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GENERALIZED MARKOV PROJECTIONS AND MATRIX SUMMABILITY

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1. **Introduction.** In [A_1] is defined a class of Markov operators on $C(X)$ (X compact T^2), called Generalized Averaging Operators (g.a.o.) which yield an easy solution to the following problem: given a fixed Markov operator T, find necessary and sufficient conditions on any other Markov operator R for the relation $\ker T \subset \ker R$ to hold. The main application of this is to inclusion relations between matrix summability methods. In this paper we introduce a more general class, called Generalized Markov Projections (g.m.p.), which have the same application to matrix summability. As in [A_1], g.m.p. are defined by the existence of an associated projection which satisfies (I), (II), and (III) below. Justification for the terminology is given in Remark 2.2(c) and Proposition 2.4.

These operators have another useful property called 'Quasi-bipositivity', which is discussed in Section 3. An operator is q.b.p. if $Tf \geq 0$ implies $Tf = Tf_0$ for some $f_0 \geq 0$. This property has already proved useful in the literature on matrix summability. We show that if T is quasi-bipositive and satisfies a rather simple extra condition, then it must be 'locally' a g.m.p. (Theorem 3.7.)

2. **Generalized Markov projections.** Throughout, X will be compact T^2, $C(X)$ the space of real continuous functions on X, and e the unit function. S and T will be Markov operators on $C(X)$, i.e., non-negative operators with $Te = Se = e$. We denote by t_x the regular Borel probability measure representing the functional $f \to Tf(x)$ ($f \in C(X)$). Thus, $Tf(x) = \int f\, dt_x$, and similarly $Sf(x) = \int f\, ds_x$. We write $\operatorname{car}(t_x)$ for the smallest closed set having t_x-measure 1.

2.1. Definition. Let $\mathbf{P}$ be the compact convex set of Borel probability measures on X, and T^* the adjoint of T. $(T^*\mathbf{P})^e$ will be the set of extreme points of the compact convex set $T^*\mathbf{P}$, and

$$\mathbf{A} = \{g \in C(X) : g \text{ is constant on } \operatorname{car}(t_x) \text{ whenever } t_x \in (T^*\mathbf{P})^e\}.$$

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A Markov operator T is called a *Generalized Markov Projection* if there exists another Markov operator S such that

(I) $TS = T$,

(II) $\ker S = \ker T$,

(III) $S(C(X)) \subset \mathbf{A}$.

2.2. Remarks. (a) In [A_1, Cor. 2.3] it is proved that (I) and (II) imply $S^2 = S$. Further, by [A_1, Theorem 2.5], if R is a Markov operator, then $\ker T \subset \ker R$ iff for all $x \in X$, $r_x \in co\{s_y : y \in \operatorname{car}(r_x)\}$, where co = weak-* closed convex hull.

(b) g.a.o. satisfy (I) and (II), but in (III) are required to satisfy the stronger condition that if $f \in C(X)$, Sf is constant on $\operatorname{car}(t_x)$ for *all* $x \in X$. Thus if T is g.a.o., then $t_x \neq t_y$ implies $\operatorname{car}(t_x) \cap \operatorname{car}(t_y) = \emptyset$. For g.m.p. this need hold only when t_x and t_y are extreme points.

(*c*) Proposition 2.3 below is an analogue of Theorem 3.2 of [A_1]. Note (in connection especially with infinite matrices) that it shows how S may be constructed from knowledge of T. Proposition 2.4 gives a functional relation between S and T which justifies the terminology 'g.m.p.' In 2.4, if $S = T$, then we have precisely Lloyd's identity $T(gTf) = T(TgTf)$ for Markov projections [L1]. In [A_1], g.a.o. are shown to satisfy the stronger identity $T(gSf) = TgTf$.

2.3. Proposition. If S and T are Markov operators, the following are equivalent:

(a) (I) and (III) hold,

(b) $t_x \in (T^*\mathbf{P})^e$ and $y \in \operatorname{car}(t_x)$ imply $s_y = t_x$.

Proof. (*a*) *implies* (*b*). Just the same as "(a) implies (b)" in [A_1, Theorem 3.2]. (*b*) *implies* (*a*). (III) is immediate. To prove (I), if t_x is extreme, then $TSf(x) = \int Sf(y)\,dt_x(y) = \int Tf(x)\,dt_x(y) = Tf(x)$, or $t_x(Sf) = t_x(f)$. By Krein–Milman this holds for all $x \in X$.

2.4. Proposition. If S and T are Markov operators for which (I) and (II) hold, then we have

(c) $T(gSf) = T(SgSf)$ for all f and g in $C(X)$.

Proof. By Remark 2.2(a), $S^2 = S$. Since S is a Markov projection, Theorem 2 of [L1] implies that $S(gSf) = S(SgSf)$ for all f and g, i.e., $gSF - SgSf \in \ker S = \ker T$, whence the result.

2.5. Example. A simple 'prototypical' example is as follows. Let $X = \{1, 2, 3\}$, so $C(X)$ is (algebraically) the same as R^3. We define T by the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

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i.e., $t_1=\delta_1$ (the Dirac mass at 1), $t_2=\frac{1}{2}(\delta_2+\delta_3)$, and $t_3=\frac{1}{2}(t_1+t_2)$. Then S is given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

2.6. Application to matrix summability. We follow the notation of Section 4 of $[A_1]$. $T=(t_{mn})$ will be a non-negative regular matrix whose mth row is denoted by t_m. To apply the concept of g.m.p. we assume T has the following special form: there exists a sequence $m(1)<m(2)<\cdots$ such that (a) if $j\neq k$, then $t_{m(k)}$ and $t_{m(j)}$ have disjoint supports, (b) every row of T is a finite convex combination of elements of $\{t_{m(k)}:k\geq 1\}$. We define the matrix $S=(s_{mn})$ as follows: if $m\in \operatorname{car}(t_{m(k)})$ for some k, then $s_m=t_{m(k)}$. If $m\notin \operatorname{car}(t_{m(k)})$ for all k, then $s_m=t_m$. As in $[A_1, 4.2]$, it is easy to check that the operators T^* and S^* induced on $C(\beta N\backslash N)$ satisfy the conditions (I) and (II) for g.m.p. Exactly the same proof as that of Theorem 4.4 of $[A_1]$ yields

Theorem. *Let the matrices T and S be as above. If R is a non-negative regular matrix, then the following are equivalent:*

(a) $C_T\subset C_R$,

(b) *for each* $f\in C^*(N)$, $\lim_{m\to\infty}[\inf\{|r_m(f)-s(f)|:s\in W_m\}]=0$, *where* $W_m=$ *convex hull of* $\{s_p:p\in \operatorname{car}(r_m)\}$.

2.7. Remark. G.m.p. have the following interesting property: if $Tf\geq 0$, then there exists $f_0\geq 0$ such that $Tf=Tf_0$.

Proof. Let $t_x\in(t^*\mathbf{P})^e$ and $y\in\operatorname{car}(t_x)$. Then $s_y=t_x$, and so $Sf(y)=Tf(x)$. Thus $Sf\geq 0$ on $\operatorname{car}(t_x)$. By Krein–Milman, the union of these is dense in $K=$ closure $\bigcup\{\operatorname{car}(t_x):x\in X\}$, so $Sf\geq 0$ on K. Let $f_0=\max(Sf,0)$. Since for each g in $C(X)$ the values of Tg are determined by the values of g on K, we have $Tf=TSf=Tf_0$, where $f_0\geq 0$.

3. **Quasi-bipositivity.** Again, T is a Markov operator on $C(X)$ and $K=$ closure $\bigcup\{\operatorname{car}(t_x):x\in X\}$. In 3.1 we define the q.b.p. property, and in 3.3 we show that if T is q.b.p., then so is T^*. In 3.5 we show that if T is 'locally' a g.m.p., then T is q.b.p. In 3.7 we give a very simple criterion for a closed range q.b.p. operator to be locally g.m.p., namely that ker T be 'locally' the kernel of a Markov projection. Finally, we discuss applications of quasi-bipositivity to matrix summability.

3.1. Definition. T is called *quasi-bipositive* if $T(C(X)^+)=(T(C(X)))^+$, i.e., if T satisfies the condition of Remark 2.7. (Note that if T is $1-1$, then it is q.b.p. iff $T^{-1}\geq 0$, i.e., iff T is 'bipositive'.)

3.2. Remark. We shall indicate briefly with a typical situation why this property is useful in practice. If S and T are closed range Markov operators, it is easy to show that the following are equivalent:

(a) $\ker T \subset \ker S$,

(b) $S^*(C(X)^*) \subset T^*(C(X)^*)$.

However if T is q.b.p., then (b) is equivalent to

(c) $S^*\mathbf{P} \subset T^*\mathbf{P}$.

[Proof that (b) implies (c): if $m \in S^*\mathbf{P}$, then $m \geq 0$ and $m = T^*n$, where $n \in c(X)^*$. By 3.3 below, T^* is q.b.p., so there exists $m_0 \geqslant 0$ with $m = T^*n = T^*m_0$. $m_0 \in \mathbf{P}$, because $m_0(e) = T^*m_0(e) = m(e) = 1$.] ($c$) is a much more effective criterion than (b) because Markov operators are frequently defined directly by the values of $T^*\delta_x \in T^*\mathbf{P}$. This is further discussed in 3.10.

The following example will show that quasi-bipositivity is really needed for the implication '(a) implies (c)'. Let $X = \{1, 2, 3\}$ and let T and S be given by the matrices

$$\begin{bmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

respectively. It is easy to check that T is not q.b.p., and that $\ker T = \ker S$, while $\delta_1 \in S^*\mathbf{P} \backslash T^*\mathbf{P}$.

3.3. Theorem. *If T has closed range, the following are equivalent:*

(a) *T is q.b.p.,*

(b) *T^* is q.b.p., and e is an interior point of the cone $T(C(X)^+)$ (considered as a cone in $T(C(X))$).*

Proof. (a) *implies* (b). To prove e is an interior point of $T(C(X)^+)$, let $G = \{f \in T(C(X)) : \|f - e\| < 1\}$. If $f = Tg \in G$, then $f \geq 0$, so by (a) there exists $g_0 \in C(X)^+$ with $f = Tg_0$. Hence $f \in T(C(X)^+)$, and so $G \subset T(C(X)^+)$.

Now let $T^*m \geq 0$. If $f \in T(C(X))^+ = T(C(X)^+)$, then for some $g \geq 0$, $f = Tg$, whence $m(f) = (T^*m)(g) \geq 0$. Thus, m restricted to $T(C(X))$ is a positive linear functional. Let $m_0 = m(e)^{-1}m$, and let m_1 be a Hahn–Banach extension to all $C(X)$ of the restriction of m_0 to $T(C(X)$, having norm 1. Then m_1 is a probability, and we want to show that $T^*m_1 = T^*m_0$ (whence $T^*m = T^*(m(e)m_0)$). But $m_1 - m_0$ vanishes on $T(C(X))$, so for $f \in C(X)$, $0 = (m_1 - m_0)(Tf) = T^*m_1(f) - T^*m_0(f)$.

(b) *implies* (a). Suppose $f_0 \in T(C(X))^+ \backslash T(C(X)^+)$. Since $T(C(X)^+)$ is a positive cone with interior (relative to $T(C(X))$), there exists m in $T^*(C(X)^*)$ such that $m(f_0) = -1$, but for all $g \in C(X)^+$, $(T^*m)(g) = m(Tg) \geqslant 0$ [D, page 22, Theorem 6]. By Hahn–Banach, we can assume m is defined on all $C(X)$. Hence $T^*m \geq 0$, and $m(f_0) = -1$. If $f_0 = Tg_0$, then $-1 = m(f_0) = (T^*m)(g_0)$. By hypothesis there exists $m_1 \geq 0$ such that $T^*m_1 = T^*m$, whence $-1 = (T^*m)(g_0) = (T^*m_1)(g_0) = m_1(Tg_0) = m_1(f_0)$. But this contradicts $m_1 \geq 0$ and $f_0 \geq 0$.

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3.4. Definitions. We recall

$$\mathbf{A}=\{g\in C(X): g \text{ constant on } \operatorname{car}(t_x) \text{ when } t_x \text{ extreme in } T^*\mathbf{P}\},$$

and further define

$$\mathbf{A}_0=\{g\in C(X): f\in\ker T \text{ implies } gf\in\ker T\}.$$

(This concept originates in matrix summability–see, e.g., [P].) If $L\subset C(X)$, we write $L_K=\{f\mid K: f\in L\}$, where $f\mid K$ is the restriction of f to K.

3.5. Proposition. If $C(K)=\ker T_K\oplus\mathbf{A}_K$, then T is q.b.p.

Proof. Suppose $Tf\geq 0$. Write $f\mid K=g\mid K+h\mid K$, where $g\in\ker T$ and $h\in\mathbf{A}$. Then $Tf=Th$ and, as in 2.7, it is easy to see that $h\geq 0$ on K. Let f_0 be a non-negative continuous extension of $h\mid K$ to all X. Then $Tf=Tf_0$ and $f_0\geq 0$.

3.6. Lemma. *If T is* q.b.p. *with closed range, then* $\mathbf{A}=\mathbf{A}_0$.

Proof. Clearly, $\mathbf{A}\subset\mathbf{A}_0$. Let $f\in\mathbf{A}_0$. Since T has closed range, $(\ker T)^\perp=T^*(C(X)^*$, and by [L, page 36, lemma 2_R], f is constant on $\operatorname{car}(m)$ whenever m is an extreme point of $\operatorname{ball}(T^*C(X)^*)$. Since T is q.b.p., 3.3 implies $T^*(C(X)^*)\cap\mathbf{P}=T^*\mathbf{P}$. Thus if $t_x\in(T^*\mathbf{P})^e$, then clearly $t_x\in(\operatorname{ball}(T^*C(X)^*)^e$, and so f is constant on $\operatorname{car}(t_x)$. Hence $f\in\mathbf{A}$, and we have proved $\mathbf{A}_0\subset\mathbf{A}$.

3.7. Theorem. *If T is* q.b.p. *with closed range, the following are equivalent:*

(a) $C(K)=\ker T_K\oplus\mathbf{A}_K$,

(b) $\ker T_K$ *is the kernel of some Markov projection on* $C(K)$.

Proof. (*a*) *implies* (*b*). Let Q be the projection on $C(K)$ with kernel $\ker T_K$ and range $\mathbf{A}_K$. We shall prove Q is Markov. If e_K is the unit function in $C(K)$, then $e_K\in\mathbf{A}_K$, and hence $Qe_K=e_K$. To prove $Q\geq 0$, let $f\in C(K)^+$. Then $f=g+h$ with $g\in\ker T_K$ and $h\in\mathbf{A}_K$, and we must prove $h\geq 0$. Let $g_0\in\ker T$ and $h_0\in\mathbf{A}$ such that $g_0\mid K=g$ and $h_0\mid K=h$. If $f_0=h_0+g_0$, then $f_0\mid K=f$, and if $t_x\in(T^*\mathbf{P})^e$, then $0\leq Tf_0(x)=Th_0(x)=\int h_0\,dt_x$. Since $h_0\in\mathbf{A}$, it is constant on $\operatorname{car}(t_x)$, whence $h_0\geq 0$ on $\operatorname{car}(t_x)$. By Krein–Milman, $h\geq 0$.

(*b*) *implies* (*a*). Let Q be a Markov projection on $C(K)$ with kernel $\ker T_K$. We must prove range $Q=\mathbf{A}_K$. Now Q satisfies Lloyd's identity [L1, Theorem 2] $Q((g-Qg)Qh)=0$ for all g and h in $C(K)$. Hence if $f\in\ker T_K=\ker Q$, then $Q(fQh)=0$, i.e., $Qh\in\mathbf{A}_{0K}$ By Lemma 3.6, $Qh\in\mathbf{A}_K$.

3.8. Remark. Similarly, one can prove that a q.b.p. T is a g.m.p. iff $\ker T$ is the kernel of some Markov projection on $C(X)$.

3.9. Example. We shall define a g.b.p. on $C(X)$, where $X=\{1,2,3\}$, which does not satisfy 3.7. T is given by the matrix

$$\begin{bmatrix}\frac12 & 0 & \frac12\\ 0 & \frac12 & \frac12\\ \frac14 & \frac14 & \frac12\end{bmatrix}$$

To prove T is q.b.p., note that its range consists of functions of the form $f(1)=a$, $f(2)=b$, $f(3)=2^{-1}(a+b)$ (which we write for convenience as row vectors $(a, b, 2^{-1}(a+b))$, despite the fact that they are actually operated on by T as column vectors). One checks that $T(2a, 2b, 0)=(a, b, 2^{-1}(a+b))$. If $(a, b, 2^{-1}(a+b))\geq(0,0,0)$, then $a\geq 0$ and $b\geq 0$, so $(2a, 2b, 0)\geq(0,0,0)$. Now $\ker T$ is the one-dimensional linear span of the vector $(1,1,-1)$, and $\mathbf{A}$ consists of the constant functions. Since $K=X$, the direct sum decomposition of 3.7 fails. It is an amusing exercise to check directly that $\ker T$ is not the kernel of any Markov projection on $C(X)$—in fact if $\ker T\subset\ker Q$, then range $(Q)=$ constants, so that $\ker Q$ is 2-dimensional.

3.10. Applications to summability. We cite some applications which have already appeared in the literature.

Theorem. *Let $S=(s_{mn})$ and $T=(t_{mn})$ be non-negative regular matrices, and assume that the operator T^* induced by T on $C(\beta N\backslash N)$ is* q.b.p. *Then*

(a) *$C_T\subset C_S$ iff* $\lim(n\to\infty)d(S_n, R_m)=0$ *for all m, where S_n is the* nth *row of S, and R_m is the norm-closed convex hull of $\{T_m, T_{m+1},\ldots\}$.* [A_3].

(b) *Let V_S be the space of 'S-almost convergent functions' (i.e., functions assigned the same value by all S-invariant means). Then $C_T\subset V_S$ "consistently" iff* $\lim(n, p\to\infty)d(S_{n,p}, R_m)=0$, *where R_m is as in* (a), *and $S_{n,p}$ is the* pth *row of the matrix* $S_n=(1/n)(S+\cdots+S^n)$. [A_2].

Finally, operators satisfying the conditions of Theorem 3.7 appear in [A_4] under the name "good operators".

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