ON CERTAIN IDENTITIES INVOLVING BASIC SPINORS AND CURVATURE SPINORS

H. A. BUCHDAHL

(received 22 May 1961)

1. Introduction

The spinor analysis of Infeld and van der Waerden [1] is particularly well suited to the transcription of given flat space wave equations into forms which constitute possible generalizations appropriate to Riemann spaces [2]. The basic elements of this calculus are (i) the skew-symmetric spinor $\gamma_{\mu\nu}$, (ii) the hermitian tensor-spinor $\sigma^{k\dot{\mu}\nu}$ (generalized Pauli matrices), and (iii) the curvature spinor $P^{\mu}{}_{\nu kl}$. When one deals with wave equations in Riemann spaces V_4 one is apt to be confronted with expressions of somewhat bewildering appearance in so far as they may involve products of a large number of σ-symbols many of the indices of which may be paired in all sorts of ways either with each other or with the indices of the components of the curvature spinors. Such expressions are generally capable of great simplification, but how the latter may be achieved is often far from obvious. It is the purpose of this paper to present a number of useful relations between basic tensors and spinors, commonly known relations being taken for granted [3], [4], [5]. That some of these new relations appear as more or less trivial consequences of elementary identities is largely the result of a diligent search for their simplest derivation, once they had been obtained in more roundabout ways.

Certain relations take a particularly simple form when the V_4 is an Einstein space, and some necessary and sufficient conditions relating to Einstein spaces and to spaces of constant Riemannian curvature are considered in the last section.

2. Basic Relations and definitions

(a) The defining equations for the $\sigma^{k\dot{\mu}\nu}$ may be taken to be (reference [4], p. 20)

(2.1) $$\sigma^{k\dot{\lambda}\mu}\sigma^{l}{}_{\dot{\lambda}\nu} = \tfrac{1}{2}\delta^{\mu}_{\nu}g^{kl} - \tfrac{1}{2}ie^{klmn}\sigma_{m}{}^{\dot{\lambda}\mu}\sigma_{n\dot{\lambda}\nu}$$

and

(2.2) $$\sigma^{k\dot{\mu}\nu} = \sigma^{k\nu\dot{\mu}},$$

the latter expressing the hermiticity of the "Pauli matrices". In (2.1) e^{klmn} is a tensor derived from the contravariant alternating tensor density of Levi-Civita (see reference [2], p. 99, footnote):

(2.3) $$e^{klmn} = \tilde{\omega}(-g)^{-1/2}\varepsilon^{klmn}.$$

Then

(2.4) $$e^{klmn}e_{abcd} = -\,\delta^{klmn}_{abcd},$$

where the tensor on the right is a generalized Kronecker delta. From (2.1) it follows that [1]

(2.5) $$\sigma^{(k\dot{\lambda}\mu}\sigma^{l)}{}_{\dot{\lambda}\nu} = \tfrac{1}{2}\delta^{\mu}_{\nu}g^{kl}.$$

It is convenient to define the tensor

(2.6) $$\eta^{kl}{}_{st} = \delta^{k}{}_{[s}\delta^{l}{}_{t]} - \tfrac{1}{2}ie^{kl}{}_{st},$$

and the tensor-spinor

(2.7) $$S^{kl\mu}{}_{\nu} = \sigma^{[k\dot{\lambda}\mu}\sigma^{l]}{}_{\dot{\lambda}\nu}.$$

Then (2.1) yields

(2.8) $$S^{kl\mu}{}_{\nu} = -\tfrac{1}{2}ie^{klst}S_{st}{}^{\mu}{}_{\nu},$$

or

(2.9) $$\bar{\eta}^{kl}{}_{st}S^{st\mu}{}_{\nu} = 0,$$

where the bar denotes complex conjugation. (2.9) expresses the self-duality of $S^{st\mu}{}_{\nu}$ (in any fixed spin frame). Notice that $S^{st\mu\nu}$ is symmetric in its spinor indices. (2.9) may also be written

(2.10) $$\eta^{kl}{}_{st}S^{st\mu}{}_{\nu} = 2S^{kl\mu}{}_{\nu}.$$

From (2.1) and (2.8)

(2.12) $$S^{kl\mu}{}_{\nu} = \sigma^{k\dot{\lambda}\mu}\sigma^{l}{}_{\dot{\lambda}\nu} - \tfrac{1}{2}\delta^{\mu}_{\nu}g^{kl}.$$

(b) For later use the following known relations need to be quoted.

(2.12) $$\sigma^{k}{}_{\dot{\lambda}\nu}\sigma^{l\dot{\lambda}\alpha}\sigma^{m}{}_{\dot{\mu}\alpha} = \tfrac{1}{2}(g^{kl}\sigma^{m}{}_{\dot{\mu}\nu} + g^{lm}\sigma^{k}{}_{\dot{\mu}\nu} - g^{km}\sigma^{l}{}_{\dot{\mu}\nu} + ie^{klmr}\sigma_{r\dot{\mu}\nu})$$

(2.13) $$= (\tfrac{1}{2}g^{lm}g^{kr} - \eta^{klmr})\sigma_{r\dot{\mu}\nu},$$

from which one gets by transvection with $\sigma^{n\dot{\mu}\nu}$

(2.14) $$\sigma^{k}{}_{\dot{\lambda}\nu}\sigma^{l\dot{\lambda}\alpha}\sigma^{m}{}_{\dot{\mu}\alpha}\sigma^{n\dot{\mu}\nu} = \tfrac{1}{2}(g^{kl}g^{mn} + g^{kn}g^{lm} - g^{km}g^{ln} + ie^{klmn}),$$

(2.15) $$= \tfrac{1}{2}g^{kl}g^{mn} - \eta^{klmn}.$$

[1] The symmetrizing and alternating brackets always act on only one kind of indices, i.e. tensor indices, *or* undotted spinor indices, *or* dotted spinor indices.

From the relations given so far one easily infers the following:

(2.16) $$\sigma^{s\dot{\alpha}\beta} S_{sk\mu\nu} = -\sigma_k{}^{\dot{\alpha}}{}_{(\mu}\delta_{\nu)}{}^{\beta},$$

(2.17) $$\sigma_{s\dot{\mu}\alpha} S^{kl\alpha}{}_{\nu} = \eta^{kl}{}_{st}\sigma^{t}{}_{\dot{\mu}\nu},$$

(2.18) $$S^{ks\mu\nu} S_{kt\alpha\beta} = \sigma^{s\dot{\lambda}(\mu}\sigma_{t\dot{\lambda}(\alpha}\delta_{\beta)}{}^{\nu)},$$

and from this in particular

(2.19) $$S^{kl\mu\nu} S_{kl\alpha\beta} = 2\delta^{(\alpha}{}_{\mu}\delta^{\beta)}{}_{\nu}.$$

Further

(2.20) $$S^{ks\mu\nu} S_{kt}{}^{\dot{\alpha}\dot{\beta}} = -\sigma^{s(\dot{\alpha}(\mu}\sigma_t{}^{\nu)\dot{\beta})},$$

(2.21) $$S^{kl\mu\nu} S_{ab\mu\nu} = \eta^{kl}{}_{ab}.$$

In particular one has from (2.20)

(2.22) $$S^{kl\mu\nu} S_{kl\dot{\alpha}\dot{\beta}} = 0.$$

Finally, write simply S^{kl} for $S^{kl\mu}{}_{\nu}$, S^{kl} being regarded as a 2×2 matrix, so that

(2.23) $$(S^{kl} S^{ab})^{\mu}{}_{\nu} = S^{kl\mu}{}_{\alpha} S^{ab\alpha}{}_{\nu}.$$

Then the commutator of S^{kl} and S^{ab} is given by (reference [4], p. 30)

(2.24) $$[S^{kl}, S^{ab}] = 4g^{[k[a}S^{b]l]}.$$

(c) (i) As regards the Riemann tensor R_{klmn} and the curvature spinor $P^{\mu}{}_{\nu kl}$, their algebraic properties alone are of interest in the present context [disregarding section 6(a) for the moment]. The symmetries of the former are

(2.25—29) $R_{kl(mn)} = 0, \quad R_{(kl)mn} = 0, \quad R_{k[lmn]} = 0, \quad R_{klmn} - R_{mnkl} = 0.$

The trace-free part of the Ricci tensor R_{kl} will be denoted by E_{kl}, i.e.

(2.30) $$E_{kl} = R_{kl} - \tfrac{1}{4}g_{kl}R.$$

The V_4 is an Einstein space if

(2.31) $$E_{kl} = 0,$$

whilst if it is a space of constant Riemannian curvature

(2.32) $$R_{klmn} = \tfrac{1}{6}g_{k[n}g_{m]l}R, \quad (R = \text{const.}).$$

The conformal curvature tensor of the V_4 is defined as [6]

(2.33) $$C_{klmn} = R_{klmn} - 2g_{[k[n}(R_{m]l]} - \tfrac{1}{6}g_{m]l]}R),$$

and it is entirely trace-free, i.e. if a, b is any pair of indices selected from amongst k, l, m, n, then

https://doi.org/10.1017/S1446788700026951 Published online by Cambridge University Press

(2.34) $$g^{ab}C_{klmn} = 0.$$

(ii) Taking the imaginary part of the contraction $\Gamma^{\lambda}{}_{\lambda k}$ of the linear spinor connection to be zero one has (reference [5], p. 726)

(2.35) $$P^{\mu}{}_{\nu kl} = \tfrac{1}{2}S^{mn\mu}{}_{\nu}R_{klmn},$$

so that incidentally

(2.36) $$P_{[\mu\nu]kl} = 0.$$

(2.35) may be inverted by means of (2.21) and gives

(2.37) $$\eta^{ab}{}_{kl}R_{abmn} = -2S_{kl}{}^{\mu}{}_{\nu}P^{\nu}{}_{\mu mn},$$

which may be resolved into its real and imaginary parts. In a space of constant Riemannian curvature (2.32) and (2.35) imply

(2.38) $$P^{\mu}{}_{\nu kl} = -\tfrac{1}{12}S_{kl}{}^{\mu}{}_{\nu}R.$$

3. Tensor identities involving R_{klmn}

(a) Consider the tensor $e^{klab}e_{mncd}R_{ab}{}^{cd}$. Use (2.4), write the generalized Kronecker delta as a determinant of simple Kronecker deltas, and expand this determinant. Then the tensor under consideration evidently becomes a linear sum of the components of the Riemann tensor and its contractions. In this way one is immediately led to the *identity*

(3.1) $$e^{klab}e^{mncd}R_{abcd} = -4R^{klmn} + 16g^{[k[n}E^{m]l]}.$$

From this it follows that

(3.2) $$e^{mncd}R^{kl}{}_{cd} - e^{klcd}R^{mn}{}_{cd} = 4e^{klcd}\delta^{[m}{}_{c}E^{n]}{}_{d}.$$

(b) Next, contemplate the tensor $\eta^{ab}{}_{kl}\bar{\eta}^{cd}{}_{mn}R_{abcd}$. Using (2.6) and multiplying the two first factors out explicitly one obtains just terms of the form of the left hand members of (3.1) and (3.2). Hence finally

(3.3) $$\eta^{ab}{}_{kl}\bar{\eta}^{cd}{}_{mn}R_{abcd} = 4\eta_{kla[m}E_{n]}{}^{a}.$$

(c) It may be noted in passing that the cyclic identity (2.27) may be reexpressed trivially but usefully in the form

(3.4) $$\eta^{kbcd}R_{lbcd} + R^{k}{}_{l} = 0.$$

4. Relations not involving $P^{\mu}{}_{\nu kl}$

(a) By transvection of (2.12) with $\sigma_{l\dot{\rho}\sigma}$. One obtains relations which differ from the former only trivially but whose explicit availability is often useful. One has at once

https://doi.org/10.1017/S1446788700026951 Published online by Cambridge University Press

(4.1) $\sigma^a{}_{\dot{\alpha}\beta}\sigma^b{}_{\dot{\gamma}\delta} = \frac{1}{2}(\sigma^a{}_{\dot{\alpha}\delta}\sigma^b{}_{\dot{\gamma}\beta} + \sigma^b{}_{\dot{\alpha}\delta}\sigma^a{}_{\dot{\gamma}\beta} + g^{ab}\gamma_{\dot{\alpha}\dot{\gamma}}\gamma_{\beta\delta} - ie^{abcd}\sigma_{c\dot{\alpha}\delta}\sigma_{d\dot{\gamma}\beta}).$

From this there follow at once the relations

$$\sigma^{(a}{}_{\dot{\alpha}[\beta}\sigma^{b)}{}_{\delta]\dot{\gamma}} = \frac{1}{4}g^{ab}\gamma_{\dot{\alpha}\dot{\gamma}}\gamma_{\beta\delta}, \tag{4.2}$$

and

$$\sigma^{[a}{}_{\dot{\alpha}\beta}\sigma^{b]}{}_{\dot{\gamma}\delta} = -\frac{1}{2}ie^{abcd}\sigma_{c\dot{\alpha}\delta}\sigma_{d\dot{\gamma}\beta}. \tag{4.3}$$

(4.3) is remarkable in that it is a stronger form of (2.8). A similar remark applies to the comparison of (4.2) with (2.5), though the situation is more trivial in this case for the following reason. If the elementary identity [reference [5], p. 717)

$$\gamma^{[\mu\nu}\gamma^{\rho]\sigma} = 0 \tag{4.4}$$

be transvected with any spinor $t_{\nu\rho}{}^{\cdots}{}_{\cdots}$ of arbitrary valence one gets

$$t^{[\mu\nu]}{}^{\cdots}{}_{\cdots} = \frac{1}{2}\gamma^{\mu\nu}t_\lambda{}^{\lambda\cdots}{}_{\cdots}. \tag{4.5}$$

(4.5) in conjunction with (2.5) then gives (4.2) at once. Note that (4.3) may also be written as

(4.6—7) $\sigma^{[a}{}_{\dot{\mu}(\nu}\sigma^{b]}{}_{\beta)\dot{\alpha}} = \frac{1}{2}\eta^{ab}{}_{cd}\sigma^c{}_{\dot{\mu}\nu}\sigma^d{}_{\dot{\alpha}\beta},\quad \sigma^{[a}{}_{\dot{\mu}[\nu}\sigma^{b]}{}_{\beta]\dot{\alpha}} = \frac{1}{2}\bar{\eta}^{ab}{}_{cd}\sigma^c{}_{\dot{\mu}\nu}\sigma^d{}_{\dot{\alpha}\beta}.$

If (4.3) be transvected with $\sigma_{b\dot{\mu}\nu}$ one gets

$$\sigma^a{}_{\dot{\alpha}\beta}\gamma_{\dot{\mu}\dot{\gamma}}\gamma_{\nu\delta} - \sigma^a{}_{\dot{\gamma}\delta}\gamma_{\dot{\mu}\dot{\alpha}}\gamma_{\nu\beta} = -ie^{abcd}\sigma_{b\dot{\mu}\nu}\sigma_{c\dot{\alpha}\delta}\sigma_{d\dot{\gamma}\beta}, \tag{4.8}$$

and from this in turn one infers that

$$ie^{abcd}\sigma_{a\dot{\alpha}\beta}\sigma_{b\dot{\gamma}\delta}\sigma_{c\dot{\mu}\nu}\sigma_{d\dot{\rho}\sigma} = \gamma_{\dot{\mu}\dot{\gamma}}\gamma_{\dot{\alpha}\dot{\rho}}\gamma_{\nu\beta}\gamma_{\delta\sigma} - \gamma_{\dot{\mu}\dot{\alpha}}\gamma_{\dot{\gamma}\dot{\rho}}\gamma_{\nu\delta}\gamma_{\beta\sigma}. \tag{4.9}$$

(b) Some useful identities arise when one considers the product of one η-symbol and one S-symbol with only one index from each paired. Thus, suppressing the spinor indices of $S^{kl\mu}{}_\nu$ whenever convenient, one has because of (2.6)

$$\eta^{klca}S^b{}_c = g^{a[k}S^{l]b} - \frac{1}{2}ie^{klca}S^b{}_c.$$

Now

$$\begin{aligned} e^{klca}S^b{}_c &= -\frac{1}{2}ie^{klca}e^b{}_{cmn}S^{mn}, \quad \text{by (2.8)},\\ &= \frac{1}{2}ig^{bd}\delta^{klca}_{dcmn}S^{mn}, \quad \text{by (2.4)},\\ &= -\frac{1}{2}ig^{bd}\delta^{kla}_{dmn}S^{mn} = -i(g^{kb}S^{la} + g^{bl}S^{ak} + g^{ab}S^{kl}). \end{aligned}$$

Hence

$$\eta^{klca}S^b{}_c = -2g^{[k[a}S^{b]l]} - \frac{1}{2}g^{ab}S^{kl}. \tag{4.10}$$

Restoring spinor superscripts μ, ν and transvecting with $S_{mn\mu\nu}$ one obtains in view of (2.21) the following useful relation involving the η-tensor alone:

(4.11) $$\eta^{kla}{}_c\eta^{bc}{}_{mn} = 2g^{[k[a}\eta^{b]l]}{}_{mn} + \tfrac{1}{2}g^{ab}\eta^{kl}{}_{mn}.$$

In particular,

(4.12) $$\eta^{kl}{}_c{}^{(a}\eta^{b)c}{}_{mn} = -\tfrac{1}{2}g^{ab}\eta^{kl}{}_{mn}.$$

In exactly the same way one gets

(4.13) $$\bar{\eta}^{klca}S^b{}_c = -2g^{[k(a}S^{b)l]} + \tfrac{1}{2}g^{ab}S^{kl},$$

and

(4.14) $$\bar{\eta}^{kla}{}_c\eta^{bc}{}_{mn} = 2g^{[k(a}\eta^{b)l]}{}_{mn} - \tfrac{1}{2}g^{ab}\eta^{kl}{}_{mn},$$

(see also eq. (4.18)).

(c) The "integrability conditions on the Lorentz group" (2.24) are often quoted in the literature. It is desirable to consider, more generally, the matrix product $S^{kl}S^{ab}$ from which (2.24) *and* the anti-commutator $[S^{kl}, S^{ab}]_+$ may be read off directly. Write

$$(S^{kl}S^{ab})^\mu{}_\nu = \tfrac{1}{2}S^{kl\mu}{}_\alpha(\sigma^{a\dot{\rho}\alpha}\sigma^b{}_{\dot{\rho}\nu} - \sigma^{b\dot{\rho}\alpha}\sigma^a{}_{\dot{\rho}\nu})$$

and apply (2.17) to each term on the right. It then follows directly that

$$\begin{aligned}(S^{kl}S^{ab})^\mu{}_\nu &= \eta^{klc[a}\sigma^{b]}{}_{\dot{\rho}\nu}\sigma_c{}^{\dot{\rho}\mu}\\ &= \tfrac{1}{2}\eta^{klc[a}(\delta^{b]}{}_c\delta^\mu{}_\nu - 2S^{b]}{}_c{}^\mu{}_\nu), \quad \text{by (2.11)},\end{aligned}$$

so that

$$S^{kl}S^{ab} = -\tfrac{1}{2}\eta^{klab} - \eta^{klc[a}S^{b]}{}_c.$$

Drawing upon (4.10) one has therefore the result in the desired form

(4.15) $$S^{kl}S^{ab} = 2g^{[k[a}S^{b]l]} - \tfrac{1}{2}\eta^{klab}.$$

From this (2.24) may be read off at once. On the other hand

(4.16) $$[S^{kl}, S^{ab}]_+ = -\eta^{klab}.$$

If (4.14) be transvected with $S_{ab}{}^{\gamma\delta}$ the resulting relation may be simplified by means of (2.6, 8, 18, 19) to give

$$S^{kl[\mu(\alpha}\gamma^{\beta)\nu]} = \tfrac{1}{2}\gamma^{\mu\nu}S^{kl\alpha\beta}.$$

This, however, is nothing new: on the contrary the stronger relation

$$S^{kl[\mu|\alpha}\gamma^{\beta|\nu]} = \tfrac{1}{2}\gamma^{\mu\nu}S^{kl\alpha\beta}$$

follows at once from (4.5) if one takes $t^{\mu\nu}{}_{\dots}{}^{\dots} = S^{kl\mu\nu}$. On the other hand (4.15) may be transvected with $S^{cd\nu}{}_\mu$ in which case one finds

(4.17) $$tr(S^{kl}S^{ab}S^{cd}) = -2g^{[k[a}\eta^{b]l]cd},$$

from which it follows incidentally that

https://doi.org/10.1017/S1446788700026951 Published online by Cambridge University Press

(4.18) $$g^{[k[a}{}_{}\eta^{b]l]}{}_{cd} = -\,\delta^{[k}{}_{[c}\eta_{d]}{}^{l]ab},$$

whence (4.11) may be written in a slightly different way.

(d) A useful transformation of the outer product $S^{kl\dot{\mu}}{}_{\dot{\nu}}S^{mn\alpha}{}_{\beta}$ may be obtained as follows. Transvect with $\sigma^{a\dot{\nu}\beta}$, giving

$$S^{kl\dot{\mu}}{}_{\dot{\nu}}S^{mn\alpha}{}_{\beta}\sigma^{a\dot{\nu}\beta} = \eta^{mnas}\bar{\eta}^{kl}{}_{st}\sigma^{t\dot{\mu}\alpha},$$

by double application of (2.17). Transvecting now with $\sigma_{a\dot{\nu}'\beta'}$

(4.19) $$S^{kl\dot{\mu}}{}_{\dot{\nu}}S^{mn\alpha}{}_{\beta} = \eta^{mnas}\bar{\eta}^{kl}{}_{st}\sigma^{t\dot{\mu}\alpha}\sigma_{a\dot{\nu}\beta},$$

where, incidentally, the product of the η-symbols on the right is of the form which occurs in eq. (4.14).

5. Relations involving $P^{\mu}{}_{\nu kl}$

(a) From (2.37) one has directly on transvecting with g^{lm}

(5.1) $$R_{kl} = S_{k}{}^{m\nu}{}_{\mu}P^{\mu}{}_{\nu lm} + S_{k}{}^{m\dot{\nu}}{}_{\dot{\mu}}P^{\dot{\mu}}{}_{\dot{\nu}lm}.$$

However, consider eq. (3.4). Using (2.21) to remove the η-tensor it becomes

$$-\,R_{k}{}^{l} = S^{lc\alpha\beta}S^{mn}{}_{\alpha\beta}R_{kcmn} = 2S^{lb\alpha\beta}P_{\alpha\beta kb}, \quad \text{by (2.35)},$$

i.e.

(5.2) $$R_{kl} = 2S_{k}{}^{m\nu}{}_{\mu}P^{\mu}{}_{\nu lm}.$$

Comparison with (5.1) shows that the validity of the cyclic identity (2.28) is equivalent to the *reality* of $S_{k}{}^{m\nu}{}_{\mu}P^{\mu}{}_{\nu lm}$. It is occasionally useful to rewrite (5.2) by means of (2.16) as

(5.3) $$\sigma^{m}{}_{\dot{\lambda}\mu}P^{\mu}{}_{\nu km} = \tfrac{1}{2}\sigma^{m}{}_{\dot{\lambda}\nu}R_{km}.$$

Transvecting (5.2) with g^{kl} and then using (4.5) one has as a special case of the former

(5.4) $$S^{kl\mu}{}_{\alpha}P^{\nu\alpha}{}_{kl} = -\,\tfrac{1}{4}\gamma^{\mu\nu}R.$$

The following point may be noted. The quantity on the left of (5.4) is skew in μ and ν, and is a scalar for a given spin frame. It therefore follows at once that it must be a numerical multiple of $\gamma^{\mu\nu}R$. This kind of reasoning sometimes allows the reduction of more or less complicated expressions in an elegant manner, (cf. section 5(c)).

(b) An important identity follows directly from (2.37) and (3.3), viz.

$$\bar{\eta}^{cd}{}_{mn}S^{kl\mu}{}_{\nu}P^{\nu}{}_{\mu cd} = -\,2\eta^{kl}{}_{a[m}E_{n]}{}^{a}.$$

Transvecting this with $S_{kl\alpha\beta}$ one has

(5.5) $$\bar{\eta}^{cd}{}_{mn} P^{\alpha\beta}{}_{cd} = 2S_{a[m}{}^{\alpha\beta} E_{n]}{}^{a}.$$

Transvecting this in turn with $S^{mn\dot{p}\dot{\sigma}}$ and using (2.10) one gets

(5.6) $$S^{cd\dot{p}\dot{\sigma}} P^{\alpha\beta}{}_{cd} = S^{mn\dot{p}\dot{\sigma}} S_{am}{}^{\alpha\beta} E_{n}{}^{a}.$$

Using (2.20) on the right and keeping the symmetries of $P^{\alpha\beta}{}_{cd}$ and E_{ab} in mind one finds

(5.7) $$S^{cd\dot{p}\dot{\sigma}} P^{\alpha\beta}{}_{cd} = \sigma^{a\dot{p}\alpha} \sigma^{b\dot{\sigma}\beta} E_{ab},$$

which is essentially a relation derived elsewhere (reference [2], p. 100). If, starting with (5.7) therefore, one transvects the latter with $S_{kl\dot{p}\dot{\sigma}}$ one gets

(5.8) $$\bar{\eta}^{cd}{}_{kl} P^{\alpha\beta}{}_{cd} = \sigma_{[k\dot{p}}{}^{\lambda} \sigma_{l]\dot{\sigma}\lambda} \sigma^{a\dot{p}\alpha} \sigma^{b\dot{\sigma}\beta} E_{ab} = \bar{\eta}^{cb}{}_{kl} S^{a}{}_{c}{}^{\alpha\beta} E_{ab},$$

using first (2.12) and then (2.11). This does not look like (5.5). However, it may immediately be reduced to the latter by means of (4.13), keeping in mind that E_{kl} is trace-free. Incidentally, (5.5) implies that

(5.9) $$S_{kl}{}^{\mu\nu} P_{\mu\nu mn} - S_{mn}{}^{\mu\nu} P_{\mu\nu kl} = ie_{kla[m} E_{n]}{}^{a}.$$

(c) The identity

(5.10) $$ie^{klam} S^{b}{}_{m\lambda}{}^{\rho} P^{\lambda}{}_{\rho kl} = R^{ab} - \tfrac{1}{2} g^{ab} R$$

is of interest because of the appearance of the Einstein tensor on the right. It may be inferred for instance by using (2.35), (2.6), (2.28), (2.4), in this order. If, however one draws upon an analytical property of the Riemann tensor, viz. the Bianchi identity

(5.11) $$R_{kl[mn;s]} = 0,$$

then one may invoke the reasoning outlined at the end of section 5(a) to arrive at (5.10) more elegantly.

First, observe that (5.11), upon transvection with $S^{kl\mu}{}_{\nu}$, at once implies

(5.12) $$P^{\mu}{}_{\nu[kl;m]} = 0.$$

If t^{ab} denote the left hand member of (5.10) then, by inspection,

$$t^{ab}{}_{;a} = 0,$$

in view of (5.12). However, the only divergence-free fundamental tensor of a V_4 which is a linear homogeneous function of the components of the Riemann tensor is $R^{ab} - \frac{1}{2}g^{ab}R$ to within a constant factor, c, say. Now

$$t_{a}{}^{a} = -2S^{kl\rho}{}_{\lambda} P^{\lambda}{}_{\rho kl} = -2R,$$

because of (2.8) and (5.2). It follows that $c = 1$, which establishes the result (5.10).

6. A relation involving the conformal curvature tensor

Finally I consider an identity involving products of the components of the curvature spinor with the S-tensor-spinor whose indices are *undotted*, (as contrasted with (4.19) for instance). Originally I obtained this identity by a very clumsy route, requiring many of the preceding relations. It turns out, however, that the result follows immediately, indeed almost trivially, by repeated application of (4.5). Consider the symmetrized product

$$S^{kl}{}_{(\rho\sigma}P^{\mu}{}_{\lambda)kl} = \tfrac{1}{3}(S^{kl}{}_{\rho\sigma}P^{\mu}{}_{\lambda kl} + S^{kl}{}_{\lambda\rho}P^{\mu}{}_{\sigma kl} + S^{kl}{}_{\sigma\lambda}P^{\mu}{}_{\rho kl}). \tag{6.1}$$

The second term on the right is, in view of (4.5),

$$S^{kl}{}_{\rho\lambda}P^{\mu}{}_{\sigma kl} = S^{kl}{}_{\rho\sigma}P^{\mu}{}_{\lambda kl} + \gamma_{\sigma\lambda}S^{kl}{}_{\rho}{}^{\epsilon}P^{\mu}{}_{\epsilon kl};$$

similarly the third term on the right is

$$S^{kl}{}_{\sigma\lambda}P^{\mu}{}_{\rho kl} = S^{kl}{}_{\rho\sigma}P^{\mu}{}_{\lambda kl} + \gamma_{\rho\lambda}S^{kl}{}_{\sigma}{}^{\epsilon}P^{\mu}{}_{\epsilon kl}.$$

Applying (5.4) to the last members of the two preceding equations and collecting terms, one has

$$S^{kl}{}_{(\rho\sigma}P^{\mu}{}_{\lambda)kl} = S^{kl}{}_{\rho\sigma}P^{\mu}{}_{\lambda kl} + \tfrac{1}{6}\delta^{\mu}{}_{(\rho}\gamma_{\sigma)\lambda}R. \tag{6.2}$$

From (2.33)

$$S^{kl}{}_{\rho\sigma}S^{mn\mu\nu}C_{klmn} = S^{kl}{}_{\rho\sigma}S^{mn\mu\nu}(R_{klmn} - 2g_{kn}R_{ml} + \tfrac{1}{3}g_{kn}g_{ml}R).$$

The second term on the right is

$$\begin{aligned} 2S^{l}{}_{n\rho\sigma}S^{mn\mu\nu}R_{ml} &= 2\sigma^{m\dot{\varepsilon}(\mu}\delta^{l}{}_{\dot{\varepsilon}(\rho}\delta^{\nu)}{}_{\sigma)}R_{ml}, \quad \text{by (2.18)},\\ &= \delta_{\rho}{}^{(\mu}\delta_{\sigma}{}^{\nu)}R, \quad \text{by (2.5)}. \end{aligned}$$

Again, the third term on the right is

$$-\tfrac{1}{3}S^{kl}{}_{\rho\sigma}S_{kl}{}^{\mu\nu}R = -\tfrac{2}{3}\delta^{(\mu}{}_{\rho}\delta^{\nu)}{}_{\sigma}R, \quad \text{by (2.19)}.$$

Rewriting the first term by means of (2.35) one therefore has

$$S^{kl}{}_{\rho\sigma}S^{mn\mu\nu}C_{klmn} = 2S^{kl}{}_{\rho\sigma}P^{\mu\nu}{}_{kl} + \tfrac{1}{3}\delta^{(\mu}{}_{\rho}\delta^{\nu)}{}_{\sigma}R.$$

Transvecting with $\gamma_{\nu\lambda}$ and comparing with (6.2) there follows the desired result

$$S^{kl}{}_{(\rho\sigma}P^{\mu}{}_{\lambda)kl} = \tfrac{1}{2}S^{kl}{}_{\rho\sigma}S^{mn\mu}{}_{\lambda}C_{klmn}. \tag{6.3}$$

One may replace the conformal curvature tensor on the right by the conformal curvature spinor introduced elsewhere [7]. Thus

$$S^{kl}{}_{(\rho\sigma}P^{\mu}{}_{\lambda)kl} = S^{kl}{}_{\rho\sigma}\Gamma^{\mu}{}_{\lambda kl}. \tag{6.4}$$

7. Einstein spaces and spaces of constant curvature

Identities relating specifically to an S_4, i.e. a space of constant Riemannian curvature, are redundant in the sense that, in view of (2.38), any of the relations of section 4 which involve at least one S-symbol will at the same time constitute an identity of the kind contemplated here. However, one may pose a rather more interesting problem, viz. one may enquire into the implications of the vanishing of certain expressions involving $P^{\mu}{}_{\nu kl}$. I shall consider two such cases which are of particular interest.

(i) Suppose the V_4 is such that

$$S^{kl}{}_{(\rho\sigma}P^{\mu}{}_{\lambda)kl} = 0. \tag{7.1}$$

Then the right hand member of (6.3) vanishes, and, transvecting it with $S_{ab}{}^{\rho\sigma}S_{cd\mu}{}^{\lambda}$, this implies that

$$\eta^{kl}{}_{ab}\eta^{mn}{}_{cd}C_{klmn} = 0. \tag{7.2}$$

The vanishing of the real part of the tensor on the left requires

$$C_{abcd} - \tfrac{1}{4}e^{kl}{}_{ab}e^{mn}{}_{cd}C_{klmn} = 0.$$

The second term on the right may now be treated exactly along the lines of section 3(a), since R_{abcd} and C_{abcd} have the same symmetries. Keeping (2.34) in mind, the tensor corresponding to E_{kl} in (3.1) is a zero tensor; so that

$$e^{kl}{}_{ab}e^{mn}{}_{cd}C_{klmn} = -4C_{abcd}. \tag{7.4}$$

Hence (7.3) implies $C_{klmn} = 0$. This in turn means that (reference [6], p. 92) the V_4 is an S_4. Conversely, if the V_4 is an S_4 then $C_{klmn} = 0$, and (7.1) is satisfied. Hence

a necessary and sufficient condition that a V_4 be a space of constant Riemannian curvature is that $S^{kl}{}_{(\rho\sigma}P^{\mu}{}_{\lambda)kl}$ be a zero spinor.

(ii) Suppose now that the V_4 is an Einstein space, i.e. that (2.31) is satisfied. Then (5.5) shows that

$$\bar{\eta}^{kl}{}_{mn}P^{\alpha}{}_{\beta kl} = 0. \tag{7.5}$$

Thus,

in an Einstein space $P^{\alpha}{}_{\beta kl}$ is a self dual tensor,

(the spin frame being fixed but arbitrary). Contemplating the condition

$$S^{cd\dot\rho\dot\sigma}P^{\alpha}{}_{\beta cd} = 0, \tag{7.6}$$

which is equivalent to (7.5), one sees at once from (5.7) that it implies that the V_4 is an Einstein space. Hence *a necessary and sufficient condition that a* V_4 *be an Einstein space is that* $S^{kl}{}_{\dot{\rho}\dot{\sigma}} P^{\mu}{}_{\lambda kl}$ *be a zero spinor.*

Evidently (7.6) is a weaker condition than (7.1); a conclusion which casual inspection of the respective expressions involved would scarcely suggest.

Bibliography

[1] Infeld, L. and van der Waerden, B.L., Die Wellengleichung des Elektrons in der allgemeinen Relativitätstheorie, Sitz. Preu. Akad. Wiss., **9** (1930), 380—401.

[2] E.g. Buchdahl, H. A., On the compatibility of relativistic wave equations for particles of higher spin in the presence of a gravitational field, Nuovo Cimento, **10** (1958), 96—103.

[3] Harish-Chandra, A note on the σ-symbols, Proc. Ind. Acad. Sci., **23** (1946), 152—163.

[4] Corson, E.M., Tensors, spinors, and relativistic wave equations. Blackie, London (1953), Chap. II, §§ 6—13.

[5] Bade, W. L., and Jehle, H., An introduction to spinors, Revs. Mod. Phys., **25** (1953), 714—729.

[6] Eisenhart, L. P., Riemannian geometry, Princeton University Press (1949), Chap. 2, p. 90.

[7] Buchdahl, H. A., On extended conformal transformations of spinors and spinor equations, Nuovo Cimento, **11** (1959), 496—506.

Physics Department, University of Tasmania,
Hobart, Tasmania