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SEMICRITICAL RINGS AND THE QUOTIENT PROBLEM

BY
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ABSTRACT. The purpose of this paper is to examine the relationship between the quotient problem for right noetherian nonsingular rings and the quotient problem for semicritical rings. It is shown that a right noetherian nonsingular ring R has an artinian classical quotient ring iff certain semicritical factor rings R/K_i, $i=1,\ldots,n$, possess artinian classical quotient rings and regular elements in R/K_i lift to regular elements of R for all i. If R is a two sided noetherian nonsingular ring, then the existence of an artinian classical quotient ring is equivalent to each R/K_i possessing an artinian classical quotient ring and the right Krull primes of R consisting of minimal prime ideals. If R is also weakly right ideal invariant, then the former condition is redundant. Necessary and sufficient conditions are found for a nonsingular semicritical ring to have an artinian classical quotient ring.

1. **Introduction and definitions.** A ring R is *semicritical* provided there are right ideals J_i, $i=1,\ldots,m$, such that R/J_i is a critical R-module for all i and $\bigcap_{i=1}^{m} J_i=0$. According to [1, 2], the class of semicritical rings properly contains the class of semiprime rings with Krull dimension. The purpose of this paper is to examine the relationship between the quotient problem for right noetherian, nonsingular rings and the quotient problem for semicritical rings.

In section 2, we show that a right noetherian, right nonsingular ring R has an artinian classical quotient ring iff certain semicritical factor rings R/K_i, $i=1,\ldots,n$ possess artinian classical quotient rings, and regular elements of R/K_i lift to regular elements of R for all i. The ideals K_i that we consider are contained in the prime radical N. Furthermore, the rings R/K_i may not be semiprime.

The remainder of section 2 is devoted to examining the quotient problem for two sided noetherian, nonsingular rings, Let rt-K prime $(R)=\{\operatorname{ass}(X_j/X_{j-1}) \mid j=1,\ldots,s\}$ where $0=X_0<\cdots<X_s=R$ is a critical composition series for R. It is shown that a two sided noetherian, nonsingular ring R has an artinian classical quotient ring iff rt-Kprime(R) consists of minimal prime ideals, and R/K_i has an artinian classical quotient ring for all i. As a corollary, when R is smooth, R has an artinian classical quotient ring iff R/K_1

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has an artinian classical quotient ring. For a two sided noetherian, nonsingular, weakly right ideal invariant ring R, the existence of an artinian classical quotient ring is equivalent to rt-Kprime(R) consisting of minimal prime ideals.

In section 3, we examine the quotient problem for right nonsingular, semicritical rings. Our main result gives necessary and sufficient conditions for such a ring to possess an artinian classical quotient ring, and extends a result in [2] concerning the quotient problem for α-primitive rings.

Unless otherwise stated, 'module' will mean unitary right R-module, and $|M|$ will denote the *Krull dimension* of a module M. A ring R is called α-*primitive* provided there exists a faithful, critical module C with $|C| = \alpha$. An ideal D of a ring R is α-*coprimitive* provided R/D is a α-primitive ring. A module M is called *smooth* provided $|N| = |M|$ for all non-zero submodules $N \leq M$. We use the notation $N \leq_e M$ to mean that N is an *essential* submodule of a module M. The *right annihilator* in R of a subset S of a module M will be denoted by rt(S). If C is a critical module, then the *assassinator* of C, denoted ass(C), is the set of elements of R which annihilate a non-zero submodule of C. The *critical socle* SM of a module M is the sum in M of all critical submodules of M. An indecomposable injective module E is called a α-*indecomposable injective* provided E contains a α-critical submodule. The unmodified terms 'noetherian' and 'nonsingular' will mean that these conditions hold on both sides. Finally, 'quotient ring' will mean right quotient ring.

2. **The quotient problem.** In this section, R will always denote a right noetherian, right nonsingular ring with Krull dimension sequence $0 \leq \alpha_n < \cdots < \alpha_1 = |R|$. Here, the ordinals α_i are the ordinals that occur as the Krull dimension of non-zero right ideals of R. For each i, let S_i be the unique maximal right ideal of R with $|S_i| = \alpha_i$, and let $A_i = \mathrm{rt}(S_i)$. Then S_i and A_i are two sided ideals for all i and $0 \neq S_n < \cdots < S_1 = R$ and $0 = A_1 \leq A_2 \leq \cdots \leq A_n$.

Recall that a submodule N of a module M is *closed* in M provided N has no proper essential extensions in M.

2.1 PROPOSITION. *For all i, R/A_i is a nonsingular right R-module and $|R/A_i| = \alpha_i$.*

Proof. Since S_i is nonsingular, $\mathrm{rt}(y_1, \ldots, y_s)$ is closed in R for all $y_1, \ldots, y_s \in S_i$. By [8; 3.14], closed submodules of R satisfy the descending chain condition. It follows that $A_i = \mathrm{rt}(x_1, \ldots, x_t)$ for some $x_1, \ldots, x_t \in S_i$. Thus, there is a 1–1 map $R/A_i \to S_i^t$ and hence, R/A_i is a nonsingular right R-module with $|R/A_i| \leq \alpha_i$. Since S_i is an R/A_i-module, $|R/A_i| \geq |S_i| = \alpha_i$. Therefore, $|R/A_i| = \alpha_i$.

An ideal D of R is called a α_i,A_i-coprimitive ideal of R provided $D \geq A_i$ and D/A_i is a α_i-coprimitive ideal of R/A_i. Central to our results is the following characterization of the minimal α_i,A_i-coprimitive ideals of R.

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2.2 Proposition. *An ideal D of R is a minimal α_i,A_i-coprimitive ideal iff there is a α_i-indecomposable injective R/A_i-module E, unique up to isomorphism, such that $D=\mathrm{rt}(SE)$. Furthermore, for all i, minimal α_i,A_i-coprimitive ideals of R exist and are finite in number.*

Proof. The first statement follows from [2; 2.2, 2.4].

Since $|R/A_i|=\alpha_i$ by 2.1, R/A_i contains a prime ideal P/A_i such that $|R/P|=\alpha_i$ by [9; 7.5]. Let $C\leq R/P$ be a critical right ideal. Clearly, C is a α_i-critical R/A_i-module. Existence then follows by [2; 2.4]. That there are only finitely many minimal α_i, A_i-coprimitive ideals follows by [2; 2.5].

For each i, let $K_i=N\cap D_1'\cap\cdots\cap D_k'$ where N is the prime radical of R and $D_1',\ldots,D_k'$ are all of the minimal α_i,A_i-coprimitive ideals of R. Then by [1; 3.6], R/K_i is a semicritical ring for all i. We will show that the quotient problem reduces to the quotient problem for each of the rings R/K_i.

In the following, $C(J)$ will denote the set of elements of R which are regular modulo a two sided ideal J. Note that by [8; 3.33], the right regular elements of R are regular, and $C(0)=\{x\in R\mid xR\leq_e R\}$.

2.3 Lemma. *Let S be a ring with $|S|=\alpha$.*

(a) *If S is a α-primitive ring, then any smooth right S-module M with $|M|=\alpha$ is nonsingular.*

(b) *Suppose S is a α-primitive ring. Then a right ideal J of S is essential in S iff $|S/J|<\alpha$.*

(c) *If D is a α-coprimitive ideal of S, then $|S/D|=\alpha$.*

(d) *If D is a α-coprimitive ideal of S, then $C(0)\subseteq C(D)$.*

Proof. If S is α-primitive and $|S|=\alpha$, then there exists a faithful, nonsingular, α-critical right S-module by [1; 3.5]. Statements (a) and (b) then follow from [1; 4.1(2), 2.6].

(c) Since D is a α-coprimitive ideal, S/D has a faithful α-critical module C. Thus, $\alpha=|S|\geq|S/D|\geq|C|=\alpha$ and hence, $|S/D|=\alpha$.

For (d), let $c\in C(0)$. Then $|S/cS|<\alpha$ and therefore, $|S/cS+D|<\alpha$. By (c) and (b), $cS+D/D\leq_e S/D$. According to [1; 4.1(3)], S/D is smooth. Thus, by (a), S/D is right nonsingular. Therefore, $c\in C(D)$.

2.4 Lemma. *For all i, $C(0)\subseteq C(K_i)$.*

Proof. Let $c\in C(0)$. Then $cR\leq_e R$ and, since A_i is closed in R by 2.1, $cR+A_i/A_i\leq_e R/A_i$. Thus, $c\in C(A_i)$. Let D be a minimal α_i,A_i-coprimitive ideal of R. Then by 2.1 and 2.3(d), $c+A_i\in C(D/A_i)$. Therefore, $c\in C(D)$ for all minimal α_i,A_i-coprimitive ideals D. Now, let $r=R$ with $cr\in K_i$. By [7; 2.3, 2.5], $C(0)\subseteq C(N)$. Then $cr\in N$ and hence, $r\in N$. Also, $cr\in D$ and hence, $r\in D$ for all minimal α_i, A_i-coprimitive ideals D. Thus, $r\in K_i$. Similarly, $rc\in K_i$ implies that $r\in K_i$. Therefore, $c\in C(K_i)$.

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2.5 Theorem. *The following are equivalent*:

(1) *R has an artinian classical quotient ring.*

(2) *For all i, R/K_i has an artinian classical quotient ring and $C(0) = C(K_i)$.*

(3) *For all i, R/K_i has an artinian classical quotient ring and $C(0) = \bigcap_{i=1}^{n} C(K_i)$.*

Proof. (1) implies (2): The prime radical of R/K_i is N/K_i. Let $c + K_i \in C(N/K_i)$. Then $c \in C(N) = C(0)$. By 2.4, $c \in C(K_i)$. Therefore, R/K_i has an artinian classical quotient ring by Small [11].

For the second statement, let $c \in C(K_i)$. By [7; 2.3, 2.5], $c + K_i \in C(N/K_i)$ and hence, $c \in C(N)$. Thus, $c \in C(0)$ by Small [11]. Therefore, $C(K_i) \subseteq C(0)$. Equality follows by 2.4.

(2) implies (3): Clear.

(3) implies (1): Let $c \in C(N)$. Clearly, $c + K_i \in C(N/K_i)$ for all i. Since R/K_i has an artinian classical quotient ring, $c \in C(K_i)$ for all i. Therefore, $c \in C(0)$ and hence, R has an artinian classical quotient ring by Small [11].

The following demonstrates that even when R has an artinian classical quotient ring, the rings R/K_i may not be semiprime. Thus, 2.5 is different than Small's result [11].

2.6 Example. Let Z denote the integers, Q the rationals and $Q[x]$ the ordinary ring of polynomials over Q. Let

$$R = \begin{bmatrix} Z & Q & Q[x] & Q[x] \\ 0 & Q & Q[x] & Q[x] \\ 0 & 0 & Q[x] & Q[x] \\ 0 & 0 & 0 & Q[x] \end{bmatrix}$$

be the indicated ring of matrices with the usual operations. Then R has an artinian classical quotient ring. According to [3; 3.6], $|R| = 1$ and the minimal 1-coprimitive ideals of R are:

$$D_1 = \begin{bmatrix} 0 & Q & Q[x] & Q[x] \\ 0 & Q & Q[x] & Q[x] \\ 0 & 0 & Q[x] & Q[x] \\ 0 & 0 & 0 & Q[x] \end{bmatrix}, \quad D_2 = \begin{bmatrix} Z & Q & Q[x] & Q[x] \\ 0 & 0 & 0 & Q[x] \\ 0 & 0 & 0 & Q[x] \\ 0 & 0 & 0 & Q[x] \end{bmatrix},$$

$$D_3 = \begin{bmatrix} Z & Q & Q[x] & Q[x] \\ 0 & Q & Q[x] & Q[x] \\ 0 & 0 & Q[x] & Q[x] \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

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It is easy to see that

$$K_1 = \begin{bmatrix} 0 & Q & Q[x] & Q[x] \\ 0 & 0 & 0 & Q[x] \\ 0 & 0 & 0 & Q[x] \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

and that K_1 is properly contained in the prime radical of R.

The next example show that in general, the two conditions of 2.5(2) are necessary for R to have an artinian classical quotient ring.

2.7 EXAMPLE. By [2; 4.1], there is a 2-primitive ring S that *does not* have an artinian classical quotient ring. Let T be any prime ring with $|T|=3$. Consider $R=S\times T$ with coordinate-wise addition and multiplication. Then R has Krull dimension sequence $2<3=|R|$. Also, $K_2=S\times 0\cap N(s)\times 0=N(S)\times 0$. Thus, as rings, $R/K_2\simeq S/N(S)\times T$ and therefore, R/K_2 has an artinian classical quotient ring. Let $c\in C(N(S))-C(0)\subseteq S$. Then $(c+N(S),\ 1)$ is regular in R/K_2, but $(c, 1)$ is not regular in R. Therefore, $C(0)\neq C(K_2)$. On the other hand, $K_1=0$ and R/K_1 does not have an artinian classical quotient ring. Obviously, $C(0)=C(K_1)$.

Next, we examine the quotient problem for two sided noetherian, nonsingular rings. An example from [6] provides an example of such a ring which does not possess an artinian classical quotient ring.

2.8 EXAMPLE. Let $Z_p=Z/pZ$ where p is a prime number. Let

$$R = \begin{bmatrix} Z & 0 & Z_p \\ Z_p & Z_p & Z_p \\ 0 & 0 & Z_p \end{bmatrix}.$$

Then R is a ring under the usual matrix operations. According to [6; p. 382], R is a two sided noetherian *PI* ring without an artinian classical quotient ring. If I is an essential right ideal of R, then

$$J = \begin{bmatrix} pmZ & 0 & Z_p \\ 0 & 0 & Z_p \\ 0 & 0 & Z_p \end{bmatrix} \leq I$$

for some $m\neq 0$. It is easily checked that J and therefore, I is not the right annihilator of any non-zero element of R. Thus, R is right nonsingular. If I is an essential left ideal of R, then

$$\begin{bmatrix} pkZ & 0 & 0 \\ Z_p & Z_p & Z_p \\ 0 & 0 & 0 \end{bmatrix} \leq I$$

for some $k\neq 0$. A similar argument shows that R is left nonsingular.

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Since R is right noetherian, there is a series of right ideals $0 = X_0 < \cdots < X_s = R$ such that for all j, X_j/X_{j-1} is a β_j-critical module for a sequence of ordinals $\beta_1 \leq \cdots \leq \beta_s$. Such a series is called a *critical composition series of R*. The distinct ordinals among the β_j are just the ordinals α_i in the Krull dimension sequence of R. Furthermore, the ideals S_i appear in every critical composition series of R. Thus, if X_j/X_{j-1} is α_i-critical, then $|X_j| = \alpha_i$. Therefore, when X_j/X_{j-1} is α_i-critical, X_j/X_{j-1} is a R/A_i-module.

In 2.9, 2.10 and 2.11, E_j will denote the R/A_i-injective hull of X_j/X_{j-1} when X_j/X_{j-1} is α_i-critical, and $D_j = \text{rt}(SE_j)$ and $Q_j = \text{ass}(X_j/X_{j-1})$ for all j. By 2.2, D_j is a minimal α_i,A_i-coprimitive ideal. Since the R-injective hulls of the critical factors of any two critical composition series are pair-wise isomorphic after a suitable permutation of indices, the corresponding R/A_i-injective hulls are isomorphic. It follows that the modules E_j and the ideals D_j and Q_j are independent of the choice of such a series.

2.9 LEMMA. *For* $j = 1, \ldots, s$,

(a) $D_j \leq Q_j$

(b) $|R/D_j| = |R/Q_j|$

(c) *There are critical submodules* $C' \leq C \leq SE_j$ *such that* $D_j = \text{rt}(C)$ *and* $Q_j = \text{ass}(C')$.

Proof. Statement (a) is obvious.

(b) Since D_j/A_i is a α_i-coprimitive ideal of R/A_i for some i, $|R/D_j| = \alpha_i = |R/A_i|$ by 2.1 and 2.3(c). Then by [1; 4.2], $|R/Q_j| = \alpha_i = |R/D_j|$.

(c) Since D_j/A_i is a α_i-coprimitive ideal of R/A_i for some i and $|R/A_i| = \alpha_i$ by 2.1, D_j/A_i is the right annihilator in R/A_i of some α_i-critical R/A_i-module $C \leq SE_j$ by [2; 2.2]. Considering C as an R-module, $D_j/A_i = \text{rt}(C)/A_i$. Thus, $D_j = \text{rt}(C)$. Then $Q_j = \text{ass}(C)$ and hence, there is a submodule $C' \leq C$ for which $Q_j = \text{rt}(C')$.

The *right Krull primes of R*, denoted by rt-Kprime(R), is the set of prime ideals $\{Q_j \mid j = 1, \ldots, s\}$.

2.10 LEMMA. *Suppose* rt-Kprime(R) *consists of minimal prime ideals. If* $|R/D_j| = \alpha_i$, *then* $C(K_i) \subseteq C(D_j)$.

Proof. Let $K_i = D'_1 \cap \cdots \cap D'_t \cap P_1 \cap \cdots \cap P_q$ be an irredundant intersection where each D'_k is a minimal α_i,A_i-comprimitive ideal, and each P_k is a minimal prime ideal. Let $D = D_j$ and let $Q = Q_j$. Since $K_i \leq Q$ and Q is a prime ideal, $D'_k \leq Q$ for some $k = 1, \ldots, t$ or $P_k \leq Q$ for some $k = 1, \ldots, q$. Now, by 2.9(c), there are critical submodules $C' \leq C \leq SE_j$ such that $D = \text{rt}(C)$ and $Q = \text{rt}(C')$. If $D'_k \leq Q$, then $C'D'_k = 0$. By [2; 2.1], $D'_k = D$. If $P_k \leq Q$, then $P_k = Q$ since Q is a minimal prime ideal. Thus, $D \leq Q = P_k$ by 2.9(a). In either case, the intersection representing K_i can be rewritten as an irredundant intersection $K_i = D'_1 \cap \cdots \cap D'_p \cap P_1 \cap \cdots \cap P_r$ where $D'_1 = D$.

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Now, D/K_i is closed in R/K_i. If not, then $D/K_i \leq_e J/K_i$ for some right ideal $J \neq D$. Thus, $C(J/K_i) = CJ \neq 0$ and hence, there is a onto map $J/K_i \to C''$ where $0 \neq C'' \leq C$. Since $D/K_i \leq_e J/K_i$ and each D'_k/K_i is a two sided ideal, $\bigcap_{h\neq 1} D'_h \cap P_1 \cap \cdots \cap P_r \leq \text{rt}(J/K_i) \leq \text{rt}(C'')$. Also, since $Q = \text{ass}(C)$ and $0 \neq C'' \leq C$, $\text{rt}(C'') \leq Q$. Then Q a prime ideal implies that $D'_h \leq Q$ for some $h \neq 1$ or $P_s \leq Q$ for some $s = 1, \ldots, r$. In the first case, $C'D'_h = 0$. By [2; 2.2], $D'_h = D = D'_1$. If $P_s \leq Q$, then $P_s = Q$ since Q is a minimal prime ideal. Thus, $D \leq P_s$. In either case, the intersection representing K_i is no longer irredundant. Therefore, D/K_i is closed in R/K_i.

Let $c \in C(K_i)$. Then $cR + K_i/K_i \leq_e R/K_i$, and by closure of D/K_i, $cR + D/D \leq_e R/D$. Therefore, $c \in C(D)$.

2.11 Theorem. *Let R be noetherian and nonsingular. Then R has an artinian classical quotient ring iff* rt-Kprime(R) *consists of minimal prime ideals and R/K_i has an artinian classical quotient ring for all i.*

Proof. Necessity: For the first statement, let $Q = Q_j$ be a member of rt-Kprime(R). By [7; 2.14], it suffices to show that $Q \cap C(N)$ is empty. Let $c \in C(N)$. Since R has an artinian classical quotient ring, $c \in C(0)$ by Small [11]. Then $cR \leq_e R$. Now, $D = D_j \leq Q$ by 2.9, and D is a minimal α_i, A_i-coprimitive ideal of R for some i. By 2.1, A_i is closed in R. Therefore, $cR + A_i/A_i \leq_e R/A_i$ and hence, $c \in C(A_i)$. By 2.3(d), $c \in C(D)$. Thus, $cR + D/D \leq_e R/D$. According to 2.3(b) and 2.9(b), $|R/cR + Q| \leq |R/cR + D| < |R/D| = |R/Q|$. Therefore, by 2.3(b), $cR + Q/Q \leq_e R/Q$. Then $c \in C(Q)$ and therefore, $c \notin Q$.

The second statement follows by 2.5.

Sufficiency: By 2.5, it suffices to show that $\bigcap_{i=1}^n C(K_i) = C(0)$. Let $c \in \bigcap_{i=1}^n C(K_i)$ and suppose $rc = 0$ for some $r \in R$. Let $0 = X_0 < \cdots < X_s = R$ be a critical composition series of R as in the comment preceeding 2.9. If $r \in X_k$, then $r + X_{k-1} \in X_k/X_{k-1}$. Now, X_k/X_{k-1} is a α_i-critical module for some i. Since $|R/D_k| = \alpha_i$ by 2.3(c), X_k/X_{k-1} is a nonsingular R/D_k-module by 2.3(a). Since $c \in \bigcap_{i=1}^n C(K_i)$, $c \in C(D_k)$ by 2.10. Then $(r + X_{k-1})(c + D_k) = 0$ and X_k/X_{k-1} nonsingular over R/D_k implies that $r \in X_{k-1}$. Eventually, $r \in X_0 = 0$. Thus, c is left regular. Then R left noetherian and left nonsingular implies that $c \in C(0)$. By 2.4, $C(0) \subseteq \bigcap_{i=1}^n C(K_i)$. Therefore, $C(0) = \bigcap_{i=1}^n C(K_i)$.

2.12 Corollary. *Let R be noetherian and nonsingular. If R has an artinian classical quotient ring, then R/K_i is a right nonsingular ring for all i.*

Proof. Let $K_i = D'_1 \cap \cdots \cap D'_t \cap P_1 \cap \cdots \cap P_q$ be an irredundant intersection as in 2.10. We will first show that for all k, the ideals D'_k/K_i and P_k/K_i are closed in R/K_i. Suppose $D'_k/K_i \leq_e J/K_i$ for some right ideal $J \neq D'_k$. Let $D'_k = \text{rt}(C)$ and let $P = \text{ass}(C)$ where C is a α_i-critical module. Since the intersection representing K_i is irredundant, there are right ideal I_j, $j = 1, \ldots, p$ such that

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each R/I_j is a critical module, and $K_i = I_1 \cap \cdots \cap I_p$ is an irredundant intersection, and for some s, R/I_s contains a non-zero submodule which is subisomorphic to C. Now, the canonical map $R/K_i \to \bigoplus_{j=1}^{p} R/I_j$ is an essential monomorphism. Thus, R/K_i contains a non-zero critical submodule which is subisomorphic to R/I_s. Consequently, R/K_i contains a non-zero critical submodule which is subisomorphic to C. Therefore, P/K_i equals the assassinator in R/K_i of a critical right ideal of R/K_i. By 2.11, R/K_i has an artinian classical quotient ring. Then according to [6; 3], P/K_i is a minimal prime ideal of R/K_i. As in the proof of 2.10, $D'_h \leq P$ for some $h \neq k$ or $P_j \leq P$ for some $j = 1, \ldots, q$. If $P_j \leq P$, then $P_j/K_i \leq P/K_i$ and hence, $P_j = P$. As in 2.10, both cases supply a contradiction. On the other hand, suppose $P_k/K_i \leq_e J/K_i$. Since R/P_k is a prime ring, $P_k = \mathrm{rt}(U)$ where $U \leq R/P_k$ is a critical module. If $UJ \neq 0$, then there is an epimorphism $J \to U'$ for some submodule $0 \neq U' \leq U$. Since $P_k/K_i \leq_e J/K_i$, $D_1 \cap \cdots \cap D_t \cap (\bigcap_{h \neq k} P_h) \leq \mathrm{rt}(J) \leq \mathrm{rt}(J) \leq \mathrm{rt}(U') = P_k$ which violates the irredundancy of the intersection representing K_i.

Let $L/K_i \leq_e R/K_i$ and suppose $xL \leq K_i$ for some $x \in R$. By closure of D'_k/K_i, $L + D'_k/D'_k \leq_e R/D'_k$ for $k = 1, \ldots, t$. Now, $xL \leq D'_k$ for all k. Since R/D'_k is a right nonsingular ring by 1; 4.1(3), $x \in D'_k$ for all k. Similarly, $x \in P_k$ for $k = 1, \ldots, q$. Thus, $x \in K_i$. Therefore, R/K_i is a right nonsingular ring.

The following example shows the necessity of the condition of 2.11 that R be left and right nonsingular.

2.13 Example. Let

$$R = \begin{bmatrix} Z & Z_p \\ 0 & Z_p \end{bmatrix}$$

be the indicated ring of matrices with the usual operations. Then R has Krull dimension sequence $0 < 1 = |R|$,

$$S_2 = \begin{bmatrix} 0 & Z_p \\ 0 & Z_p \end{bmatrix} < S_1 = R$$

and

$$0 = A_1 < A_2 = \begin{bmatrix} Z & Z_p \\ 0 & 0 \end{bmatrix}.$$

Also, S_2 is the only minimal 1-coprimitive ideal of R, A_2 is the only minimal 0, A_2-coprimitive ideal of R, and the prime radical of R is

$$N = \begin{bmatrix} 0 & Z_p \\ 0 & 0 \end{bmatrix}.$$

Thus, $K_1 = K_2 = N$. The series

$$0 < \begin{bmatrix} 0 & 0 \\ 0 & Z_p \end{bmatrix} < \begin{bmatrix} 0 & Z_p \\ 0 & Z_p \end{bmatrix} < R$$

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is a critical composition series for R. Therefore, rt-Kprime$(R)=\{A_2, S_2\}$ consists of minimal prime ideals. However, by [7; p. 239], R is not left nonsingular and R does not possess an artinian classical quotient ring.

Note that the ring of 2.8 is a two sided noetherian, nonsingular ring without an artinian classical quotient ring and rt-Kprime(R) does not consist of minimal prime ideals.

If R is smooth and Q is a right Krull prime of R, then $|R/Q|=|R|$ by 2.9(b) and 2.3(c). Thus, rt-Kprime(R) consists of minimal prime ideals. The following is then an immediate consequence of 2.11.

2.14 Corollary. *Let R be noetherian, nonsingular and smooth. Then R has an artinian classical quotient ring iff R/K_1 has an artinian classical quotient ring.*

An ideal T of R is *weakly right ideal invariant* provided $|T/JT|<|R/T|$ for all right ideals J with $|R/J|<|R/T|$. A ring is called *weakly right ideal invariant* provided every ideal is weakly right ideal invariant. For a weakly right ideal invariant, right nonsingular ring, the notions of semicritical and semiprime coincide.

2.15 Proposition. *Let R be weakly right ideal invariant.*

(a) *If D is a β-coprimitive ideal with $|R/D|=\beta$, then D is a prime ideal.*

(b) *If R is semicritical, then R is semiprime.*

Proof. (a) Let D be a β-comprimitive ideal with $|R/D|=\beta$. It is easily checked that R/D is a weakly right ideal invariant ring. Since $|R/D|=\beta$, the zero ideal of R/D is a prime ideal of R/D by [5; 2.9]. Therefore, D is a prime ideal of R.

(b) According to [1; 3.6(2)], there exists ideals D_i, $i=1,\ldots,m$, such that $\bigcap_{i=1}^m D_i=0$ and for all i, D_i is a β_i-coprimitive ideal with $|R/D_i|=\beta_i$ for some ordinals $\beta_1,\ldots,\beta_m$. By (a), D_i is a prime ideal for all i. Therefore, R is a semiprime ring.

2.16 Corollary. *Let R be noetherian, nonsingular and weakly right ideal invariant. Then R has an artinian classical quotient ring iff* rt-Kprime(R) *consists of minimal prime ideals.*

Proof. By 2.1, 2.3(c) and 2.15(a), every α_i,A_i-coprimitive ideal of R is a prime ideal for all i. Therefore, R/K_i is a semiprime ring for all i. The result follows by 2.11.

Remark. Muller [10] has shown that a Krull symmetric noetherian ring R has an artinian classical quotient ring iff lt-Kprime$(R)\cup$rt-Kprime(R) consists of minimal prime ideals.

3. **Semicritical rings.** According to 2.11, a necessary condition for a noetherian, nonsingular ring R to have an artinian classical quotient ring is that each of the semicritical factor rings R/K_i have an artinian classical quotient ring.

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Furthermore, by 2.12, it is necessary that R/K_i be a right nonsingular ring for all i. In this section, we determine when a right nonsingular, semicritical ring has an artinian classical quotient ring.

As before, *R will always denote a right noetherian, right nonsingular ring with Krull dimension sequence* $0 \le \alpha_n < \cdots < \alpha_1 = |R|$.

3.1 LEMMA. *If R is a semicritical ring, then for each i, R contains a α_i-critical right ideal.*

Proof. Let J be a right ideal of R with $|J| = \alpha_i$. Then there are critical right ideals C_j, $j = 1, \ldots, m$, such that $C_1 \oplus \cdots \oplus C_m \le_e J$. By [1; 2.1(4), 2.4], $|J| = |C_1 \oplus \cdots \oplus C_m| = |C_k| = \alpha_i$ for some k. Therefore, C_k is a α_i-critical right ideal of R.

If R contains a α_i-critical right ideal, let $T_i = \cap \{\mathrm{rt}(SE(C)) \mid C$ is a α_i-critical right ideal of $R\}$ where $E(C)$ denotes the R-injective hull of C.

3.2 LEMMA. *If R is semicritical, then for all i:*
(a) *R/T_i is smooth and semicritical with $|R/T_i| = \alpha_i$.*
(b) *R/T_i is a nonsingular right R-module.*
(c) $C(0) \subseteq C(T_i)$.

Proof. (a) and (b) Let C be a α_i-critical right ideal for some i. Since $SE(C)$ is nonsingular, $\mathrm{rt}(y_1, \ldots, y_s)$ is closed in R for all $y_1, \ldots, y_s \in SE(C)$. By [8; 3.14], closed submodules of R satisfy the descending chain condition. It follows that there exists $x_1, \ldots, x_t \in SE(C)$ such that $\mathrm{rt}(SE(C)) = \mathrm{rt}(x_1, \ldots, x_t)$. Thus, since R has finite right uniform dimension, R/T_i is semicritical and there is a 1–1 map $R/T_i \to \bigoplus_{j=1}^{k} U_j$ for some nonsingular α_i-critical modules $U_1, \ldots, U_k$. Consequently, R/T_i is nonsingular and smooth with $|R/T_i| = \alpha_i$.
(c) Let $c \in C(0)$. Then $cR + T_i \le_e R$. By (b), T_i is closed in R. Thus, $cR + T_i/T_i \le_e R/T_i$. Then since R/T_i is a right nonsingular ring, $c \in C(T_i)$.

According to 3.2(a) and [9; 7.5], for each i, there is a prime ideal $P_i \ge T_i$ such that $|R/P_i| = \alpha_i$. The behavior of these prime ideals determines when a right nonsingular, semicritical ring has an artinian classical quotient ring.

3.3 THEOREM. *Let R be semicritical, and for all i, let $P_i \ge T_i$ be prime ideals with $|R/P_i| = \alpha_i$. Then R has an artinian classical quotient ring iff for all $c \in C(N)$, we have $|P_i/cP_i + T_i| < \alpha_i$ for all i.*

Proof. For the forward implication, let $c \in C(N)$. Then $c \in C(0)$ and hence, $c \in C(T_i)$ by 3.2(c). Thus, $cR + T_i/T_i \cap P_i/T_i = cP_i + T_i/T_i \le_e P_i/T_i$. Then by 3.2(a) and by [1; 2.1(4), 2.4], $|P_i/cP_i + T_i| < \alpha_i$.

For the reverse implication, let $c \in C(N)$. Then $cR + N/N \le_e R/N$ and so $cR + N \le_e R$. Thus, $cR + P_i \le_e R$ for all i. Now, R/P_i is a prime ring. Therefore R/P_i is smooth as a right R-module. Since $|R/P_i| = |R/T_i| = \alpha_i$ and R/T_i is smooth and semicritical by 3.2(a), P_i/T_i is closed in R/T_i by [1; 2.4].

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Then T_i closed in R implies that P_i is closed in R. Thus, $cR+P_i/P_i \leq_e R/P_i$. Therefore, $c \in C(P_i)$ and $|R/cR+P_i| < \alpha_i$. Now, $|R/cR+T_i| = \sup\{|cR+P_i/cR+T_i|, |R/cR+P_i|\}$ and $|cR+P_i/cR+T_i| = |P_i/(T_i+(P_i \cap cR))| = |P_i/cP_i+T_i| < \alpha_i$. Thus, $|R/cR+T_i| = |(R/T_i)/(cR+T_i/T_i)| < \alpha_i$. Then since R/T_i is smooth, $cR+T_i/T_i \leq_e R/T_i$. Therefore, $c \in C(T_i)$ for all i. Now, $\bigcap_{i=1}^n T_i = 0$ by [4; 3.1]. Clearly, this implies that $c \in C(0)$. Therefore, R has an artinian classical quotient ring.

3.4 Corollary. *Let R be smooth and semicritical, and let P be a prime ideal with $|R/P| = |R| = \alpha$. Then R has an artinian classical quotient ring iff $|P/cP| < \alpha$ for all $c \in C(N)$.*

Proof. Since R is smooth, $T_1 = 0$. The result follows by 3.3.

3.5 Corollary [2; 4.4]. *Let R be a α-primitive ring with $|R| = \alpha$ and let P be the prime ideal of R with $|R/P| = \alpha$. Then R has an artinian classical quotient ring iff $|P/cP| < \alpha$ for all $c \in C(N)$.*

References

1. A. K. Boyle and E. H. Feller, *Semicritical Modules and k-Primitive Rings*, in Module Theory, Springer-Verlag Lecture Notes No. 700, 1979, 57–74.

2. ——, *α-Coprimitive Ideals and α-Indecomposable Injectives*, Comm. in Alg., Vol. 8, 1980, 1151–1167.

3. ——, *α-Injectives and the Semicritical Socle Series*, Preprint.

4. A. K. Boyle; *The Large Condition for Rings with Krull Dimension*, Proc. A.M.S., Vol. 27, 1978, 27–32.

5. E. H. Feller, *Rings where the Annihilator of α-Critical Modules are Prime Ideals*, Pac. J. Math., Vol. 93, 1981, 299–306.

6. A. W. Goldie and G. Krause; *Artinian Quotient Rings of Ideal Invariant Noetherian Rings*, J. Alg., Vol. 63, No. 2, 1980, 374–388.

7. A. W. Goldie; *The Structure of Noetherian Rings*, in Lectures on Rings and Modules, Springer-Verlag Lecture Notes No. 246, 1972, 213–321.

8. K. R. Goodearl; *Ring Theory*, Marcel-Dekker, 1976.

9. R. Gordon and J. C. Robson; *Krull Dimension*, Memoirs A.M.S., No. 133, 1973.

10. B. J. Muller; *The Quotient Problem for Noetherian Rings*, Can. J. Math., Vol. 33, No. 3, 1981, 734–748.

11. L. W. Small; *Orders in Artinian Rings*, J. Alg., Vol. 4, 1966, 13–41.

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