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CLASS-NUMBER PROBLEMS FOR CUBIC NUMBER FIELDS

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1. Introduction

Let $\mathbf{M}$ be any number field. We let D_{M}, d_{M}, h_{M}, ζ_{M}, $\mathbf{A}_{\mathrm{M}}$ and $\mathrm{Reg}_{\mathrm{M}}$ be the discriminant, the absolute value of the discriminant, the class-number, the Dedekind zeta-function, the ring of algebraic integers and the regulator of $\mathbf{M}$, respectively. We set $c = \dfrac{3+2\sqrt{2}}{2}$. If q is any odd prime we let $(\cdot/q)$ denote the Legendre's symbol. We let D_P and d_P be the discriminant and the absolute value of the discriminant of a polynomial P.

LEMMA A (See [Sta, Lemma 3] and [Hof, Lemma 2]). *Let $\mathbf{M}$ be any number field. Then, ζ_{M} has at most one real zero in*

$$\left[1 - \frac{1}{c \log d_{\mathrm{M}}}, 1\right[;$$

if such a zero exists, it is simple and is called a Siegel zero.

LEMMA B (See [Lou 2]). *Let $\mathbf{M}$ be a number field of degree $n = r_1 + 2r_2$ where $\mathbf{M}$ has r_1 real conjugate fields and $2r_2$ complex conjugate fields. Let $s_0 \in [(1/2), 1[$ be such that $\zeta_{\mathrm{M}}(s_0) \leq 0$. Then,*

$$\mathrm{Res}_{s=1}(\zeta_{\mathrm{M}}) \geq (1-s_0) d_{\mathrm{M}}^{(s_0-1)/2}\left(1 - \frac{2r_1}{d_{\mathrm{M}}^{s_0/2n}} - \frac{2\pi r_2}{d_{\mathrm{M}}^{s_0/n}}\right).$$

2. Lower bounds for class-numbers of cubic number fields

Let $\mathbf{K}$ be a cubic number field. If $\mathbf{K}/\mathbf{Q}$ is normal then $\mathbf{K}$ is a cyclic cubic number field. Let $f_{\mathbf{K}}$ be its conductor. Then $d_{\mathbf{K}} = f_{\mathbf{K}}^2$ and $\zeta_{\mathbf{K}}(s) = \zeta(s)L(s,\chi)L(s,\bar{\chi})$ where χ is a primitive cubic Dirichlet character modulo $f_{\mathbf{K}}$. Hence, we get

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$\zeta_{\mathbf{K}}(s) \leq 0$, $s \in [0,1[$. With $s_0 = 1 - (2/\log d_{\mathbf{K}})$ Lemma B provides the following lower bound which improves the one given in [Let]

$$\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}}) \geq \frac{1}{3\log(f_{\mathbf{K}})}, f_{\mathbf{K}} \geq 10^6.$$

Lettl used his lower bound to determine all the simplest cubic number fields with small class-numbers. Here, the simplest cubic number fields are real cubic cyclic numbers fields defined as being the splitting fields of the polynomials $P(X) = X^3 - aX^2 - (a+3)X - 1$ $(a \geq 0)$ whose discriminants $d_P = (a^2 + 3a + 9)^2$ are square of a prime $p = a^2 + 3a + 9$, which implies $\mathbf{A_K} = \mathbf{Z}[\varepsilon]$ where $\varepsilon > 1$ is the only real root of $P(X)$ greater than one (we have $\varepsilon \in]a+1, a+2[$). He found that there are 7 simplest cubic fields with class-number one, and none with class-number two or three.

In the same spirit, Lazarus determined all the simplest quartic fields with class-number 1 or 2. Here, the simplest quartic number fields are real quartic cyclic number fields defined as being the splitting fields of the polynomials $P(X) = X^4 - 2aX^3 - 6X^2 + 2aX + 1$ $(a \geqslant 0)$ whose discriminants $d_P = 256 d_a^3$ are such that $d_a = a^2 + 4$ is odd-square-free. He found that there are 6 simplest quartic fields with class-number one, and 3 with class-number two.

From now on, we assume that $\mathbf{K}$ is not normal. Thus, the normal closure $\mathbf{N}$ of $\mathbf{K}$ is a sextic number field with Galois group the symmetric group $\mathcal{S}_3$.

LEMMA C. *Let $\mathbf{K}/\mathbf{Q}$ be a non-normal cubic extension with normal closure $\mathbf{N} = \mathbf{KL}$ where $\mathbf{L} = \mathbf{Q}(\sqrt{D_{\mathbf{K}}})$ is quadratic. It holds $\zeta_{\mathbf{N}}\zeta^2 = \zeta_{\mathbf{K}}^2\zeta_{\mathbf{L}}$. Hence, $d_{\mathbf{N}} = d_{\mathbf{K}}^2 d_{\mathbf{L}}$, and $d_{\mathbf{N}}$ divides $d_{\mathbf{K}}^3$. Finally, $\zeta_{\mathbf{K}}$ does not have any real zero in the range $[1 - (1/(3c\log d_{\mathbf{K}})), 1[$.*

Proof. The first point is proved page 227 of [Cas-Fro] using Artin L-series formalism. The second point follows from the first one using the functional equations satisfied by these zeta-functions (see [Lou 1]). According to the first point, any real zero in $]0,1[$ of $\zeta_{\mathbf{K}}$ is a multiple zero of $\zeta_{\mathbf{N}}$. Hence, the fourth point follows from Lemma A. It remains to prove the third point. According to Stickelberger's theorem, we have $D_{\mathbf{K}} \equiv 0,1 \pmod 4$. Hence, $D_{\mathbf{L}}$ divides $D_{\mathbf{K}}$, and we may define $f \geq 1$ by means of $D_{\mathbf{K}} = f^2 D_L$. □

According to Lemmata A, B and C, we get:

THEOREM 1. *Let* $\mathbf{K}$ *be a non-normal cubic number field. It holds*

$$h_{\mathbf{K}}\,\mathrm{Reg}_{\mathbf{K}} \geq \frac{1}{55}\frac{\sqrt{d_{\mathbf{K}}}}{\log d_{\mathbf{K}}},\ d_{\mathbf{K}} \geq 4\cdot 10^{5}.$$

3. Computation of the class-number of a non-abelian cubic number field K with negative discriminant

We make use of the results of [Bar-Lox-Wil] and [Bar-Wil-Ban]. Set $\Phi(s) = (\zeta_{\mathbf{K}}/\zeta)(s) = \sum_{j\geq 1}\alpha(j)j^{-s}$. Then,

$$h_{\mathbf{K}}\,\mathrm{Reg}_{\mathbf{K}} = \frac{\Phi(1)}{C} = \sum_{j\geq 1}\frac{\alpha(j)}{jC}e^{-jC} + \sum_{j\geq 1}\alpha(j)E(jC)$$

where

$$E(y) = \int_{y}^{\infty}\frac{e^{-x}}{x}\,dx = -\log x - \gamma - \sum_{j\geq 1}\frac{(-1)^{j}}{j(j!)}x^{j},$$

where $\gamma = 0.577215664901\ldots$ is the Euler's constant, and where $C = 2\pi/\sqrt{d_{\mathbf{K}}}$.

Now, $j \mapsto \alpha(j)$ is a multiplicative function such that $\alpha(p^{n}) = F(p^{n}) - F(p^{n-1})$ where $F(k)$ is the number of distinct ideals of $\mathbf{K}$ with norm $k \geq 1$. Moreover, if p does not divide $d_{\mathbf{K}}$, then

$$(p) = \begin{cases} (p) & \text{which implies } (D_{\mathbf{K}}/p) = +1 \text{ (Type I)},\\ \mathscr{P}_1\mathscr{P}_2 & \text{if and only if } (D_{\mathbf{K}}/p) = -1 \text{ (Type II)},\\ \mathscr{P}_1\mathscr{P}_2\mathscr{P}_3 & \text{which implies } (D_{\mathbf{K}}/p) = +1 \text{ (Type III)}.\end{cases}$$

If p divides $d_{\mathbf{K}}$, then with f as in the proof of Lemma C we have

$$(p) = \begin{cases} \mathscr{P}_1^{2}\mathscr{P}_2 & \text{if } p \text{ does not divide } f \text{ (Type IV)},\\ \mathscr{P}^{3} & \text{if } p \text{ divides } f \text{ (Type V)}.\end{cases}$$

If the ring of algebraic integers of $\mathbf{K}$ is generated by an algebraic integer $x_{\mathbf{K}}$ and if $P_{\mathbf{K}}(X)$ is the minimum polynomial over $\mathbf{Q}$ of $x_{\mathbf{K}}$, then we are in the Type I or Type III cases according as $P_{\mathbf{K}}(X)$ does not have or has at least one root modulo p, and we are in the Type IV or Type V cases according as $P_{\mathbf{K}}(X)$ has a double or a triple root modulo p.

4. Explicit class-number problem for non-normal cubic number fields

First example. In the same way we got Theorem 1, we get

THEOREM 2. (a) (See [Fro-Tay, Chapter 5]) *Let* $l \geq 1$ *be an integer. Set* $P_l(X) = X^3 + lX - 1$. *Then* $P_l(X)$ *is irreducible in* $\mathbf{Q}[X]$, *has negative discriminant* $D_l = -d_l = -(4l^3 + 27)$, *and has exactly one real root* x_l. *Set* $\varepsilon_l = 1/x_l$. *Then,* $l < \varepsilon_l < l + 1$. *Set* $\mathbf{K}_l = \mathbf{Q}(x_l) = \mathbf{Q}(\varepsilon_l)$. *Then* $\mathbf{K}_l$ *is a real cubic number field with negative discriminant and* ε_l *is the fundamental unit greater than one of the cubic order* $\mathbf{Z}[x_l] = \mathbf{Z}[\varepsilon_l]$.

(b) *Assume that the ring of algebraic integers* $\mathbf{A}_l$ *of* $\mathbf{K}_l$ *is equal to* $\mathbf{Z}[\varepsilon_l]$. *Then,*

$$h_l \geq \frac{\sqrt{d_l}}{20 \log^2(d_l)} \text{ when } d_l \geq 2 \cdot 10^5, \text{ and } h_l > 3 \text{ when } l > 400$$

(*here* h_l *is the class-number of* $\mathbf{K}_l$). *Hence, there are* 5 *such* $\mathbf{K}_l$ *with class-number one, namely* $\mathbf{K}_1$, $\mathbf{K}_2$, $\mathbf{K}_3$, $\mathbf{K}_5$ *and* $\mathbf{K}_{11}$; *there are* 2 *such* $\mathbf{K}_l$ *with class-number two, namely* $\mathbf{K}_4$ *and* $\mathbf{K}_7$; *there are* 3 *such* $\mathbf{K}_l$ *with class-number three, namely* $\mathbf{K}_6$, $\mathbf{K}_{15}$ *and* $\mathbf{K}_{17}$.

LEMMA D. (a) *The ring of algebraic integers of* $\mathbf{K}_l$ *is equal to* $\mathbf{Z}[\varepsilon_l]$ *if and only if* 3^2 *does not divide* l, *and* p^2 *does not divide* $d_l = 4l^3 + 27$ *for any prime* $p \geq 5$.

(b) *Under this assumption, we have*:

(i) *For any prime* $p \neq 3$ *that divides* d_l *we have* $F(p^n) = n + 1$ *and* $\alpha(p^n) = 1$, $n \geq 1$.

(ii) *If* $p = 3$ *divides* d_l, *then* $F(3^n) = 1$ *and* $\alpha(3^n) = 0$, $n \geq 1$.

(iii) *If* p *does not divide* d_l *and* $(-d_l/p) = -1$, *then* $\alpha(p^n) = (1 + (-1)^n)/2$.

(iv) *If* p *does not divide* d_l *and* $(-d_l/p) = +1$, *then*

$$\alpha(p^n) = \begin{cases} n+1 & \text{if } P_l(X) \text{ has at least one root modulo } p, \\ (n+1/3) = \begin{cases} 1 & \text{if } n \equiv 0 \pmod 3, \\ -1 & \text{if } n \equiv 1 \pmod 3, \\ 0 & \text{if } n \equiv 2 \pmod 3, \end{cases} & \text{if } P_l(X) \text{ has no real root modulo } p. \end{cases}$$

Proof. Point (b) follows from Section 3 (see [Bar-Lox-Wil, Table 2]).

Now, we proceed with the proof of point (a). We note that the integral bases of all cubic fields are determined in [Alb]. Though, we give a simple proof for this special case. Let t be an integer and set

$$y_{p,l}(t) = \frac{x_l^2 + tx_l + t^2 + l}{p}.$$

Then $y_{p,l}(t)$ is a root of the following polynomial

$$Y^3 - \frac{P_l'(t)}{p} Y^2 + \frac{3tP_l(t)}{p^2} Y - \frac{P_l(t)^2}{p^3}$$

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(being the characteristic polynomial of the linear map $z \mapsto y_{p,l}(t)z$ of the three $\mathbf{Q}$-dimensional vector space $\mathbf{K}_l$, then $y_{p,l}(t)$ is indeed a root of this polynomial). Hence, $y_{p,l}(t)$ is an algebraic integer provided that $P_l(t) \equiv 0 \pmod{p^2}$ and $P_l'(t) \equiv 0 \pmod{p}$.

Now, assume that $\mathbf{Z}[x_l]$ is the ring of algebraic integers of $\mathbf{K}_l$ and let p be any prime which divides $d_l = 4l^3 + 27$. Hence p is odd.
First, assume that $p \neq 3$. Then p does not divide l and we take $t = t_{l,p}$ such that $2lt_{l,p} \equiv 3 \pmod{p}$ and where we hence write $2lt_{l,p} = 3 + ap$. Then,

$$4l^2 P_l'(t_{p,l}) = 3(3+ap)^2 + 4l^3 \equiv d_l \equiv 0 \pmod{p},$$

$$8l^3 P_l(t_{p,l}) = (3+ap)^2 + 4l^3(3+ap) - 8l^3 \equiv d_l + apd_l \equiv d_l \pmod{p^2}.$$

Hence, if p^2 would divide d_l then $y_{p,l}(t_{p,l})$ would be an algebraic integer of $\mathbf{K}_l$ and p would divide the index of $\mathbf{Z}[x_l]$ in $\mathbf{A}_l$. Contradiction.
Second, assume that $p = 3$. If 3^2 would divide l, then $y_{3,l}(1)$ would be an algebraic integer. Hence, 3 would divide the index of $\mathbf{Z}[x_l]$ in $\mathbf{A}_l$. Contradiction.

Conversely, first assume that if $p \neq 3$ divides d_l then p^2 does not divides d_l. Since $d_l = (\mathbf{A}_l : \mathbf{Z}[x_l])^2 d_{\mathbf{K}_l}$, then p does not divides this index $(\mathbf{A}_l : \mathbf{Z}[x_l])$ which is thus a 3-power. Note that if 3 divides this index, then 3 divides d_l, i.e., 3 divides l.
Second, it is known that if x is an algebraic integer whose minimum polynomial is p-Eisenstein for a prime p, then p does not divides the index of $\mathbf{Z}[x]$ in the ring of algebraic integers of the number field $\mathbf{Q}(x)$ (see [Nar, Lemma 2.2, page 60]). Hence, if we can find an integer a such that $P_l(a) \equiv P_l'(a) \equiv 0 \pmod{3}$ and $P_l(a) \not\equiv 0 \pmod{3^2}$, then the minimum polynomial $Q_a(Y) = P_l(Y+a) = Y^3 + 3aY^2 + P_l'(a)Y + P_l(a)$ of $y_l(a) = x_l - a$ is then 3-Eisenstein. Hence, 3 does not divide the index $(\mathbf{A}_l : \mathbf{Z}[x_l])$, which implies the desired result. Now, assume that 3 divides l but that 3^2 does not divide l. Then, $Q_1(Y)$ being 3-Eisenstein we get the desired result. □

The computational method of the third section and Lemma D provide the following table of class-numbers h_l of these number fields $\mathbf{K}_l$ for the first values of $l \geq 1$ such that $\mathbf{Z}[\varepsilon_l]$ is the ring of algebraic integers of $\mathbf{K}_l$. According to the more extensive class-number computation of h_l for $l \leq 400$, we easily get Theorem 2(b).

l	d_l	h_l	l	d_l	h_l	l	d_l	h_l	l	d_l	h_l
1	31	1	13	8815	5	24	55323	18	37	202639	8
2	59	1	14	11003	8	25	62527	6	38	219515	23
3	135	1	15	13527	3	26	70331	8	39	237303	15
4	283	2	16	16411	8	28	87835	32	40	256027	28
5	527	1	17	19679	3	29	97583	6	41	275711	6
6	891	3	19	27463	7	30	108027	15	42	296379	39
7	1399	2	20	32027	10	31	119191	12	43	318055	16
10	4027	6	21	37071	6	32	131099	10	44	340763	18
11	5351	1	22	42619	12	34	157243	28	46	389371	27
12	6939	6	23	48695	4	35	171527	12	47	415319	8

□

Second example. Let $\mathbf{K} = \mathbf{Q}(\sqrt[3]{d})$, $d \geq 2$, be a real pure cubic number field. Assume that d is cube-free and define a and b by means of $(a, b) = 1$ and $d = ab^2$. Then, $D_{\mathbf{K}} = -3(ab)^2$ or $D_{\mathbf{K}} = -27(ab)^2$ according as $d \equiv \pm 1 \pmod 9$ or not. Now, $\mathbf{L} = \mathbf{Q}(\sqrt{-3})$ and $\zeta_{\mathbf{L}}$ does not have any real zero in $]0,1[$. Hence, we get $\zeta_{\mathbf{N}}(s) \leq 0$, $s \in]0,1[$. According to Lemma C, we have $\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}}) = (\mathrm{Res}_{s=1}(\zeta_{\mathbf{K}}))^2 \mathrm{Res}_{s=1}(\zeta_{\mathbf{L}})$ and we may apply Lemma B with $s_0 = 1 - (2/\log d_{\mathbf{N}})$, which provides an optimal lower bound on $\mathrm{Res}_{s=1}(\zeta_{\mathbf{N}})$. Since $\mathrm{Res}_{s=1}(\zeta_{\mathbf{L}}) = \pi/3\sqrt{3}$, we get the following lower bound that improves the one given in Theorem 1:

THEOREM 3 (See [Bar-Lou]). *Let* $\mathbf{K}$ *be a real pure cubic number field. Then,*

$$h_{\mathbf{K}} \,\mathrm{Reg}_{\mathbf{K}} \geq \frac{1}{9}\sqrt{\frac{d_{\mathbf{K}}}{\log d_{\mathbf{K}}}},\ d_{\mathbf{K}} \geq 3\cdot 10^4.$$

THEOREM 4. *When* $m \geq 1$ *is such that* $m^3 \pm 1$ *is cube-free, we set* $\mathbf{K}_{\pm m} \overset{def}{=} \mathbf{Q}(\sqrt[3]{m^3 \pm 1})$ *which is a pure cubic real number field. There are* 2 *such* $\mathbf{K}_{\pm m}$ *with class-number one, namely* $\mathbf{K}_{+1}$ *and* $\mathbf{K}_{+2}$; *there does not exist any such* $\mathbf{K}_{\pm m}$ *with class-number two; and there are* 3 *such* $\mathbf{K}_{\pm m}$ *with class-number three, namely* $\mathbf{K}_{-2}$, $\mathbf{K}_{-3}$ *and* $\mathbf{K}_{+3}$.

Proof. Set $d_{\pm m} = m^3 \pm 1$ and $\omega_{\pm m} = \sqrt[3]{d_{\pm m}}$. Then, $\varepsilon_{\pm m} = \pm/(\omega_{\pm m} - m) = \omega_{\pm m}^2 + m\omega_{\pm m} + m^2$ is a unit of $\mathbf{A}_{\mathbf{K}_{\pm m}}$, and we have $1 < \varepsilon_{\pm m} \leq 4\omega_{\pm m}^2 = (8d_{\pm m})^{2/3}$. Hence,

$$\mathrm{Reg}_{\mathbf{K}_{\pm m}} \leq \log \varepsilon_{\pm m} \leq \frac{2}{3}\log(8d_{\pm m}) \leq \frac{2}{3}\log(3d_{\mathbf{K}_{\pm m}}),$$

and according to Theorem 3 we get

$$h_{\mathbf{K}_{\pm m}} \geq \frac{1}{6}\sqrt{\frac{d_{\mathbf{K}_{\pm m}}}{\log^3(3d_{\mathbf{K}_{\pm m}})}},\ d_{\mathbf{K}_{\pm m}} \geq 3\cdot 10^4.$$

which implies $h_{\mathbf{K}_{\pm m}} > 3$ for $d_{\mathbf{K}_{\pm m}} > 1.1\cdot 10^6$. Note that $m > 72$ implies $d_{\mathbf{K}_{\pm m}} \geq 3d_m > 1.1\cdot 10^6$. Now, the computation of a and b for $d_{\pm m} = m^3 \pm 1$ and $1 \leq m \leq 72$ yields $d_{\mathbf{K}_{\pm m}} \leq 1.1\cdot 10^6$ if and only if $1 \leq m \leq 6$ when $d_{\pm m} = m^3 + 1$, and $2 \leq m \leq 7$ when $d_{\pm m} = m^3 - 1$ (note that $d_{\pm m}$ must be cube-free). The class-numbers of these number fields may be found in [Hos-Wad], and they provide the desired result. □

5. Another class-number problem

Let $\mathbf{K}$ be a real quadratic number field of discriminant $D > 0$. The ring of algebraic integers $\mathbf{A_K} = \mathbf{Z}\left[\frac{D+\sqrt{D}}{2}\right]$ of $\mathbf{K}$ writes $\mathbf{A_K} = \mathbf{Z}[\varepsilon]$ where $\varepsilon = \frac{u+v\sqrt{D}}{2} > 1$ is a unit of $\mathbf{A_K}$ if and only if $v = 1$, hence if and only if $D = m^2 \pm 4$, $m \geq 1$. In that case, $\varepsilon_D \leq \varepsilon = \frac{m+\sqrt{D}}{2} \leq \sqrt{D+4}$ where ε_D is the fundamental unit of $\mathbf{K}$. According to the Brauer-Siegel theorem, there are only finitely many real quadratic number fields with discriminants $D = m^2 \pm 4$ of given class-number. Up to now, no one knows how to make the Brauer-Siegel effective in the real quadratic case, without assuming a suitable generalized Riemann hypothesis (see [Mol-Wil]. In contrast to the real quadratic case, let $\mathbf{K}$ be a cubic number field with negative discriminant whose ring of algebraic integers $\mathbf{A_K}$ is generated by a unit ε, i.e. such that $\mathbf{A_K} = \mathbf{Z}[\varepsilon]$. This clearly amounts to saying that $\mathbf{A_K} = \mathbf{Z}[\varepsilon_{\mathbf{K}}]$, where $\varepsilon_{\mathbf{K}} > 1$ is the fundamental unit of $\mathbf{K}$. According to Theorem 1 and the following Proposition 5, we get the effective Corollary 6 that would enable one to explicitly determine all the cubic number fields of negative discriminants whose have small class-numbers and whose rings of algebraic integers are generated by units.

PROPOSITION 5. *A polynomial $P(X)$ is said of type (T) if it is a monic irreducible cubic polynomial with integral coefficients (say of the form $P(X) = X^3 - aX^2 + bX - 1$) with exactly one real root ε_P (i.e. $D_P < 0$) which satisfies $\varepsilon_P > 1$ (i.e. we have*

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$b \leq a-1$), *Then, there exists an effective constant* $c_1 > 0$ *such that for any polynomial of type* (*T*) *we have* $d_P = |D_P| \geq c_1 \varepsilon_P^{3/2}$.

COROLLARY 6. *Let* h *be a positive integer. Then, there are only finitely many cubic number fields* $\mathbf{K}$ *of negative discriminants* $D_{\mathbf{K}}$ *which have class-number* h *and whose rings of algebraic integers write* $\mathbf{A_K} = \mathbf{Z}[\varepsilon_{\mathbf{K}}]$, *where* $\varepsilon_{\mathbf{K}} > 1$ *is the fundamental unit of* $\mathbf{K}$. *Moreover, there exists* $c_2 > 0$ *effective such that* $|D_{\mathrm{K}}| \leq c_2 h^2 \log^4 h$ *for these number fields.*

Proo of Proposition 5. Set $\varepsilon = \varepsilon_P$, $x_1 = \varepsilon$, and let $x_2 = \alpha + i\beta$ and $x_3 = \alpha - i\beta$ be the two complex conjugate roots of this polynomial. Then,

$$\begin{aligned}
1 &= x_1x_2x_3 = \varepsilon(\alpha^2 + \beta^2),\\
b &= x_1x_2 + x_1x_3 + x_2x_3 = 2\varepsilon\alpha + (\alpha^2 + \beta^2) = 2\varepsilon\alpha + (1/\varepsilon),\\
a &= x_1 + x_2 + x_3 = \varepsilon + 2\alpha.
\end{aligned}$$

Hence, we have $|\alpha| \leq 1/\sqrt{\varepsilon} < 1$ and $\varepsilon < \alpha + 2$, which implies $a \geq 0$. Moreover, we have $|b| \leq 2\sqrt{\varepsilon} + 1 \leq 2\sqrt{a+2} + 1$, which implies that there are only finitely many polynomials of type (T) with $0 \leq a \leq 17$. Hence, we may assume $a \geq 18$.

We have $-D_P = 4(a^3 + b^3) - a^2b^2 - 18ab + 27$.

First, we assume $\alpha \geq 0$. Since $0 < 2\varepsilon\alpha + (1/\varepsilon) < 2\sqrt{\varepsilon + 2\alpha}$ (since $0 \leq \alpha \leq 1/\sqrt{\varepsilon}$ and $\varepsilon > 1$), we get $1 \leq b < 2\sqrt{a}$. Now $b \mapsto f(b) = -D_P = 4b^3 - a^2b^2 - 18ab + 4a^3 + 27$ is decreasing in the range $[1, 2\sqrt{a}]$ (since $a \geq 9$). So, we write $4a = m^2 + r$, with $m \geq 0$ and $0 \leq r \leq 2m$, which provides $16d_p \geq 16f(m)$ if $r \geq 1$, and $16d_P \geq 16f(m-1)$ if $r = 0$. Noticing that $16f(b) = (4a)^2(4a - b^2) - 72b(4a - b^2) - 8b^3 + 432$, we thus get

$$16d_P \geq \begin{cases} r(m^2+r)^2 - 72rm - 8m^3 + 432 \geq m^4 - 8m^3 + 2m^2 - 72m + 433 & \text{if } r \geq 1,\\ 2m^5 - m^4 - 8m^3 - 120m^2 + 192m + 368 & \text{if } r = 0.\end{cases}$$

Since $4a \leq m^2$ and $\varepsilon < a + 2$, we get the desired result.

Second, we assume $\alpha \leq 0$. Then, $b = 2\varepsilon\alpha + (1/\varepsilon) \leq 1/\varepsilon < 1$, i.e. $b \leq 0$. We set $B = -b$. Now $g(B) = -D_P = -4B^3 - a^2B^2 + 18aB + 4a^3 + 27$ is decreasing on $[1, +\infty[$ (since $a \geq 9$), and $g(\sqrt{4a+1}) < 0$ (since $a \geq 16$). Hence, we get $B < \sqrt{4a+1}$. So, we write $4a + 1 = m^2 + r$, with $m \geq 0$ and $0 \leq r \leq 2m$. Since $g(0) \geq g(1)$ (since $a \geq 18$), we have $16d_P \geq 16g(m)$ if $r \geq 1$, and $16d_P \geq 16g(m-1)$ if $r = 0$. We thus get

$$16d_P \geq \begin{cases} (r-1)(m^2+r-1)^2+72m(r-1)+8m^3+432 \geq 8m^3+432 & \text{if } r \geq 1, \\ 2m^5-2m^4+4m^3+124m^2-262m+566 & \text{if } r=0. \end{cases}$$

As in the first case, we get the desired result.

Let us note that when $P(X) = X^3 - M^2X^2 - 2MX - 1$ we have $d_P = 4M^3 + 27$ and $M^2 < \varepsilon_P < M^2 + 1$ ($M \geq 2$) which implies $d_P \approx 4\varepsilon_P^{3/2}$. □

6. Conclusion

Let h and n be two given positive integers. Are there only finitely many number fields $\mathbf{K}$ of degree n with class-number h such that their rings of algebraic integers are generated by units? More precisely, let $n \geq 1$ be a given positive integer and let $\mathbf{K} = \mathbf{Q}(x)$ be a number field of degree n where x is an algebraic unit which is a root of any irreducible monic polynomial of the form $P(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_1X \pm 1$. If we assume that the ring of algebraic integers of $\mathbf{K}$ is equal to $\mathbf{Z}[x]$ and that $\mathbf{K}$ has a unit group of rank 1, is that true that the class-number of $\mathbf{K}$ tends to infinity with $d_{\mathrm{K}} = d_P$?

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