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ON THE RANK NUMBERS OF AN ARC

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0. Introduction. The kth *rank number*, $\text{rank}_k B$, of a differentiable arc B in real projective n-space is the least upper bound of the number of osculating k-spaces of B which meet an $(n - k - 1)$-flat, $k = 0, 1, \ldots, n - 1$. The number $\text{rank}_0 B$ is called the *order* of B; cf. 1.1–1.3. It has been conjectured by Peter Scherk that

$$\text{rank}_k B \geqq (k + 1)(n - k), \tag{0.1}$$

equality holding if and only if B has the order n; cf. [**2**, p. 396]. In this paper we prove the following results.

THEOREM 1. *If B is a differentiable elementary arc, then* (0.1) *holds for* $k = 0, 1, \ldots, n - 1$.

THEOREM 2. *If B is a differentiable elementary arc and* $\text{order}\, B > n$, *then* $\text{rank}_k B > (k + 1)(n - k)$ *for* $k = 1, \ldots, n - 2$.

By a theorem of Park [**3**, p. 38], every differentiable arc contains a subarc of order n. This eliminates the assumption that B is elementary from Theorem 1. We do not know whether it can be dropped from Theorem 2.

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1. Prerequisites. We first list some definitions and known results which will be used throughout the paper. Unless otherwise stated, they are quoted from [**4**].

1.1. We consider arcs in real projective n-space R_n. An *arc* B is the continuous image of an open interval. Thus the points of B depend continuously on a real parameter s. The point corresponding to the parameter s will also be denoted by s.

The image of a neighbourhood of the parameter s on the parameter interval is a *neighbourhood* of the point s on B. If a sequence of parameter values converges to the parameter s, we say that the corresponding sequence of points on B also converges to the point s.

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1.2. The *order* of B is the least upper bound of the number of points that B can have in common with any hyperplane in R_n. Clearly, the order of B is not less than n. An arc of order n has end points.

An arc is *elementary* if it is the finite union of arcs of order n and of their end points.

The *order of a point* s on B is defined to be the order of a sufficiently small neighbourhood of s on B. A *point* s is called *regular* if it has order n. An elementary arc has only finitely many *singular*, i.e., non-regular, points. An *arc* is *regular* if all its points are regular.

1.3. We call a point s of B *differentiable* if all the linear *osculating spaces* $L_k^n(s)$ exist, $k = -1, 0, 1, \ldots, n$. We construct them inductively. Define $L_{-1}^n(s) = \emptyset$. Suppose that we have defined the osculating k-space $L_k^n(s)$ and postulated its existence. Then we postulate that:

(i) if $t \neq s$ is a point of B sufficiently close to s, then $tL_k^n(s)$ is a $(k+1)$-space (here, $tL_k^n(s)$ denotes the linear subspace spanned by t and $L_k^n(s)$; a similar notation will be used throughout).

(ii) this $(k+1)$-space converges as $t \to s$. Then we define

$$L_{k+1}^n(s) = \lim_{t \to s} tL_k^n(s).$$

Thus $L_0^n(s)$ is the point s itself. We call $L_{n-1}^n(s)$ the *osculating hyperplane* of B at s. If a hyperplane contains $L_k^n(s)$ but not $L_{k+1}^n(s)$, we say that it contains $L_k^n(s)$ *exactly*, $-1 \leqq k \leqq n-2$.

We say that B is *differentiable* if each of its points is differentiable.

1.4. Let φ denote the projection of R_n from a point P.

(a) If B is differentiable in R_n, then φB is differentiable in R_{n-1}.

(b) If B has order n and $P \in B$, then φB has order $n-1$.

(c) If B has order n and P is an arbitrary point in space, then φB is an arc of order n or $n-1$. By a theorem of Haupt, every differentiable arc of order n in R_{n-1} is elementary; cf. [**2**, p. 249]. Hence, the projection of an elementary arc is also elementary.

(d) If B is regular and P does not lie on any osculating hyperplane of B, then φB is regular.

From now on, "arc" means "differentiable elementary arc".

1.5. A duality maps the family of the osculating k-spaces of an arc B into a family of $(n-k-1)$-spaces $M_{n-k-1}^n(s)$ in the dual n-space. In particular, the osculating hyperplanes of B are mapped onto a family C of points. This family C is an arc and $M_{n-k-1}^n(s)$ is the osculating $(n-k-1)$-space of C at s, $k = 0, 1, \ldots, n-1$.

1.6. Let B be an arc of order n; $s \in B$. If a hyperplane contains $L_k^n(s)$ exactly, count s with the *multiplicity* $k+1$ as a point of contact of B with this hyperplane. Then *the sum of the multiplicities of the points of contact of B with a hyperplane is at most n.*

Dually, if a point P lies on $L_{n-k}{}^n(s)$ but not on $L_{n-k-1}^n(s)$, count $L_{n-1}{}^n(s)$ as passing through P with the multiplicity k. Then *the sum of the multiplicities with which the osculating hyperplanes pass through P is at most n.*

These statements remain valid if one but not both end points are added to B.

1.7. The class of any arc B is the least upper bound of the number of osculating hyperplanes of B passing through a point P in R_n. The statements of 1.6 imply that *B has order n if and only if it has class n.*

1.8. If $k + 1$ points of an arc B of order n converge to a point s of B, then the k-space spanned by them converges to $L_k{}^n(s)$ and, by duality, the intersection of their osculating hyperplanes is an $(n - k - 1)$-space which converges to $L_{n-k-1}^n(s)$ (strong differentiability and strong dual differentiability).

These statements also hold if we take into account the multiplicities described in 1.6. For instance, if s_1 and s_2 converge to s and $0 \leqq j \leqq k - 1$, $s_1 \neq s_2$, then the k-space $L_j{}^n(s_1)L_{k-j-1}^n(s_2)$ converges to $L_k{}^n(s)$.

In particular, if all the $k + 1$ points are identified, i.e. if one point is counted with the multiplicity $k + 1$, we obtain the statement that *the osculating spaces $L_k{}^n(s)$ of an arc of order n vary continuously with s.* Clearly, this last property extends to all our elementary arcs.

1.9. Dualizing the projection of the dual of B, we obtain the dual projection φ^* of B. Then $\varphi^* B$ is an arc in $E = R_{n-1}$ whose points are given by

$$\varphi^*(s) = \begin{cases} s & \text{if } L_1{}^n(s) \subset E, \\ E \cap L_1{}^n(s) & \text{otherwise.} \end{cases}$$

This dual projection has the following properties; cf. 1.4.

(a) If B has order n and E is an osculating hyperplane of B, then $\varphi^* B$ has order $n - 1$.

(b) If B has order n and E is an arbitrary hyperplane, then $\varphi^* B$ is an arc of order n or $n - 1$.

(c) If B is regular and E does not meet B, then $\varphi^* B$ is regular.

2. Lower bounds for the rank numbers.

2.1. LEMMA. *Let B be a regular arc in R_n, $s_0 \in B$. Let l be a straight line which is not contained in any osculating hyperplane of B. Consider the mapping*

$$\tau(s) = l \cap L_{n-1}{}^n(s)$$

of B into l. If $\tau(s)$ changes its direction at s_0, then

$$\tau(s_0) = l \cap L_{n-2}{}^n(s_0).$$

Proof. Since B is elementary and regular, every point of B is strongly differentiable and strongly dually differentiable.

The arc $\tau(B)$ on l may be considered as the result of repeated dual projections. Hence $\tau(B)$ is elementary and $\tau(s)$ changes its direction only finitely many times.

Since $\tau(s)$ changes its direction at s_0, there are sequences s_i and s_i', both converging monotonically to s_0, such that s_0 lies between s_i and s_i' on B, for every i, and

$$\tau(s_i) = \tau(s_i') = \tau_i,$$

say. Thus

$$\tau_i \in L_{n-1}^n(s_i) \cap L_{n-1}^n(s_i') \cap l.$$

Let $i \to \infty$. Then

$$L_{n-1}^n(s_i) \cap L_{n-1}^n(s_i') \to L_{n-2}^n(s_0),$$

by the strong dual differentiability of s_0. Hence

$$\tau_i \to \tau(s_0) = L_{n-2}^n(s_0) \cap l.$$

2.2. The following lemma is a slight generalization of a result due to Derry [**1**, p. 161].

LEMMA. *Let B be a regular arc in R_n. Let P be a point of R_n lying on k osculating hyperplanes of B, say*

$$P \in L_{n-1}^n(s_1) \cap \ldots \cap L_{n-1}^n(s_k),$$

where $s_1 < s_2 < \ldots < s_k$. If Q is a point of R_n which does not lie on any osculating hyperplane of B, and φ is the projection of R_n from Q, then φP lies on at least $k - 1$ osculating hyperplanes of φB, say

$$\varphi P \in L_{n-2}^{n-1}(t_1) \cap \ldots \cap L_{n-2}^{n-1}(t_{k-1}),$$

where

$$s_1 < t_1 < s_2 < \ldots < t_{k-1} < s_k.$$

Proof. Since Q does not lie on any osculating hyperplane of B, the intersection

$$\tau(s) = PQ \cap L_{n-1}^n(s)$$

is uniquely defined for all $s \in B$. Since

$$\tau(s_i) = \tau(s_{i+1}) = P$$

for $i = 1, \ldots, k - 1$ and $\tau(s)$ is always distinct from Q, there exists at least one point t_i on B with $s_i < t_i < s_{i+1}$, where $\tau(s)$ changes its direction. By 2.1,

$$PQ \cap L_{n-2}^n(t_i) \neq \emptyset.$$

The statement follows.

2.3. LEMMA. *For a fixed value of k, $0 \leqq k \leqq n - 2$, let $B_1, \ldots, B_{n-k}$ be regular arcs in R_n and let $P_1, \ldots, P_{n-k}$ be points of R_n such that*

(i) *$P_1, \ldots, P_{n-k}$ are independent, i.e.,*

$$\dim(P_1 \ldots P_{n-k}) = n - k - 1,$$

(ii) *P_i lies on n osculating hyperplanes of B_i,*

(iii) *for all h with $1 \leqq h \leqq n - k - 1$ and every $t_j \in B_j$,*

$$\dim(L_{n-h}^n(t_{j_0}) P_{j_1} \ldots P_{j_h}) = n$$

for every choice of the $(h + 1)$*-tuple* $j_0, \ldots, j_h$ *from* $1, \ldots, n - k$. *Then* $P_1 \ldots P_{n-k}$ *meets the osculating k-spaces of* $k + 1$ *points of each of* $B_1, \ldots, B_{n-k}$.

Proof. For $i = 1, \ldots, n - k$, let φ_i be the projection of R_n from P_i. Then, by property (iii) and 1.4(d), $\varphi_h\varphi_{h-1} \ldots \varphi_3\varphi_2 B_1$ is a regular arc in $R_{n-(h-1)}$ and $\varphi_h \ldots \varphi_2 P_{h+1}$ does not lie on any osculating hyperplane of $\varphi_h \ldots \varphi_2 B_1$, $1 \leqq h \leqq n - k - 2$ (for $h = 1$, the φs do not appear). Hence by 2.2, $\varphi_2 P_1$ lies on $n - 1$ osculating hyperplanes of $\varphi_2 B_1$, $\varphi_3\varphi_2 P_1$ lies on $n - 2$ osculating hyperplanes of $\varphi_3\varphi_2 B_1$, and, in general, $\varphi_{h+1} \ldots \varphi_2 P_1$ lies on $n - h$ osculating hyperplanes of $\varphi_{h+1} \ldots \varphi_2 B_1$, $h = 1, \ldots, n - k - 1$. Thus $\varphi_{n-k} \ldots \varphi_2 P_1$ lies on $k + 1$ osculating hyperplanes of $\varphi_{n-k} \ldots \varphi_2 B_1$. But this means that $P_1 \ldots P_{n-k}$ meets $k + 1$ osculating k-spaces of B_1. Symmetrically, it meets $k + 1$ osculating k-spaces of each of $B_2, \ldots, B_{n-k}$.

2.4. LEMMA. *Let* $0 \leqq k \leqq n - 2$. *Suppose that* $s_1, \ldots, s_{n-k}$ *are regular points of B with the following properties:*

$$(2.1) \qquad s_1, \ldots, s_{n-k} \text{ are independent}$$

and

$$(2.2) \qquad \dim(L_{n-h}{}^n(s_{j_0})s_{j_1} \ldots s_{j_h}) = n \qquad (h = 1, \ldots, n - k - 1)$$

for every choice of the $(h + 1)$*-tuple* $j_0, \ldots, j_h$ *from* $1, \ldots, n - k$. *Then for* $i = 1, \ldots, n - k$, *there exists a closed neighbourhood* N_i *of* s_i *in* R_n *containing* s_i *in its interior and such that, if* P_i *is any point of* N_i *and* t_i *is any point of a neighbourhood* B_i *of* s_i *on* B, $B_i \subset N_i$, *then*

$$(2.3) \qquad P_1, \ldots, P_{n-k} \text{ are independent}$$

and

$$(2.4) \qquad \dim(L_{n-h}{}^n(t_{j_0})P_{j_1} \ldots P_{j_h}) = n \qquad (h = 1, \ldots, n - k - 1)$$

for every choice of the $(h + 1)$*-tuple* $j_0, \ldots, j_h$ *from* $1, \ldots, n - k$.

Proof. Suppose (2.4) were false. Then there would exist an $(h + 1)$-tuple of indices $j_0, \ldots, j_h$ from $1, \ldots, n - k$ and a sequence of $(h + 1)$-tuples

$$t_{j_0}{}^\lambda, P_{j_1}{}^\lambda, \ldots, P_{j_h}{}^\lambda, \qquad \lambda = 1, 2, \ldots,$$

such that

$$(2.5) \qquad \lim t_{j_0}{}^\lambda = s_{j_0}, \lim P_{j_1}{}^\lambda = s_{j_1}, \ldots, \lim P_{j_h}{}^\lambda = s_{j_h}$$

and that

$$L_{n-h}{}^n(t_{j_0}{}^\lambda),\ P_{j_1}{}^\lambda, \ldots, P_{j_h}{}^\lambda$$

lie in a hyperplane E^λ. We may assume that the E^λ converge to a hyperplane E. Since $L_{n-h}{}^n(s)$ is continuous, (2.5) implies that

$$L_{n-h}{}^n(s_{j_0}),\ s_{j_1}, \ldots, s_{j_h}$$

lie in E, contradicting (2.2).

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The proof of (2.3) is even simpler.

2.5. *Proof of Theorem* 1. Without loss of generality, we may assume that $n \geqq 3$, $1 \leqq k \leqq n - 2$, and that B has order n.

Let $s_1, \ldots, s_{n-k}$ be any $n - k$ points of B. By 1.6, they satisfy conditions (2.1) and (2.2). Hence there exist closed neighbourhoods $N_1, \ldots, N_{n-k}$ with the properties (2.3) and (2.4).

Let P_i be a point of N_i lying on n osculating hyperplanes of a neighbourhood B_i of s_i on B, $B_i \subset N_i$, $i = 1, \ldots, n - k$. Such points always exist by the strong dual differentiability of B; cf. 1.8. Then the points P_i and the subarcs B_i satisfy the assumptions of 2.3. Therefore the $(n - k - 1)$-flat $P_1 \ldots P_{n-k}$ meets the osculating k-spaces of at least $k + 1$ points of each of $B_1, \ldots, B_{n-k}$, i.e., altogether it meets at least $(k + 1)(n - k)$ osculating k-spaces of B.

3. Two lemmas.

3.1. LEMMA. *Let B be an arc of order greater than n in R_n. Let Σ^n be any finite set of points of B containing all the singular points of B. Then there exist a hyperplane E and $n + 1$ points $s_1, \ldots, s_{n+1}$ of B such that*

$(1)_n$: $$E \cap \Sigma^n = \emptyset,$$

$(2)_n$: $$E \text{ contains } s_1, \ldots, s_{n+1} \text{ exactly},$$

$(3)_n$: $$s_1, \ldots, s_n \text{ span } E,$$

$(4)_n$: $$s_1, \ldots, s_{n-1}, s_{n+1} \text{ span } E,$$

$(5)_n$: $$\dim(L_{n-h}{}^n(s_{j0})s_{j1} \ldots s_{jh}) = n \qquad (h = 1, \ldots, n - 3),$$

for every choice of the $(h + 1)$-tuple $j_0, \ldots, j_h$ from $1, \ldots, n - 2$.

Note that the parameters s_n and s_{n+1} are distinct, but that the corresponding points in R_n may coincide.

Proof. We note that $(5)_n$ is void for $n \leqq 3$.

The case $n = 1$ is trivial. Suppose that the statement is true up to $n - 1$.

Some hyperplane meets B in more than n points. We may assume that these points span the hyperplane. Hence at least one of them, say s_0, has the property that the projection $\varphi_0 B$ of B from s_0 has order $> n - 1$. With B, $\varphi_0 B$ is an elementary arc; cf. 1.4(c).

Let $\Sigma_0{}^n$ be the union of Σ^n with s_0 and all the points of B which coincide with s_0.

Let $\Sigma_0{}^{n-1}$ be the set consisting of $\varphi_0\Sigma_0{}^n$, the points of $\varphi_0 B$ coinciding with points of $\varphi_0\Sigma_0{}^n$, and the singular points of $\varphi_0 B$.

By our induction hypothesis, there exists a hyperplane E_0 through s_0 and through n points $s_{01}, \ldots, s_{0n}$ of B such that

$(1)_{n-1}$: $\varphi_0 E_0 \cap \Sigma_0^{n-1} = \emptyset$, and thus $\{s_{01}, \ldots, s_{0n}\} \cap \Sigma_0^n = \emptyset$,

$(3)_{n-1}$: $\varphi_0 s_{01}, \ldots, \varphi_0 s_{0,n-1}$ span $\varphi_0 E_0$,

$(4)_{n-1}$: $\varphi_0 s_{01}, \ldots, \varphi_0 s_{0,n-2}, \varphi_0 s_{0n}$ span $\varphi_0 E_0$.

Hence, except possibly for the pair $s_{0,n-1}, s_{0n}$, no two of the points $s_{01}, \ldots, s_{0,n-1}, s_{0n}$ can coincide.

If $n = 2$ and s_{01} and s_{02} coincide, put $s_2 = s_{01}$ and $s_3 = s_{02}$. Each of the conditions

(i) $s_1 \notin L_1^2(s_2) \cup L_1^2(s_3)$,

(ii) $s_2 \notin L_1^2(s_1)$,

(iii) $s_1 s_2 \cap \Sigma^2 = \emptyset$

excludes only a finite number of points. Hence there is a point s_1 satisfying all three of them. Then s_1, s_2, and s_3 satisfy our requirements.

If $n = 2$ and s_{01} and s_{02} do not coincide or if $n > 2$, then we put $s_1 = s_{01}$. Then s_1 does not coincide with any of $s_{02}, \ldots, s_{0n}$, so that s_1 has the following properties:

$(a)_1$: $s_1 \notin \Sigma^n$,

$(b)_1$: if φ_1 is the projection from s_1, then order $\varphi_1 B > n - 1$, since E_0 meets B in $s_0, s_1, s_{02}, \ldots, s_{0n}$.

Define

$$\Sigma_1^n = \Sigma^n \cup \{s \in B | s \in L_{n-1}^n(s_1)\} \cup \{s \in B | s_1 \in L_{n-1}^n(s)\}.$$

Let Σ^{n-1} consist of $\varphi_1 \Sigma_1^n$ and all the points of $\varphi_1 B$ coinciding with any point of $\varphi_1 \Sigma_1^n$. Then $\varphi_1 s_1 \in \Sigma^{n-1}$ since $s_1 \in L_{n-1}^n(s_1)$.

By the induction assumption, there exists a hyperplane E_1 through s_1 and through n points $s_{12}, \ldots, s_{1,n+1}$ on B such that

$(1)_{n-1}$: $\varphi_1 E_1 \cap \Sigma^{n-1} = \emptyset$, and thus $\{s_{12}, \ldots, s_{1,n+1}\} \cap \Sigma_1^n = \emptyset$,

$(3)_{n-1}$: $\varphi_1 s_{12}, \ldots, \varphi_1 s_{1n}$ span $\varphi_1 E_1$,

$(4)_{n-1}$: $\varphi_1 s_{12}, \ldots, \varphi_1 s_{1,n-1}, \varphi_1 s_{1,n+1}$ span $\varphi_1 E_1$.

If $n = 2$, then by the definition of Σ^{n-1}, $E = s_1 s_{12} s_{13}$ has the required properties. From now on we may assume that $n \geqq 3$.

Except perhaps for the pair $s_{1n}, s_{1,n+1}$, no two of the points $s_1, s_{12}, \ldots, s_{1,n+1}$ coincide. Put $s_2 = s_{12}$. Then the points s_1, s_2 have the following properties:

$(a)_2$: $s_1 s_2 \cap \Sigma^n = \emptyset$,

$(b)_2$: if φ_i is the projection from s_i,

$i = 1, 2$, then order $\varphi_i B > n - 1$ and order $\varphi_2 \varphi_1 B > n - 2$,

$(c)_2$: $\dim(s_1 s_2) = 1$,

$(d)_2$: $\dim(L_{n-1}^n(s_{j_0}) s_{j_1}) = n$ for any permutation j_0, j_1 of 1, 2.

Now suppose that we have k points $s_1, \ldots, s_k$ for some fixed k, $2 \leqq k \leqq n-3$, such that

(a)$_k$: $$s_1 \ldots s_k \cap \Sigma^n = \emptyset,$$

(b)$_k$: if φ_i is the projection from s_i $(i = 1, \ldots, k)$, then
$$\text{order } \varphi_{j_h} \ldots \varphi_{j_1} B > n - h \qquad (h = 1, \ldots, k),$$
where $j_1, \ldots, j_h$ is any h-tuple from $1, \ldots, k$,

(c)$_k$: $$\dim(s_1 \ldots s_k) = k - 1,$$

(d)$_k$: $\dim(L_{n-h}{}^n(s_{j_0})s_{j_1} \ldots s_{j_h}) = n$ for any $(h+1)$-tuple $j_0, \ldots, j_h$ from $1, \ldots, k$ $(h = 1, \ldots, k-1)$.

Define

$$\begin{aligned}\Sigma_k{}^n = \Sigma^n &\cup \{s \in B \mid \dim(s\, s_1 \ldots s_k) < k\} \\ &\cup \{s \in B \mid \dim(L_{n-h}{}^n(s_{j_0})s_{j_1} \ldots s_{j_h}) < n \\ &\qquad \text{for some } h,\ 1 \leqq h \leqq k, \text{ and some } (h+1)\text{-tuple} \\ &\qquad\qquad s_{j_0}, \ldots, s_{j_h} \text{ from } s, s_1, \ldots, s_k\}.\end{aligned}$$

Put $\Psi_k = \varphi_k \ldots \varphi_1$. Let Σ^{n-k} consist of $\Psi_k \Sigma_k{}^n$ and all the points of $\Psi_k B$ coinciding with any point of $\Psi_k \Sigma_k{}^n$. Then $\Psi_k s_i \in \Sigma^{n-k}$ since $s_i \in L_{n-k-1}^n(s_i)$, $i = 1, \ldots, k$.

Again by our induction hypothesis, there exists a hyperplane E_k through $s_1, \ldots, s_k$ and through $n-k+1$ points $s_{k,k+1}, \ldots, s_{k,n+1}$ on B such that

(1)$_{n-k}$: $\Psi_k E_k \cap \Sigma^{n-k} = \emptyset$, and thus $\{s_{k,k+1}, \ldots, s_{k,n+1}\} \cap \Sigma_k{}^n = \emptyset$,

(3)$_{n-k}$: $\Psi_k s_{k,k+1}, \ldots, \Psi_k s_{kn}$ span $\Psi_k E_k$,

(4)$_{n-k}$: $\Psi_k s_{k,k+1}, \ldots, \Psi_k s_{k,n-1}, \Psi_k s_{k,n+1}$ span $\Psi_k E_k$.

In particular, no two of the points $s_{k,k+1}, \ldots, s_{k,n+1}$ coincide, except possibly for the pair $s_{k,n}, s_{k,n+1}$. Put $s_{k+1} = s_{k,k+1}$. Then the points $s_1, \ldots, s_{k+1}$ have the properties (a)$_{k+1}$, (b)$_{k+1}$, (c)$_{k+1}$, and (d)$_{k+1}$. We have thus proved by induction the existence of $n-2$ points $s_1, \ldots, s_{n-2}$ with the corresponding properties (a)$_{n-2}$, ..., (d)$_{n-2}$.

We now define

$$\begin{aligned}\Sigma_{n-2}{}^n = \Sigma^n &\cup \{s \in B \mid \dim(s\, s_1 \ldots s_{n-2}) < n-2\} \\ &\cup \{s \in B \mid \dim(L_1{}^n(s)\, s_1 \ldots s_{i-1} L_1{}^n(s_i) s_{i+1} \ldots s_{n-2}) < n \\ &\qquad\qquad \text{for some } i,\ 1 \leqq i \leqq n-2\}.\end{aligned}$$

Put $\Psi = \varphi_{n-2} \ldots \varphi_1$. Let Σ^2 consist of $\Psi \Sigma_{n-2}{}^n$ and all the points of ΨB coinciding with some point of $\Psi \Sigma_{n-2}{}^n$. Then $\Psi s_i \in \Sigma^2$ since $s_i \in L_1{}^n(s_i)$, $i = 1, \ldots, n-2$.

Since ΨB has order > 2 (by property $(\mathrm{b})_{n-2}$), there is a hyperplane E through $s_1, \ldots, s_{n-2}$ and through three points s_{n-1}, s_n, s_{n+1} of B such that

$(1)_2$: $\quad \Psi E \cap \Sigma^2 = \emptyset$, and thus $\{s_{n-1}, s_n, s_{n+1}\} \cap \Sigma_{n-2}{}^n = \emptyset$,

$(2)_2$: $\quad \Psi E$ contains $\Psi s_{n-1}, \Psi s_n, \Psi s_{n+1}$ exactly,

$(3)_2$: $\quad \Psi s_{n-1}, \Psi s_n$ span ΨE,

$(4)_2$: $\quad \Psi s_{n-1}, \Psi s_{n+1}$ span ΨE.

We can now verify that $s_1, \ldots, s_{n+1}$ possess the properties $(1)_n, \ldots, (5)_n$.

Verification of $(1)_n$. If s lies on $s_1 \ldots s_{n-2}$, then $s \notin \Sigma^n$, by $(\mathrm{a})_{n-2}$. Hence, if $s \in E \cap \Sigma^n$, then $s \notin s_1 \ldots s_{n-2}$ and

$$\Psi s \in \Psi(E \cap \Sigma^n) \subset \Psi E \cap \Sigma^2 = \emptyset,$$

a contradiction.

To verify $(2)_n$, first let $1 \leqq i \leqq n-2$. Since $\Psi s_i \in \Sigma^2$, we have $\Psi s_i \notin \Psi E$, by $(1)_2$. Hence $L_1{}^n(s_i) \not\subset E$ and E contains s_i exactly.

If $n-1 \leqq i \leqq n+1$, then $\Psi s_i \in \Psi E$. Thus $\Psi s_i \notin \Sigma^2$, by $(1)_2$, and $s_i \notin s_1 \ldots s_{n-2}$. By $(2)_2$, $L_1{}^2(\Psi s_i) \not\subset \Psi E$. By the definition of $\Sigma_{n-2}{}^n$,

$$\dim(L_1{}^n(s_i)\, s_1 \ldots s_{n-2}) = n-1.$$

Hence $L_1{}^n(s_i) \not\subset E$.

Verification of $(3)_n$. By $(\mathrm{c})_{n-2}$, $\dim(s_1 \ldots s_{n-2}) = n-3$. Hence $\dim \Psi E = 1$. Let $s_1, \ldots, s_n$ span the subspace F of E. Since

$$s_{n-1}, s_n, s_{n+1} \notin s_1 \ldots s_{n-2}$$

and $\Psi s_{n-1}, \Psi s_n$ span ΨE, we have $\Psi F = \Psi E$. Hence $\dim \Psi F = 1$, $\dim F = n-1$ and $F = E$.

As for $(4)_n$, clearly, we can replace s_n by s_{n+1} and $(3)_2$ by $(4)_2$ in the verification of $(3)_n$ to obtain $(4)_n$.

Finally, the property $(\mathrm{d})_{n-2}$ of $s_1, \ldots, s_{n-2}$ yields $(5)_n$.
This completes the proof of 3.1.

3.2. Lemma. *Let B be an arc of order n in R_n. Let $s_0 \in B$ and*

$$P_0 \in L_{n-1}{}^n(s_0) \backslash L_{n-2}{}^n(s_0).$$

Then there exist an open neighbourhood O of P_0 in R_n and a closed neighbourhood B_0 of s_0 on B such that, if P is any point in O, then

$$P \in L_{n-1}{}^n(s) \backslash L_{n-2}{}^n(s) \tag{3.1}$$

for some $s \in B_0$.

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Proof. Since P_0 lies on at most n osculating hyperplanes of B, we can find a closed neighbourhood B_0 of s_0 on B, with endpoints, say, s_1 and s_2, such that $P_0 \notin L_{n-1}{}^n(s)$ for all $s \in B_0$, $s \neq s_0$. Let

$$\Sigma = L_{n-1}{}^n(s_1) \cup L_{n-1}{}^n(s_2) \cup \bigcup_{s \in B_0} L_{n-2}{}^n(s).$$

Suppose that there is no neighbourhood of P_0 with the desired property. Then there exists a sequence $P_1, P_2, \ldots$ of points converging to P_0 for which (3.1) does not hold. Since Σ is closed and $P_0 \notin \Sigma$, we may assume that no point of the sequence $P_1, P_2, \ldots$ is in Σ. Thus

$$(3.2) \qquad P_i \notin L_{n-1}{}^n(s) \quad \text{for all } s \in B_0, \qquad i = 1, 2, \ldots .$$

Let $l_i = P_0P_i$ and

$$(3.3) \qquad \tau_i(s) = l_i \cap L_{n-1}{}^n(s), \qquad s \in B_0.$$

Since R_n is compact and $\tau_i(s)$ is continuous, $\tau_i(B_0)$ is a closed segment on l_i containing P_0. By 2.1, the end points of $\tau_i(B_0)$ are points of Σ. Since $P_0 \in \tau_i(B_0)\backslash\Sigma$, P_0 is an interior point of $\tau_i(B_0)$.

By (3.2) and (3.3), $P_i \notin \tau_i(B_0)$ and, for all i, P_i and P_0 are separated on l_i by two points of Σ. Since Σ is closed, no sequence of points of Σ can converge to P_0. Thus the sequence P_i does not converge to P_0 either, a contradiction.

4. Proof of Theorem 2. For $n \geqq 3$, let B be an arc of order greater than n in R_n. Let E and $s_1, \ldots, s_{n+1}$ be chosen according to 3.1, with Σ^n consisting of the singular points of B.

Given k, $1 \leqq k \leqq n-2$, let Ψ denote the projection of R_n from $F = s_1 \ldots s_{n-k-1}$. By 3.1, $(3)_n$, and $(4)_n$, F does not contain any of $s_{n-k}, \ldots, s_{n+1}$. Hence ΨB is an arc of order greater than $k+1$ in $\Psi R_n = R_{k+1}$. Hence also

$$\text{class } \Psi B > k+1.$$

Let Σ^{k+1} be the finite set of points of ΨB consisting of $s_1, \ldots, s_{n-k-1}$, Σ^n, and the singular points of ΨB. Then applying duality to 3.1, we obtain $k+2$ points $q_1, \ldots, q_{k+2}$ on ΨB and a point Q in ΨR_n such that

$$(1)_{k+1}{}^*: \qquad Q \notin L_k{}^{k+1}(s) \quad \text{if } s \in \Sigma^{k+1},$$

$$(2)_{k+1}{}^*: \qquad Q \in L_k{}^{k+1}(q_j)\backslash L_{k-1}^{k+1}(q_j), \qquad j = 1, \ldots, k+2.$$

Thus $q_j \notin \Sigma^{k+1}$.

By 3.2, there exists an open neighbourhood of Q in ΨR_n, all of whose points have the above properties. Projection being continuous, the inverse image of that neighbourhood is an open set O in R_n. Thus, if T is any point in O, there are values t_j near q_j on B such that

$$\Psi T \notin L_k{}^{k+1}(s) \quad \text{if } s \in \Sigma^{k+1}$$

and

$$\Psi T \in L_k^{k+1}(t_j)\backslash L_{k-1}^{k+1}(t_j).$$

Hence

(4.1) $$T \notin FL_k^n(s) \quad \text{if } \Psi s \in \Sigma^{k+1}$$

and

(4.2) $$T \in FL_k^n(t_j)\backslash FL_{k-1}^n(t_j).$$

By $(5)_n$, the flats

$$L_{n-h-1}^n(s_{i_0})s_{i_1}\ldots s_{i_h} \qquad (h = 1, \ldots, n-k-2)$$

are hyperplanes in R_n; here $i_0, \ldots, i_h$ is any $(h+1)$-tuple from

$$1, \ldots, n-k-1.$$

Being open, O is not contained in any of these hyperplanes nor in any of the osculating hyperplanes $L_{n-1}^n(s_i)$, $i = 1, \ldots, n-k-1$. Moreover, we may choose O so small that none of these hyperplanes meets O, i.e., that if T is any point of O, then

(4.3) $$T \notin L_{n-1}^n(s_i) \qquad (i = 1, \ldots, n-k-1)$$

and

(4.4) $$T \notin L_{n-h-1}^n(s_{i_0})s_{i_1}\ldots s_{i_h} \qquad (h = 1, \ldots, n-k-2),$$

where $i_0, \ldots, i_h$ is any $(h+1)$-tuple from $1, \ldots, n-k-1$.

Let $T \in O$ and let $t_j \in B$ be fixed satisfying (4.2), $j = 1, \ldots, k+2$. Let l be any line through T such that

$$l \not\subset FL_k^n(t_j), \qquad j = 1, \ldots, k+2.$$

Consider the mapping

$$\tau(F, t) = (FL_k^n(t)) \cap l$$

defined for all t on B for which

(4.5) $$\dim(FL_k^n(t)) = n-1$$

and

(4.6) $$l \not\subset FL_k^n(t).$$

Since, by (4.1) and the definition of Σ^{k+1}, (4.5) holds for each t_j, (4.5) will be satisfied for all t sufficiently close to any t_j. For these values of t, the hyperplane $FL_k^n(t)$ will depend continuously on t, and hence (4.6) will be satisfied for all t close to t_j. Thus $\tau(F, t)$ will be defined and continuous in some neighbourhood of t_j, $j = 1, \ldots, k+2$.

Similarly, since by (4.2),

(4.7) $$\tau(F, t) \notin FL_{k-1}^n(t)$$

for $t = t_j$, this relation will still hold in some smaller neighbourhood of t_j. Thus altogether, $\tau(F, t)$ will be defined, continuous, and, by 2.1, monotonic in that smaller neighbourhood of t_j. Let σ_j denote, for each j, the image of that neighbourhood on l. Thus $T \in \sigma_j$ and there is a closed neighbourhood σ of T on l containing T in its interior and contained in all the σ_j and in O.

Let Q_1 and Q_2 denote the endpoints of σ. Then there are points t_{1j} and t_{2j} near t_j such that

$$\text{(4.8)} \qquad \tau(F, t_{1j}) = Q_1, \qquad \tau(F, t_{2j}) = Q_2, \qquad j = 1, \ldots, k + 2.$$

As t moves from t_{1j} to t_{2j}, $\tau(F, t)$ moves monotonically through σ from Q_1 to Q_2. Let $\bar{B}_j$ denote the closed neighbourhood of t_j on B bounded by t_{1j} and t_{2j}.

By properties $(3)_n$ and $(5)_n$ of $s_1, \ldots, s_{n+1}$, we may apply 2.4 to the points $s_1, \ldots, s_{n-k-1}$ of B. Thus for $i = 1, \ldots, n - k - 1$, there exists a closed neighbourhood N_i of s_i containing s_i in its interior such that if $P_i \in N_i$ and s_i' is any point of a neighbourhood B_i of s_i on B, $B_i \subset N_i$, then

$$\text{(4.9)} \qquad \dim(P_1 \ldots P_{n-k-1}) = n - k - 2$$

and

$$\text{(4.10)} \quad \dim(L_{n-h-1}{}^n(s_{i_0}')P_{i_1} \ldots P_{i_h}) = n - 1 \quad (h = 1, \ldots, n - k - 2)$$

for every choice of the $(h + 1)$-tuple $i_0, \ldots, i_h$ from $1, \ldots, n - k - 1$.

Since $\sigma \subset O$, (4.1) and the definition of Σ^{k+1} imply that $\sigma \cap F = \emptyset$. Also by (4.3),

$$\sigma \cap L_{n-1}{}^n(s_i) = \emptyset, \qquad i = 1, \ldots, n - k - 1.$$

Finally, (4.4) implies

$$\sigma \cap (L_{n-h-1}{}^n(s_{i_0})s_{i_1} \ldots s_{i_h}) = \emptyset \qquad (h = 1, \ldots, n - k - 2)$$

for every $(h + 1)$-tuple $i_0, \ldots, i_h$ from $1, \ldots, n - k - 1$. On the other hand, the flats

$$\tilde{F} = P_1 \ldots P_{n-k-1}$$

and the hyperplanes

$$L_{n-h-1}{}^n(s_i') \qquad \text{and} \qquad L_{n-h-1}{}^n(s_{i_0}')P_{i_1} \ldots P_{i_h}$$

depend continuously on the points s_i' and P_i; cf. (4.10). Hence, σ being closed, we may assume that the neighbourhoods $N_1, \ldots, N_{n-k-1}$ were taken so small that

$$\text{(4.11)} \qquad \sigma \cap \tilde{F} = \emptyset,$$

$$\text{(4.12)} \qquad \sigma \cap L_{n-1}{}^n(s_i') = \emptyset,$$

and

$$\text{(4.13)} \qquad \sigma \cap (L_{n-h-1}{}^n(s_{i_0}')P_{i_1} \ldots P_{i_h}) = \emptyset,$$

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for all choices of P_i in N_i and s_i' in a neighbourhood B_i of s_i on B, $B_i \subset N_i$, $i = 1, \ldots, n-k-1$. For the same reason, we may choose the N_i so small that the subarcs B_i are regular and that (4.5), (4.6), and (4.7) also hold for $\tilde{F}$, i.e., that

$$\dim \tilde{F}L_k^n(t) = n-1, \qquad l \not\subset \tilde{F}L_k^n(t)$$

and that

$$\tau(\tilde{F}, t) = (\tilde{F}L_k^n(t)) \cap l \not\subset \tilde{F}L_{k-1}^n(t)$$

for all $\tilde{F} = P_1 \ldots P_{n-k-1}$ and all $t \in \bigcup_{j=1}^{k+2}\bar{B}_j$. Thus $\tau(\tilde{F}, t)$ is defined on each $\bar{B}_j$ and maps it continuously and monotonically into l.

Let σ' be a closed segment on l containing T in its interior and contained in the interior of σ. Then

$$\tau(F, t_j) = T \in \sigma'$$

and, by (4.8),

$$\tau(F, t_{\alpha j}) = Q_\alpha \notin \sigma', \qquad \alpha = 1, 2.$$

Hence there are closed neighbourhoods M_i of s_i contained in N_i and such that s_i lies in the interior of M_i, $\tau(\tilde{F}, t_j) \in \sigma'$, and

$$\tau(\tilde{F}, t_{\alpha j}) \notin \sigma' \quad \text{for all } P_i \in M_i, \qquad i = 1, \ldots, n-k-1, \quad \alpha = 1, 2.$$

Choose $P_1, \ldots, P_{n-k-1}$ arbitrarily but fixed in $M_1, \ldots, M_{n-k-1}$, respectively. Then, $\tilde{F}$ is fixed and, as t moves on $\bar{B}_j$ from t_{1j} through t_j to t_{2j}, $\tau(\tilde{F}, t)$ moves continuously and monotonically from $\tau(\tilde{F}, t_{1j}) \notin \sigma'$ through $\tau(\tilde{F}, t_j) \in \sigma'$ to $\tau(\tilde{F}, t_{2j}) \notin \sigma'$. Hence $\sigma' \subset \tau(\tilde{F}, \bar{B}_j)$ and for each $Q \in \sigma'$, there exists a $t_j \in \bar{B}_j$ such that

$$Q = \tau(\tilde{F}, t_j), \qquad j = 1, \ldots, k+2.$$

Thus the $(n-k-1)$-flat $\tilde{F}Q$ meets the osculating k-space of one point of each of $\bar{B}_1, \ldots, \bar{B}_{k+2}$; cf. (4.11).

Let B_i be a neighbourhood of s_i on B, $B_i \subset M_i$. Let φ denote the projection of R_n from a point Q of σ'. Let P_i be a point of M_i which lies on the osculating hyperplanes of n distinct points of B_i; cf. 1.8.

We next verify that the arcs φB_i and the points φP_i in φR_n satisfy all the assumptions of 2.3. By (4.12), the arcs φB_i are regular. By (4.9) and (4.11), $\dim \varphi\tilde{F} = n-k-2$ and thus the points $\varphi P_1, \ldots, \varphi P_{n-k-1}$ are independent. By (4.12) and 2.2, the points φP_i lie on $n-1$ osculating hyperplanes of φB_i. Finally, by (4.10) and (4.13),

$$\dim \varphi(L_{n-h-1}^n(s_{i_0}')P_{i_1} \ldots P_{i_h}) = n-1,$$

for every choice of the $(h+1)$-tuple $i_0, \ldots, i_h$ from $1, \ldots, n-k-1$, $h = 1, \ldots, n-k-2$. Therefore by 2.3, the $(n-k-2)$-flat $\varphi P_1 \ldots \varphi P_{n-k-1}$ meets the osculating k-spaces of $k+1$ points of each of $\varphi B_1, \ldots, \varphi B_{n-k-1}$. Hence the $(n-k-1)$-flat $\tilde{F}Q$ meets the osculating k-spaces of $k+1$ points of each of $B_1, \ldots, B_{n-k-1}$ and of one point of each of $\bar{B}_1, \ldots, \bar{B}_{k+2}$, altogether at least

$$(n - k - 1)(k + 1) + (k + 2) > (n - k)(k + 1)$$

osculating k-spaces of B. This completes our proof.

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