad \text{and } \alpha_4 = (s-1)^4.$$

It can easily be shown that θ_0, θ_1, θ_2, θ_3, and θ_4 are the characteristic roots of NN' with multiplicities 1, α_1, α_2, α_3, and α_4 respectively.

4. **Analysis.** The usual intrablock analysis is based upon the additive model

$$Y_{ij} = \mu + t_i + b_j + e_{ij} \tag{4.1}$$

in which Y_{ij} is the yield of the plot in the jth block to which the ith treatment is applied, μ is the general effect, t_i is the effect of the ith treatment, b_j is the effect of the jth block, and the e_{ij} are independent $N(0, \sigma^2)$ variates. If $\hat{\mathbf{t}}$ is the column vector of the intrablock estimates of the treatment effects, then the reduced normal equations may be written

$$\mathbf{Q} = \mathbf{C}\hat{\mathbf{t}} \tag{4.2}$$

where $\mathbf{Q}=\mathbf{T}-(1/k)N\mathbf{B}$ and $\mathbf{C}=rI_v-(1/k)NN'$, $\mathbf{T}$ and $\mathbf{B}$ being the column vectors of the treatment and block totals respectively.

The characteristic roots of C are $\phi_0=0$, ϕ_1, ϕ_2, ϕ_3, and $\phi_4(\phi_i=r-(1/k)\theta_i$, $i=1, 2, 3, 4)$ with multiplicities 1, α_1, α_2, α_3, and α_4 respectively. The spectral decomposition of C into a linear combination of rational, symmetric, mutually orthogonal, idempotent matrices can then be obtained as

$$C = \phi_1 A_1 + \phi_2 A_2 + \phi_3 A_3 + \phi_4 A_4 \tag{4.3}$$

where, if $F_0 = I_{s^4}$,

$$\begin{aligned} F_1 &= I_{s^3} \otimes E_{ss} + I_{s^2} \otimes E_{ss} \otimes I_s + I_s \otimes E_{ss} \otimes I_{s^2} + E_{ss} \otimes I_{s^3}, \\ F_2 &= I_{s^2} \otimes E_{s^2s^2} + I_s \otimes E_{s^2s^2} \otimes I_s + E_{s^2s^2} \otimes I_{s^2} + E_{ss} \otimes I_{s^2} \otimes E_{ss} \\ &\quad + I_s \otimes E_{ss} \otimes I_s \otimes E_{ss} + E_{ss} \otimes I_s \otimes E_{ss} \otimes I_s, \\ F_3 &= I_s \otimes E_{s^3s^3} + E_{ss} \otimes I_s \otimes E_{s^2s^2} + E_{s^2s^2} \otimes I_s \otimes E_{ss} + E_{s^3s^3} \otimes I_s, \end{aligned} \tag{4.4}$$

and

$$F_4 = E_{s^4s^4},$$

then

$$\begin{aligned} A_1 &= (1/s^4)(sF_3 - 4F_4), \\ A_2 &= (1/s^4)(s^2F_2 - 3sF_3 + 6F_4), \\ A_3 &= (1/s^4)(s^3F_1 - 2s^2F_2 + 3sF_3 - 4F_4), \\ A_4 &= (1/s^4)(s^4F_0 - s^3F_1 + s^2F_2 - sF_3 + F_4). \end{aligned} \tag{4.5}$$

Following Shah [4], the solution of the reduced normal equations can be written, for a any real number, as

$$\hat{\mathbf{t}} = (\mathbf{C} + aE_{vv})^{-1}\mathbf{Q} = \{(E_{vv}/as^8) + \sum_{i=1}^{4} (A_i/\phi_i)\}\mathbf{Q}. \tag{4.6}$$

Thus

$$\begin{aligned} \hat{t}_i &= (1/\phi_4)Q_i + (1/s)(4Q_i + \sum Q_{i1})(1/\phi_3 - 1/\phi_4) \\ &\quad + (1/s^2)(6Q_i + 3\sum Q_{i1} + \sum Q_{i2})(1/\phi_2 - 2/\phi_3 + 1/\phi_4) \\ &\quad + (1/s^3)(4Q_i + 3\sum Q_{i1} + 2\sum Q_{i2} + \sum Q_{i3})(1/\phi_1 - 3/\phi_2 + 3/\phi_3 - 1/\phi_4) \end{aligned} \tag{4.7}$$

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in which $\sum Q_{ik}$ is the sum of the Q_j for all treatments that are kth associates of the treatment numbered i. Using the fact that $\sigma^2(\mathbf{C}+aE_{vv})^{-1}$ is the variance-covariance matrix of the design, where σ^2 is the intrablock error variance, it follows that the variance of a treatment contrast $\hat{t}_i-\hat{t}_j$ is

$$(4.8)\quad \operatorname{Var}(\hat{t}_i-\hat{t}_j) = \begin{cases} (2\sigma^2/s^3)[1/\phi_1+3(s-1)/\phi_2+3(s-1)^2/\phi_3+(s-1)^3/\phi_4], & \text{or} \\ (2\sigma^2/s^3)[2/\phi_1+(5s-6)/\phi_2+2(s-1)(2s-3)/\phi_3 & \\ \qquad +(s-1)^2(s-2)/\phi_4], & \text{or} \\ (2\sigma^2/s^3)[3/\phi_1+3(2s-3)/\phi_2+(2s-3)^2/\phi_3 & \\ \qquad +(s-2)(s^2-3s+3)/\phi_4], & \text{or} \\ (2\sigma^2/s^3)[4/\phi_1+6(s-2)/\phi_2+4(s^2-3s+3)/\phi_3 & \\ \qquad +(s-2)(s^2-2s+2)/\phi_4] & \end{cases}$$

according as treatments i and j are first, second, third, or fourth associates. The average variance of all such elementary treatment contrasts is then

$$(4.9)\qquad \{2\sigma^2/(1+s+s^2+s^3)\}[4/\phi_1+6(s-1)/\phi_2+4(s-1)^2/\phi_3+(s-1)^3/\phi_4]$$

and the efficiency of the design as compared to the randomized block design is

$$(4.10)\qquad (1+s+s^2+s^3)[4/\phi_1+6(s-1)/\phi_2+4(s-1)^2/\phi_3+(s-1)^3/\phi_4]^{-1}r^{-1}.$$

5. **Construction of some quartic designs**. The four dimensional lattice designs in blocks of size s can be seen to be quartic designs with parameters $v=s^4$, $b=4s^3$, $k=s$, $r=4$, $\Delta_1=1$, $\Delta_2=\Delta_3=\Delta_4=0$ if the basic pattern is taken only once. Consider, for example, the case of $s=3$. The four dimensional lattice design is then just a quartic design with $v=81$ treatments arranged into $b=108$ blocks of size $k=3$ such that each treatment is replicated $r=4$ times. The characteristic roots of NN' are just $\theta_0=12$, $\theta_1=9$, $\theta_2=6$, $\theta_3=3$, and $\theta_4=0$. The efficiency of this design is 0.5769.

In accordance with the method of Raghavarao and Chandrasekhararao (1964), quartic designs can also be derived from BIB designs as follows:

THEOREM 5.1. *If M is the incidence matrix of a BIB design with parameters $v^*=s$, b^*, k^*, r^*, and Δ, then $N=M\otimes M\otimes M\otimes M$ is the incidence matrix of a quartic design with parameters $v=s^4$, $b=b^{*4}$, $k=k^{*4}$, $r=r^{*4}$, $\Delta_1=r^{*3}\Delta$, $\Delta_2=r^{*2}\Delta^2$, $\Delta_3=r^*\Delta^3$, and $\Delta_4=\Delta^4$.*

Proof. Observing that $MM'=(r^*-\Delta)I_s+\Delta E_{ss}$, it follows that

$$\begin{aligned} NN' &= MM'\otimes MM'\otimes MM'\otimes MM' \\ &= (r^*-\Delta)^4F_0+\Delta(r^*-\Delta)^3F_1+\Delta^2(r^*-\Delta)^2F_2+\Delta^3(r^*-\Delta)F_3+\Delta^4F_4. \end{aligned}$$

Comparing this with (3.1), it is easy to verify that the parameters are as given.

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As an example, consider the BIB design with $v^*=3=b^*$, $k^*=2=r^*$, $\Delta=1$ and incidence matrix

$$M = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Then

$$N = \begin{bmatrix} K & K & 0 & K & K & 0 & 0 & 0 & 0 \\ 0 & K & K & 0 & K & K & 0 & 0 & 0 \\ K & 0 & K & K & 0 & K & 0 & 0 & 0 \\ 0 & 0 & 0 & K & K & 0 & K & K & 0 \\ 0 & 0 & 0 & 0 & K & K & 0 & K & K \\ 0 & 0 & 0 & K & 0 & K & K & 0 & K \\ K & K & 0 & 0 & 0 & 0 & K & K & 0 \\ 0 & K & K & 0 & 0 & 0 & 0 & K & K \\ K & 0 & K & 0 & 0 & 0 & K & 0 & K \end{bmatrix}$$

where K is the 9×9 matrix obtained by replacing each K in N by 1. This is then the incidence matrix of a quatric design with $v=81=b$, $k=16=r$, $\Delta_1=8$, $\Delta_2=4$, $\Delta_3=2$, and $\Delta_4=1$. The efficiency of this design is 0.9430.

6. Combinatorial properties of and necessary conditions for the existence of certain quartic designs.

THEOREM 6.1. *In a quartic design with $\theta_1=0$, k is divisible by s and every block of the design contains k/s treatments of the form $(\alpha, \beta, \gamma, \delta)$ $(\beta, \gamma, \delta=1, 2, \ldots, s)$ for every fixed α $(=1, 2, \ldots, s)$. Similarly for every fixed β, for every fixed γ, and for every fixed δ.*

Proof. For a given α, let $d_{\alpha i}$ be the number of treatments of the form $(\alpha, \beta, \gamma, \delta)$ in the ith block of the design. Then

$$\sum_{i=1}^{b} d_{\alpha i} = rs^3 \tag{6.1}$$

and

$$\sum_{i=1}^{b} d_{\alpha i}(d_{\alpha i}-1) = 3s^3(s-1)\Delta_1+3s^3(s-1)^2\Delta_2+s^3(s-1)^3\Delta_3. \tag{6.2}$$

Let $d_\alpha=(1/b)\sum_{i=1}^{b} d_{\alpha i}=k/s$. Then, since $\theta_1=0$,

$$\sum_{i=1}^{b} (d_{\alpha i}-d_\alpha)^2 = s^2(s-1)\theta_1 = 0. \tag{6.3}$$

Hence $d_{\alpha 1}=d_{\alpha 2}=\cdots=d_{\alpha b}=d_\alpha=k/s$. Since $d_{\alpha i}$ must be an integer, the theorem is proved.

The following two theorems can be proved in similar fashion:

THEOREM 6.2. *In a quartic design with* $\theta_1=\theta_2=0$, *k is divisible by* s^2 *and every block of the design contains* k/s^2 *treatments of the form* $(\alpha, \beta, \gamma, \delta)$ *for every fixed pair of elements of* α, β, γ, *and* δ.

THEOREM 6.3. *In a quartic design with* $\theta_1=\theta_2=\theta_3=0$, *k is divisible by* s^3 *and every block of the design contains* k/s^3 *treatments of the form* $(\alpha, \beta, \gamma, \delta)$ *for every fixed triple of elements of* α, β, γ, *and* δ.

These theorems give at once the following corollaries.

COROLLARY 6.1. *A necessary condition for the existence of a quartic design with* $\theta_1=0$ *is that k be divisible by s.*

COROLLARY 6.2. *A necessary condition for the existence of a quartic design with* $\theta_1=\theta_2=0$ *is that k be divisible by* s^2.

COROLLARY 6.3. *A necessary condition for the existence of a quartic design with* $\theta_1=\theta_2=\theta_3=0$ *is that k be divisible by* s^3.

Quartic designs with the following parameters are impossible as a result of these corollaries.

s	v	b	r	k	Δ_1	Δ_2	Δ_3	Δ_4	Reason
2	16	64	12	3	0	1	3	6	Corollary 6.1
2	16	32	18	9	7	10	12	8	Corollary 6.1
3	81	162	48	24	4	16	15	13	Corollary 6.2
3	81	324	96	24	12	28	33	24	Corollary 6.2
3	81	108	48	36	8	28	18	23	Corollary 6.3
4	256	432	162	96	27	72	57	62	Corollary 6.3

From Connor and Clatworthy [1], the characteristic roots of an existing design cannot be negative. Thus there follows

THEOREM 6.4. *A necessary condition for the existence of a quartic design is that* $\theta_i \geq 0$ $(i=1, 2, 3, 4)$.

Quartic designs with the following parameters are impossible as a result of Theorem 6.4.

s	v	b	r	k	Δ_1	Δ_2	Δ_3	Δ_4	Reason
2	16	8	5	10	5	2	3	1	$\theta_4 = -14$
3	81	18	8	36	4	3	5	1	$\theta_2 = -9$
4	256	36	9	64	9	2	1	3	$\theta_3 = -4$
5	625	80	16	125	6	1	3	4	$\theta_1 = -90$

A quartic design with $v=b$ will be said to be symmetric. For such a design, $\sum_{i=1}^{4} n_i\Delta_i=r(r-1)$ is always even. This will be so when s is even if and only if Δ_4 is either even or zero.

THEOREM 6.5. *A necessary condition for the existence of a symmetric quartic design with s even is that Δ_4 be even or zero.*

A quartic design with $\theta_i > 0$ $(i=1, 2, 3, 4)$ will be said to be regular. From Connor and Clatworthy [1], $|NN'|$ must be a perfect square for a regular symmetric quartic design. This will always be the case for s odd. For s even, taking Theorem 6.5 into account, this becomes

THEOREM 6.6. *A necessary condition for the existence of a regular symmetric quartic design with s even is that θ_4 be the square of an odd or even integer according as r is odd or even.*

As a result of Theorem 6.6, quartic designs with the following parameters are impossible.

s	$v=b$	$r=k$	Δ_1	Δ_2	Δ_3	Δ_4	Reason
2	16	7	3	4	1	2	$\theta_4=17$
2	16	10	5	7	5	8	$\theta_4=20$
4	256	31	10	6	3	2	$\theta_4=17$
4	256	36	6	8	4	4	$\theta_4=48$
6	1296	60	2	5	3	2	$\theta_4=72$
6	1296	71	11	10	4	2	$\theta_4=73$

Further necessary conditions for the existence of quartic designs can be obtained with the help of the Hasse–Minkowski p-invariant. For a discussion of the properties of the Legendre symbol, the Hilbert norm residue symbol and the Hasse–Minkowski p-invariant, see Shrikhande and Jain [8] or Ogawa [2].

For regular symmetric quartic designs the methods of either of the above two papers may be followed. If S is a $v \times v$ matrix whose columns form a set of rational, linearly independent eigenvectors corresponding to the characteristic roots of NN', then

$$S'NN'S = \operatorname{diag}(s^4\theta_0, \theta_1 Q_1, \theta_2 Q_2, \theta_3 Q_3, \theta_4 Q_4) \tag{6.4}$$

and

$$S'S = \operatorname{diag}(s^4, Q_1, Q_2, Q_3, Q_4) \tag{6.5}$$

where $\operatorname{diag}(a_1, a_2, \ldots, a_m)$ is the $m \times m$ diagonal matrix with diagonal entries $a_1, a_2, \ldots, a_m$, and Q_i is the gramian of the rational, linearly independent vectors corresponding to the root θ_i $(i=1, 2, 3, 4)$. Since S is a square matrix, taking determinants of both sides of (6.5) gives

$$|Q_1|\cdot|Q_2|\cdot|Q_3|\cdot|Q_4| \sim 1, \tag{6.6}$$

where $a \sim b$ means that the square free parts of a and b are the same. It follows from (6.4) that

$$C_p(NN') = C_p\{\operatorname{diag}(s^4\theta_0, \theta_1 Q_1, \theta_2 Q_2, \theta_3 Q_3, \theta_4 Q_4)\} \tag{6.7}$$

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and from (6.5) that

$$\left\{\prod_{i=1}^{4} C_p(Q_i)\right\}\left\{\prod_{i<j=1}^{4} (|Q_i|, |Q_j|)_p\right\} = +1 \tag{6.8}$$

for all primes p. (All results to follow will also be true for all primes p.)

Finally, using (6.6), (6.8) and the properties of the Hilbert norm residue symbol and the Hasse–Minkowski p-invariant, (6.7) becomes

$$\begin{aligned} C_p(NN') = (-1, -1)_p &\left\{\prod_{i=1}^{4} (-1, \theta_i)_p^{\alpha_i(\alpha_i+1)/2}\right\}\left\{\prod_{i=1}^{3} (\theta_i\theta_4, |Q_i|)_p\right\} \\ &\times\left\{\prod_{i<j=1}^{4} (\theta_i, \theta_j)_p^{\alpha_i\alpha_j}\right\}. \end{aligned} \tag{6.9}$$

(6.9) may be regarded as a necessary condition for the existence of a regular symmetric quartic design, but it may be simplified by determining $|Q_1|$, $|Q_2|$, and $|Q_3|$ as follows.

Define $4s$ $s^4 \times 1$ column vectors $\rho_{\alpha 1}$, $\rho_{\beta 2}$, $\rho_{\gamma 3}$, and $\rho_{\delta 4}$ as follows: each vector has s^3 unit entries (the remaining $s^4 - s^3$ entries being zeros) in the positions numbered

(i) $(\alpha-1)s^3 + a$; $a = 1, 2, \ldots, s$, for each $\rho_{\alpha 1}(\alpha = 1, 2, \ldots, s)$;

(ii) $as^3 + (\beta-1)s^2 + b$; $a = 0, 1, \ldots, s-1$, $b = 1, 2, \ldots, s$, for each $\rho_{\beta 2}(\beta = 1, 2, \ldots, s)$;

(iii) $(a-1)s^3 + (b-1)s^2 + (\gamma-1)s + c$; $a, b, c = 1, 2, \ldots, s$, for each $\rho_{\gamma 3}(\gamma = 1, 2, \ldots, s)$; and

(iv) $(a-1)s^3 + (b-1)s^2 + (c-1)s + \delta$; $a, b, c = 1, 2, \ldots, s$, for each $\rho_{\delta 4}(\delta = 1, 2, \ldots, s)$.

Among these $4s$ vectors, only $4(s-1)+1$ are linearly independent. Also, the vector space H generated by these vectors contains E_{v1}. Using methods similar to those in Shrikhande and Jain [8], the $4(s-1)$-dimensional subspace of H orthogonal to E_{v1} can be seen to be the proper space corresponding to the root θ_1 of NN'. Hence

$$\begin{bmatrix} s^4 & \\ & Q_1 \end{bmatrix} \sim \begin{bmatrix} s^4 & s^3E_{1p} & s^3E_{1p} & s^3E_{1p} & s^3E_{1p} \\ s^3E_{p1} & s^3I_p & s^2E_{pp} & s^2E_{pp} & s^2E_{pp} \\ s^3E_{p1} & s^2E_{pp} & s^3I_p & s^2E_{pp} & s^2E_{pp} \\ s^3E_{p1} & s^2E_{pp} & s^2E_{pp} & s^3I_p & s^2E_{pp} \\ s^3E_{p1} & s^2E_{pp} & s^2E_{pp} & s^2E_{pp} & s^3I_p \end{bmatrix} \tag{6.10}$$

in which $p = s-1$. Evaluating the determinant of each side of (6.10) there follows

$$|Q_1| \sim 1. \tag{6.11}$$

By similarly finding vectors to generate the proper spaces corresponding to the characteristic roots θ_2 and θ_3 of NN', it can also be shown that

$$|Q_2| \sim 1 \quad \text{and} \quad |Q_3| \sim 1. \tag{6.12}$$

Finally, by substituting the results (6.11) and (6.12) into (6.6), it follows that

$$|Q_4| \sim 1. \tag{6.13}$$

If the square free parts so obtained are substituted into (6.9), and if it is noted that $C_p(NN') = +1$ since $NN' \sim I_v$, there follows

THEOREM 6.7. *A necessary condition for the existence of a regular symmetric quartic design is that for all primes p*

$$\left\{\prod_{i=1}^{4}(-1, \theta_i)_p\right\}\left\{\prod_{i<j=1}^{4}(\theta_i, \theta_j)_p\right\} = +1. \tag{6.14}$$

For s odd, (6.14) is always satisfied. For s even, Theorems 6.6 and 6.7 together become

THEOREM 6.8. *Necessary conditions for the existence of a regular symmetric quartic design with s even are that θ_4 be a perfect square and, if so, then $(-1, \theta_2)_p = +1$ for all primes p.*

Quartic designs with the parameters listed below are impossible as a result of Theorem 6.8. The parameters given lead to θ_4 not a perfect square and $(-1, \theta_2)_p = -1$ for some prime p. It should be noted that the examples given following Theorem 6.6 are such that $(-1, \theta_2)_p = +1$ for all primes p.

s	$v=b$	$r=k$	Δ_1	Δ_2	Δ_3	Δ_4	Reason
4	256	25	5	3	2	2	$(-1, 33)_3 = -1$
4	256	30	5	3	3	4	$(-1, 56)_7 = -1$
6	1296	50	5	4	1	2	$(-1, 124)_2 = -1$
6	1296	56	14	7	1	2	$(-1, 220)_{11} = -1$

Shrikhande, Raghavarao and Tharthare [9] have given necessary conditions for the existence of a certain class of unsymmetric PBIB designs. If their result is applied to quartic designs it becomes, applying (6.11), (6.12), and (6.13)

THEOREM 6.9. *Necessary conditions for the existence of a quartic design with $b = s^4 - a$, where a is the sum of one or more of α_1, α_2, α_3, and α_4, and with zero a root of multiplicity a, are that*

$$\prod \theta_i^{\alpha_i} \sim 1 \tag{6.15}$$

and, if (6.15) *is satisfied, then for all primes p,*

$$\prod (1, \theta_i)_p^{\alpha_i(\alpha_i+1)/2} = +1 \tag{6.16}$$

where each product is taken over all i for which θ_i is a nonzero root.

There are fourteen cases of interest with one, two, or three of θ_1, θ_2, θ_3, and θ_4 equal to zero. For example, Theorem 6.9 may be simplified as follows when $\theta_2 = 0$.

COROLLARY 6.10. *Necessary conditions for the existence of a quartic design with only* $\theta_2=0$, *with* $a=6(s-1)^2$, *and with* $b=s^4-6(s-1)^2$, *are that*

(a) *if s is odd, that* θ_0 *be a perfect square, or*

(b) *if s is even, that* $\theta_0\theta_4$ *be a perfect square and, if so, that* $(-1, \theta_4)_p=+1$ *for all primes p.*

As a result of Corollary 6.10, the following quartic designs are impossible.

s	v	b	r	k	Δ_1	Δ_2	Δ_3	Δ_4	Reason
2	16	10	5	8	2	3	2	1	$\theta_0\theta_4=320$
2	16	10	5	8	1	4	1	3	$\theta_0\theta_4=960$

7. **On the block structure of certain quartic designs.** Shah [5], [6], [7] has given several results concerning the block structure of certain PBIB designs. Results similar to his are given for two classes of quartic designs. Type A quartic designs will be those having $\theta_1=\theta_2=\theta_3=0$. These designs satisfy Theorems 6.1, 6.2, and 6.3. Type B quartic designs will be those that have the block structure specified by Theorems 6.1, 6.2, and 6.3 but that do not necessarily have $\theta_1=\theta_2=\theta_3=0$. Obviously designs that are of type A are also of type B.

THEOREM 7.1. *The number m of treatments common to any two blocks of a quartic design satisfies the inequalities*

$$\max(0, T_1) \leq m \leq \min(k, T_2),$$

where $T_1=k(r-1)/(b-1)-T$ *and* $T_2=k(r-1)/(b-1)+T$, *if the design is of*

(a) *type A with* $b\geq 1+(s-1)^4$, *in which case*

$$T=k\{(b-1)(b-r)(v-k)(b-1\ -(s-1)^4\}^{1/2}\{s^2(s-1)^2(b-1)\}^{-1},$$

or of

(b) *type B with* $\theta_4\geq r(v-k)/(b-1)$ (*or, equivalently, with* $b\geq 1+r(v-k)/\theta_4$), *in which case*

$$T=\{(k/v)(b-2)(v-k)(\theta_4(b-1)-r(v-k))\}^{1/2}(b-1)^{-1}.$$

Proof. Having numbered the blocks, let x_i be the number of treatments common to block 1 and block i $(=2, 3, \ldots, b)$. Let $x_2=m$. Then

$$\sum_{i=3}^{b} x_i = k(r-1)-m, \tag{7.1}$$

and

$$\begin{aligned}\sum_{i=3}^{b} x_i(x_i-1) = {} & (k/s^3)\{4(k-s^3)\Delta_1+6(ks-2k+s^3)\Delta_2\\ & +4(ks^2-3ks+3k-s^3)\Delta_3\\ & +(ks^3-4ks^2+6ks-4k+s^3)\Delta_4\}\\ & -k(k-1)-m(m-1).\end{aligned} \tag{7.2}$$

(7.2) is just the number of pairs of treatments in blocks other than 1 and 2 that are common with pairs of treatments in block 1. The expression in braces can be determined by considering the number of such pairs that are of first associates, that are of second associates, that are of third associates and that are of fourth associates.

For type A designs, expressions for Δ_1, Δ_2, Δ_3, and Δ_4 can be obtained by solving the four simultaneous equations $\theta_0=rk$, $\theta_1=\theta_2=\theta_3=0$ and, with these (7.2) can be reduced to

$$(7.3)\qquad \sum_{i=3}^{b} x_i(x_i-1) = k^2\{(b-r)(v-k)-(s-1)^4(v-rk)\}/\{v(s-1)^4\} \\ -k(r-1)-m(m-1).$$

For type B designs, (7.2) can be reduced to

$$(7.4)\qquad \sum_{i=3}^{b} x_i(x_i-1) = (k/v)\{\theta_4(v-k)-k(v-rk)\}-k(r-1)-m(m-1).$$

In either case,

$$(7.5)\qquad 0 \le \sum_{i=3}^{b} (x_i-\bar{x})^2 = \sum_{i=3}^{b} x_i(x_i-1)+\sum_{i=3}^{b} x_i-\left\{\sum_{i=3}^{b} x_i\right\}^2/(b-2),$$

which may be reduced to a quadratic inequality in m whose solutions are just T_1 and T_2. Since, for every PBIB design, $0\le m\le k$, the result follows.

COROLLARY 7.1. *No two blocks of a quartic design are the same set if either*

(a) *the design is of type A with* (i) $1+(s-1)^4<b<2(1+(s-1)^4)$ *or* (ii) $b=r+(s-1)^4$ *and either* $b>2(r-1)$ *or* $v=nk(n>1)$; *or*

(b) *the design is of type B with* (i) $r(v-k)/(b-1)<\theta_4<2r(v-k)/(b-2)$ *or* (ii) $\theta_4=k$ *and either* $b>2(r-1)$ *or* $v=nk(n>1)$.

THEOREM 7.2. *The number d of blocks having exactly $m\,(\le k)$ treatments in common with a given block of a quartic design satisfies the inequality*

$$(7.6)\qquad d \le b-1-\{k(r-1)-m(b-1)\}/Q,$$

where $Q=P+m^2(b-1)-2km(r-1)$, if the design is of

(a) *type A with $b>1+(s-1)^4$ and*

$$P = \frac{k^2\{(b-r)(v-k)-(s-1)^4(v-rk)\}}{s^4(s-1)^4} \quad \textit{or}$$

(b) *type B with $\theta_4>r(v-k)/(b-1)$ and*

$$P = (k/v)\{\theta_4(v-k)-k(v-rk)\}.$$

Proof. With x_i as in Theorem 7.1, assume the blocks have been numbered so that $x_i=m$ for $i=2, 3, \ldots, d+1$. Then, much as before,

$$(7.7)\qquad \sum_{i=d+2}^{b} x_i = k(r-1)-dm$$

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and

$$\sum_{i=d+2}^{b} x_i(x_i-1) = P-k(r-1)-dm(m-1). \tag{7.8}$$

Here $\sum_{i=d+2}^{b}(x_i-\bar{x})^2 \geq 0$ leads to the inequality

$$dQ \leq (b-1)Q-\{k(r-1)-m(b-1)\}^2 \tag{7.9}$$

which gives the required result, since $Q>0$ for $b>1+(s-1)^4$ or for $\theta_4>r(v-k)/(b-1)$ as the case may be. If equality holds in (7.6), then

$$\sum_{i=d+2}^{b}(x_i-\bar{x})^2 = 0$$

implies that all x_i $(i=d+2,\ldots,b)$ are equal. Then from (7.7) the result follows.

COROLLARY 7.2. *Necessary and sufficient conditions for a given block of a quartic design to have the same number of treatments in common with each of the remaining $b-1$ blocks are that $m=k(r-1)/(b-1)$ be an integer and that, if the design is of type A, then $b=1+(s-1)^4$ or, if the design is of type B, then $\theta_4=r(v-k)/(b-1)$.*

THEOREM 7.3. *The number d of blocks of a quartic design disjoint with a given block satisfies the inequality*

$$d \leq \{(b-1)P-k^2(r-1)^2\}/P$$

if the design is of type A with $b>1+(s-1)^4$ or of type B with $\theta_4>r(v-k)/(b-1)$. Also, if some block has exactly that many blocks disjoint with it, then each nondisjoint block has $m'=P/\{k(r-1)\}$ treatments in common with the given block.

Proof. Set $m=0$ in Theorem 7.2.

THEOREM 7.4. *If a block of a quartic design with $v=kt$ and $b=rt$ $(t>1)$ has $t-1$ blocks disjoint with it, then necessary and sufficient conditions for that block to have the same number of treatments in common with each of the remaining $b-t$ blocks are that k/t be an integer and that, if the design is of type A, then $b=r+(s-1)^4$ or, if the design is of type B, then $\theta_4=k$.*

Proof. Set $d=t-1$, $m=0$, $v=kt$, and $b=rt$ in (7.9).

COROLLARY 7.4.1. *If a quartic design of type A is resolvable—that is, the b blocks are divided into r subgroups of b/r blocks each such that each subgroup contains each treatment exactly once—then $b=r+(s-1)^4$. If a quartic design of type B is resolvable, then $\theta_4=k$.*

COROLLARY 7.4.2. *A necessary and sufficient condition for a resolvable quartic design of type A or B to be affine resolvable—that is, to be resolvable such that any two blocks from different subgroups have the same number of treatments in common—is that it have a block which has the same number of treatments in common with each block belonging to a different subgroup.*

THEOREM 7.5. *In a quartic design of type A with* $b=r+(s-1)^4$ *or of type B with* $\theta_4=k$, *if* $v=kt$ *then any block of the design is disjoint with at most* $t-1$ *other blocks. If* t *is not a factor of* k, *then any block of the design is disjoint with at most* $t-2$ *other blocks.*

Proof. This follows from Theorem 7.3.

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