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Decompositions of the Hilbert Function of a Set of Points in $\mathbb{P}^n$

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Abstract. Let $\mathbf{H}$ be the Hilbert function of some set of distinct points in $\mathbb{P}^n$ and let $\alpha = \alpha(\mathbf{H})$ be the least degree of a hypersurface of $\mathbb{P}^n$ containing these points. Write $\alpha = d_s + d_{s-1} + \cdots + d_1$ (where $d_i > 0$). We canonically decompose $\mathbf{H}$ into s other Hilbert functions $\mathbf{H} \leftrightarrow (\mathbf{H}'_s, \dots, \mathbf{H}'_1)$ and show how to find sets of distinct points $\mathbb{Y}_s, \dots, \mathbb{Y}_1$, lying on reduced hypersurfaces of degrees $d_s, \dots, d_1$ (respectively) such that the Hilbert function of $\mathbb{Y}_i$ is $\mathbf{H}'_i$ and the Hilbert function of $\mathbb{Y} = \bigcup_{i=1}^{s} \mathbb{Y}_i$ is $\mathbf{H}$. Some extremal properties of this canonical decomposition are also explored.

1 Introduction

In their paper [5], the authors devised an algorithm which assigned to $\mathbf{H}$, the Hilbert function of a non-degenerate set of (say) d points in $\mathbb{P}^n$, two "simpler" Hilbert functions, $\mathbf{H}_1$ and $\mathbf{H}'_1$. $\mathbf{H}'_1$ was considered simpler because it is the Hilbert function of $0 \neq d'_1 < d$ points in a codimension one *linear* subspace of $\mathbb{P}^n$ while $\mathbf{H}_1$ was considered simpler because it is the Hilbert function of a set of $d_1 = d - d'_1$ points in $\mathbb{P}^n$.

If it turned out that $\mathbf{H}_1$ was also the Hilbert function of d_1 points in a proper linear subspace of $\mathbb{P}^n$, the algorithm terminated. If not, the algorithm was then applied to the function $\mathbf{H}_1$ to construct $\mathbf{H}'_2$ and $\mathbf{H}_2$ (as above). Since both d'_1 and d_1 are less than d and any set of fewer than $n+1$ points of $\mathbb{P}^n$ always lie in a proper linear subspace of $\mathbb{P}^n$, the algorithm terminates.

We write

$$\mathbf{H} \longrightarrow (\mathbf{H}_r, \mathbf{H}'_r, \mathbf{H}'_{r-1}, \dots, \mathbf{H}'_2, \mathbf{H}'_1)$$

to refer to the string of Hilbert functions of sets of points in proper *linear* subspaces of $\mathbb{P}^n$ which the algorithm of [5] associates to $\mathbf{H}$. We think of this as a *linear decomposition* of $\mathbf{H}$. A formula was also given in [5] which showed how to calculate the values of $\mathbf{H}$ from those of the $\mathbf{H}'_i$ and $\mathbf{H}_r$.

Questions about the canonical nature of this decomposition and the possibility of having a method for taking a string of Hilbert functions of sets of points in proper linear subspaces of $\mathbb{P}^n$ and (mimicking the formula of [5]) combining them to give a function which could be the Hilbert function of the union of those points, was taken up in the paper [3]. Simple examples showed that not every string of Hilbert functions could be so combined. So, something about the decomposition of the algorithm had yet to be uncovered.

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In [3] we analyzed the algorithm of [5] in detail and discovered that the string of Hilbert functions it constructs have simple properties which are intertwined in a very precise fashion. This discovery enabled us to assert that the algorithm canoncially constructed a string of Hilbert functions with these additional properties and that any string of Hilbert functions with these properties could be combined as wanted. Moreover, the two processes of decomposition and recombination are inverse to each other. For details on this decomposition we refer the reader to [3] and [4].

We saw in [3] that this linear decomposition of $\mathbf{H}$ could be viewed as a generalization, to $\mathbb{P}^n$ $(n > 2)$, of the *numerical character* of the Hilbert function of a set of points in $\mathbb{P}^2$ (introduced by Gruson and Peskine to aid their study of curves in $\mathbb{P}^3$ [8]).

This decomposition also made evident certain *linear extremal properties* of sets of points in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$. More precisely (and using the notation above) we showed in [3] that among all sets of points in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$ there is at least one such, $\mathbb{X}$, having a subset $\mathbb{X}_1$ where

i) $\mathbb{X}_1$ lies on a hyperplane of $\mathbb{P}^n$; and
ii) $\mathbb{X}_1$ has Hilbert function $\mathbf{H}_1'$.

Moreover, among all sets $\mathbb{X}$ with Hilbert function $\mathbf{H}$ there is none which has more than $|\mathbb{X}_1|$ points in a hyperplane.

In fact, if we use the obvious point-wise partial ordering of Hilbert functions of points in $\mathbb{P}^n$ we obtained that $\mathbf{H}_1'$ is the unique maximal element in the following (finite) set of Hilbert functions $\mathcal{H}(\mathbf{H})$: where the Hilbert function $h \in \mathcal{H}(\mathbf{H}) \Leftrightarrow$ there is a linear subspace $\mathbb{L}$ of $\mathbb{P}^n$ and h is the Hilbert function of a set $\mathbb{Y} \subset \mathbb{L}$ and $\mathbb{Y}$ is a subset of a set of points $\mathbb{X} \subset \mathbb{P}^n$ which has Hilbert function $\mathbf{H}$.

In this paper we begin an exploration of other decompositions of the Hilbert function of a non-degenerate set of points in $\mathbb{P}^n$. If we define $\alpha := \alpha(\mathbf{H})$ to be the least integer α for which $\mathbf{H}(\alpha) < \binom{\alpha+n}{n}$ and we then write

$$\alpha = d_s + \cdots + d_1$$

then our main Theorem (Theorem 3.13) gives a canonical decomposition of $\mathbf{H}$ into a string $(\mathbf{H}_s', \ldots, \mathbf{H}_1')$ where $\mathbf{H}_i'$ is now the Hilbert function of a set of points on a *hypersurface* of $\mathbb{P}^n$ of degree d_i. As in the case of the linear decomposition there is a simple formula which describes the value of $\mathbf{H}$ in terms of the values of the $\mathbf{H}_i'$. Again, we discover that the string of Hilbert functions so constructed have connections with each other, which we specify (see Proposition 3.12 and (2) of Theorem 3.13) and those conditions are precisely what is required to show that a string of Hilbert functions, satisfying these conditions, and which are Hilbert functions of sets of points in distinct reduced hypersurfaces of $\mathbb{P}^n$ of degree d_i (having no common components) can be recombined to give $\mathbf{H}$ as the Hilbert function of their union (see (4) of Theorem 3.13).

As in the case of the linear decomposition of $\mathbf{H}$ referred to above (which turns out to correspond to choosing all the d_i above equal to 1) this decomposition of $\mathbf{H}$ makes evident other *extremal* aspects of the Hilbert function $\mathbf{H}$ (see Theorem 3.14 and Theorem 3.15).

We saw in [3] that there is an easily describable family of sets of points in $\mathbb{P}^n$ with given Hilbert function $\mathbf{H}$ and which have the extremal properties we observed about the linear decomposition of $\mathbf{H}$. We called these families *k-configurations of points* in $\mathbb{P}^n$. We further showed that all k-configurations (for fixed $\mathbf{H}$) have the same graded Betti numbers in the minimal free resolution of their defining ideals.

We show, in this paper (see Proposition 3.16 and the remark after it) that k-configurations exhibit this same unusual extremal property for our more general decomposition of $\mathbf{H}$.

2 Preliminary Remarks

It will be useful to recall some notions and establish some notation and terminology. Let $\mathbb{X} = \{P_1, \dots, P_s\}$ be a set of s distinct points in the projective space $\mathbb{P}^n(k)$ (where $k = \overline{k}$ is an algebraically closed field). Then $P_i \leftrightarrow \wp_i = (L_{i1}, \dots, L_{in}) \subset R = k[x_0, x_1, \dots, x_n]$ where the L_{ij}, $j = 1, \dots, n$ are n linearly independent linear forms and $\wp_i$ is the (homogeneous) prime ideal of R generated by all the forms which vanish at P_i. The ideal

$$I = I_{\mathbb{X}} := \wp_1 \cap \cdots \cap \wp_s$$

is the ideal generated by all the forms which vanish at all the points of $\mathbb{X}$.

Since $R = \bigoplus_{i=0}^{\infty} R_i$ (R_i the vector space of dimension $\binom{i+n}{n}$ generated by all the monomials in R having degree i) and $I = \bigoplus_{i=0}^{\infty} I_i$ we get that

$$A = R/I = \bigoplus_{i=0}^{\infty} (R_i/I_i) = \bigoplus_{i=0}^{\infty} A_i$$

is a graded ring. The numerical function

$$\mathbf{H}_{\mathbb{X}}(t) = \mathbf{H}(A, t) := \dim_k A_t = \dim_k R_t - \dim_k I_t$$

is called the *Hilbert function* of the set $\mathbb{X}$ (or of the ring A).

The collection of functions

$$\mathcal{H}_n := \{\mathbf{H}_{\mathbb{X}} : \mathbb{N} \to \mathbb{N} \mid \mathbb{X} \text{ is a non-degenerate finite set of points in } \mathbb{P}^n\}$$

have been much studied. For example:

I. (Macaulay) If $\mathbf{H} \in \mathcal{H}_n$, then the values of $\mathbf{H}$, *i.e.*,

$$\mathbf{H}(0) = 1, \quad \mathbf{H}(1) = n + 1, \quad \mathbf{H}(2), \quad \cdots$$

form an O-sequence (see [11] for definition).

II. If $\mathbf{H} \in \mathcal{H}_n$ and $\mathbf{H} = \mathbf{H}_{\mathbb{X}}$ for some set $\mathbb{X}$ then, for all $t \gg 0$, $\mathbf{H}(t) = |\mathbb{X}|$.

III. If $\mathbf{H} \in \mathcal{H}_n$ and we define the function $\Delta\mathbf{H}$ by:

$$\Delta\mathbf{H}(0) = 1$$

$$\Delta\mathbf{H}(t) = \mathbf{H}(t) - \mathbf{H}(t-1) \quad \text{for } t > 0,$$

then the values of $\Delta\mathbf{H}$, *i.e.*,

$$\Delta\mathbf{H}(0) = 1, \quad \Delta\mathbf{H}(1) = n, \quad \Delta\mathbf{H}(2), \quad \cdots$$

form an O-sequence which is eventually 0.

One can prove (see *e.g.* [2]) that condition III is equivalent to saying that there is a homogeneous ideal $J \subset k[x_1, \dots, x_n]$ such that

1) $J \cap (x_1, \dots, x_n)_1 = (0)$,
2) $\sqrt{J} = (x_1, \dots, x_n)$, and
3) if $B = k[x_1, \dots, x_n]/J = \bigoplus_{i=0}^{\infty} B_i$ then $\Delta\mathbf{H}(t) = \dim_k B_t$.

I.e., $\Delta\mathbf{H}$ is the Hilbert function of some artinian quotient of $k[x_1, \dots, x_n]$.

In fact, in the terminology of [5] one has the following characterization of $\mathcal{H}_n$:

$$\mathbf{H} \in \mathcal{H}_n \quad \text{(for some } n\text{) if and only if } \mathbf{H}(1) = n + 1,$$
$$\mathbf{H} \text{ is a 0-dimensional (this is II), differentiable (this is III)}$$
$$O\text{-sequence (this is I).}$$

We use III above to define the set of functions $\mathcal{H}iArt_n$:

$$\mathcal{H}iArt_n := \{\mathbf{H} \colon \mathbb{N} \to \mathbb{N} \mid \mathbf{H} \text{ is the Hilbert function of some artinian graded quotient of } k[x_1, \dots, x_n] \text{ and } \mathbf{H}(1) = n.\}$$

From what we have observed above, we can consider Δ as a function from $\mathcal{H}_n$ to $\mathcal{H}iArt_n$. Since "integration" of a function in $\mathcal{H}iArt_n$ is a left inverse to Δ, we obtain that Δ is actually a 1-1 function. It is a well-known theorem (see *e.g.* [2]) that Δ is also a surjective function. So, we often can reduce questions about $\mathcal{H}_n$ to analogous questions about $\mathcal{H}iArt_n$.

Given $\mathbf{H} \in \mathcal{H}_n$ we define:

$$\widetilde{\alpha}(\mathbf{H}) = \text{least integer } t \text{ such that } \mathbf{H}(t) < \binom{t+n}{n};$$

$$\sigma(\mathbf{H}) = \text{ least integer } t \text{ such that } \Delta\mathbf{H}(t + \ell) = 0 \text{ for all } \ell \geq 0.$$

Notice that if B (as above) is a graded artinian quotient of $k[x_1, \dots, x_n]$ and if $B_t = 0$ for some t then $B_{t+\ell} = 0$ for all $\ell \geq 0$. It follows from this observation that we could just as well have defined $\sigma(\mathbf{H})$ as:

$$\sigma(\mathbf{H}) = \text{ least integer } t \text{ such that } \Delta\mathbf{H}(t) = 0.$$

Clearly $\widetilde{\alpha}(\mathbf{H}) \leq \sigma(\mathbf{H})$ and $\mathbf{H} \in \mathcal{H}_n$ is completely known once we know the first $\sigma(\mathbf{H})$ values of $\mathbf{H}$,

$$\mathbf{H}(0), \mathbf{H}(1) = n + 1, \dots, \mathbf{H}\big(\sigma(\mathbf{H}) - 1\big).$$

We shall also need to consider degenerate sets of points in $\mathbb{P}^n$ and their Hilbert functions. In order to do that in a systematic way we define:

$$\mathcal{S}_n = \bigcup_{i \le n} \mathcal{H}_i.$$

Thus, $\mathcal{S}_n$ is the collection of Hilbert functions of all sets of points in $\mathbb{P}^n$.

Unfortunately, now if $\mathbf{H} \in \mathcal{S}_n$ the definition we gave above of $\widetilde{\alpha}(\mathbf{H})$ is not appropriate. In order to avoid the possibility of confusion we define, for $\mathbf{H} \in \mathcal{S}_n$, the following

$$\alpha(\mathbf{H}) = \begin{cases} 1, & \text{if } \mathbf{H} \in \mathcal{H}_i, i < n; \\ \widetilde{\alpha}(\mathbf{H}), & \text{if } \mathbf{H} \in \mathcal{H}_n. \end{cases}$$

Notice that the definition of $\sigma(\mathbf{H})$ doesn't depend on where we consider $\mathbf{H}$.

We now recall some definitions from [3].

Definition 2.1

1) A 0-*type vector* will be defined to be $\mathcal{T} = 1$. It is the only 0-*type vector*. We shall define $\alpha(\mathcal{T}) = -1$ and $\sigma(\mathcal{T}) = 1$.
2) A 1-*type vector* is an object of the form $\mathcal{T} = (d)$ where $d \ge 1$ is a positive integer. For such a vector we define $\alpha(\mathcal{T}) = d = \sigma(\mathcal{T})$.
3) A 2-*type vector*, $\mathcal{T}$, is

$$\mathcal{T} = \big((d_1), (d_2), \dots, (d_m) \big)$$

where $m \ge 1$, and the (d_i) are 1-*type vectors*. We also insist that $\sigma(d_i) = d_i < \alpha(d_{i+1}) = d_{i+1}$.

For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = m$ and $\sigma(\mathcal{T}) = \sigma(\ (d_m)\ ) = d_m$.

Clearly, $\alpha(\mathcal{T}) \le \sigma(\mathcal{T})$ with equality if and only if $\mathcal{T} = \big((1), (2), \dots, (m) \big)$.

Remark: For simplicity in the notation we usually rewrite the 2-*type vector* $\big((d_1), \dots, (d_m) \big)$ as $(d_1, \dots, d_m)$.

4) Now let $n \ge 2$. An n-*type vector*, $\mathcal{T}$, is an ordered collection of $(n-1)$-*type vectors*, $\mathcal{T}_1, \dots, \mathcal{T}_s$, *i.e.*,

$$\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_s)$$

for which $\sigma(\mathcal{T}_i) < \alpha(\mathcal{T}_{i+1})$ for $i = 1, \dots, s-1$.

For such a $\mathcal{T}$ we define $\alpha(\mathcal{T}) = s$ and $\sigma(\mathcal{T}) = \sigma(\mathcal{T}_s)$.

Example 2.2 Clearly $\mathcal{T}_1 = (1,2)$, $\mathcal{T}_2 = (1,3,4)$, $\mathcal{T}_3 = (1,2,3)$, and $\mathcal{T}_4 = (2,3,4,5,6)$ are all 2-*type vectors* but $(\mathcal{T}_3, \mathcal{T}_2) = \big((1,2,3), (1,3,4) \big)$ is NOT a 3-*type vector*. However, $(\mathcal{T}_2, \mathcal{T}_4)$ is a 3-*type vector*. Also,

$$(\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_4) = \big((1,2), (1,3,4), (2,3,4,5,6) \big)$$

is a 3-*type vector*. We will, from time to time, use the simpler notation

$$\big((1,2),(1,3,4),(2,3,4,5,6)\big) = (1,2;1,3,4;2,3,4,5,6)$$

for 3-*type vectors* (see [7]).

Note also that $\big((\mathcal{T}_1)\big) = \big((1,2)\big)$ is a 3-*type vector* and that $\Big(\big((1,2)\big)\Big)$ is a 4-*type vector*.

Before we begin the proof of our main theorem we want to recall a construction given in [5] which is crucial to our discussion of *n-type vectors*.

Let $\mathbf{H} = \{b_i\} \in \mathcal{H}_n$ (so $\mathbf{H}(1) = n+1$) and write $\sigma = \sigma(\mathbf{H})$. Let $\mathbf{H}_{\mathbb{P}^{n-1}}(t) = \{d_t\}$ where $d_t = \binom{t+n-1}{n-1}$ and define $c_i = b_{i+1} - d_{i+1}$. Then we have:

$$\begin{array}{lccccccccc}
\mathbf{H}: & 1 & \binom{n+1}{1} & \cdot & \binom{\alpha-1+n}{n} & b_\alpha & \cdot & b_{\sigma-2} & < b_{\sigma-1} = & b_\sigma \\
 & (0) & (1) & \cdot & (\alpha-1) & (\alpha) & \cdot & (\sigma-2) & (\sigma-1) & (\sigma) \\
\mathbf{H}_{\mathbb{P}^{n-1}}: & 1 & \binom{n}{1} & \cdot & \binom{\alpha+n-2}{n-1} & \binom{\alpha+n-1}{n-1} & \cdot & \binom{\sigma+n-3}{n-1} & \binom{\sigma+n-2}{n-1} & \binom{\sigma+n-1}{n-1} \\
 & & \begin{array}{c}1\\ \| \\ c_0\end{array} & \cdot & c_{\alpha-2} & c_{\alpha-1} & \cdot & c_{\sigma-3} & c_{\sigma-2} & c_{\sigma-1}
\end{array}$$

Since the d_i's are strictly increasing and the b_i's are eventually constant, there is a unique integer h such that

$$1 = c_0 \le c_1 \le \cdots \le c_{h-1} > c_h.$$

Theorem 2.3 ([5]) *The sequences*

$$\mathbf{H}_1 := 1\ c_1 \cdots c_{h-1} \to \quad \textit{and} \quad \mathbf{H}_1' = \{c_i'\}$$

where

$$c_i' = \begin{cases} \binom{i+n-1}{n-1}, & \textit{for } i \le h; \\ b_i - c_{h-1}, & \textit{for } i \ge h \end{cases}$$

are 0*-dimensional differentiable O-sequences.*

Lemma 2.4 (**[3, Lemma 2.5]**) *Let* $\mathbf{H} \in \mathcal{H}_n$, $\mathbf{H}_1$ *and* $\mathbf{H}_1'$ *be as above. Then* $\sigma(\mathbf{H}) = \sigma(\mathbf{H}_1')$.

Theorem 2.5 (**[3, Theorem 2.6]**) *There is a 1-1 correspondence*

$$\mathcal{S}_n \leftrightarrow \{n\text{-}type\ vectors\}$$

where if $\mathbf{H} \in \mathcal{S}_n$ *and* $\mathbf{H} \leftrightarrow \mathcal{T}$ *then* $\alpha(\mathbf{H}) = \alpha(\mathcal{T})$ *and* $\sigma(\mathbf{H}) = \sigma(\mathcal{T})$.

We give a brief summary here of the proof of Theorem 2.5 in order to establish some notation.

The proof begins by induction, starting with $\mathcal{S}_0$ and $\mathcal{S}_1$ where if $\mathbf{H} \in \mathcal{S}_1$ then the 1-type vector associated to $\mathbf{H}$ is $\mathcal{T} = (\alpha)$ where $\alpha = \alpha(\mathbf{H})$.

In general, we apply Theorem 2.3 to $\mathbf{H} \in \mathcal{S}_n$ to get $\mathbf{H}_1$ and $\mathbf{H}_1'$. Then, by induction on $\alpha(\mathbf{H})$, we get

$$\mathbf{H}_1 \to (\mathcal{T}_1, \dots, \mathcal{T}_{\alpha(\mathbf{H}_1)})$$

where the $\mathcal{T}_i$ are all $(n-1)$-type vectors and $\mathbf{H}_1' \in \mathcal{S}_{n-1}$. So, $\mathbf{H}_1' \to \mathcal{T}'$ with $\mathcal{T}'$ an $(n-1)$-type vector. One shows:

$$\mathbf{H} \to (\mathcal{T}_1, \dots, \mathcal{T}_{\alpha(\mathbf{H}_1)}, \mathcal{T}')$$

and we denote the function from $\mathcal{S}_n$ to $\{n\text{-type vectors}\}$ by χ_n, *i.e.*,

$$\chi_n(\mathbf{H}) = (\mathcal{T}_1, \dots, \mathcal{T}_{\alpha(\mathbf{H}_1)}, \mathcal{T}').$$

We also, inductively, defined an inverse to χ_n, *i.e.*,

$$\rho_n \colon \{n\text{-type vectors}\} \to \mathcal{S}_n$$

such that if $\mathcal{T} = (\alpha)$ is a 1-type vector then

$$\rho_n(\mathcal{T}) = \mathbf{H}$$

where

$$\mathbf{H}(i) = \begin{cases} i+1, & \text{for } 0 \le i \le \alpha - 1 \\ \alpha, & \text{for } \alpha - 1 \le i. \end{cases}$$

Inductively, if $\mathcal{T} = (\mathcal{T}_1, \dots, \mathcal{T}_r)$ is an n-type vector and so $\rho_{n-1}(\mathcal{T}_i) = \widetilde{\mathbf{H}}_i \in \mathcal{S}_{n-1}$ then

$$\rho_n(\mathcal{T})(t) = \widetilde{\mathbf{H}}_r(t) + \widetilde{\mathbf{H}}_{r-1}(t-1) + \dots + \widetilde{\mathbf{H}}_1\bigl(t - (r-1)\bigr)$$

(where if $j < 0$ then $\widetilde{\mathbf{H}}_i(j) = 0$).

Example 2.6 We illustrate what the procedure above does to the function $\mathbf{H} \in \mathcal{H}_2$, where

$$\mathbf{H} := 1 \quad 3 \quad 6 \quad 10 \quad 13 \quad 15 \quad 17 \quad \to$$

and $\alpha(\mathbf{H}) = 4$, $\sigma(\mathbf{H}) = 7$.

We apply the decomposition to $\mathbf{H}$, *i.e.*,

$$\begin{array}{cccccccc|cc} \mathbf{H} & : & 1 & 3 & 6 & 10 & 13 & 15 & 17 & 17 & \to \\ & & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \\ \hline & & & 1 & 3 & 6 & 8 & 9 & 10 & 9 & \end{array}$$

to get

$$\mathbf{H}_1' := 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad \to$$

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and

$$\mathbf{H}_1 := 1 \quad 3 \quad 6 \quad 8 \quad 9 \quad 10 \quad \rightarrow .$$

Since $\mathbf{H}_1 \in \mathcal{H}_2$ we decompose again:

$$\begin{array}{ccccccc|cc} \mathbf{H}_1 & : & 1 & 3 & 6 & 8 & 9 \quad 10 & 10 & \rightarrow \\ & & 1 & 2 & 3 & 4 & 5 \quad 6 & 7 & \\ \hline & & & 1 & 3 & 4 & 4 \quad 4 & 3 & \end{array}$$

to get

$$\mathbf{H}_2' := 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad \rightarrow$$

and

$$\mathbf{H}_2 := 1 \quad 3 \quad 4 \quad \rightarrow .$$

Again, $\mathbf{H}_2 \in \mathcal{H}_2$, so we decompose again

$$\begin{array}{ccccc|cc} \mathbf{H}_2 & : & 1 & 3 & 4 & 4 & \rightarrow \\ & & 1 & 2 & 3 & 4 & \\ \hline & & & 1 & 1 & 0 & \end{array}$$

to get

$$\mathbf{H}_3' := 1 \quad 2 \quad 3 \quad \rightarrow$$

and

$$\mathbf{H}_3 := 1 \rightarrow .$$

Finally we have $\mathbf{H}_3 \notin \mathcal{H}_2$ and the algorithm terminates.

We have constructed $\mathbf{H}_1'$, $\mathbf{H}_2'$, $\mathbf{H}_3'$ and $\mathbf{H}_3$ and thus $\mathbf{H} \leftrightarrow \mathcal{T} = (1, 3, 6, 7)$.

3 Decompositions of 0-dimensional Differentiable $\mathcal{O}$-Sequences

Remark 3.1 Let $R = k[x_0, \ldots, x_n]$ and I, J be two ideals of R. Then we have an exact sequence

$$0 \longrightarrow R/(I \cap J) \longrightarrow R/I \oplus R/J \longrightarrow R/(I + J) \longrightarrow 0. \tag{3.1}$$

If I and J are homogeneous, then the mappings of (3.1) are all of degree 0 and so we obtain

$$\mathbf{H}(R/I, t) + \mathbf{H}(R/J, t) = \mathbf{H}\big(R/(I \cap J), t\big) + \mathbf{H}\big(R/(I + J), t\big) \tag{3.2}$$

for $t \geq 0$.

If $\mathbf{H}\big(R/(I + J), s\big) = 0$ for some s (and hence $\mathbf{H}\big(R/(I + J), i\big) = 0$ for $i \geq s$), then this is equivalent to $V(I) \cap V(J) = \varnothing$ (as schemes) which, in turn, is equivalent to $\sqrt{I + J} = (x_0, \ldots, x_n)$.

Lemma 3.2 **([5, Lemma 3.9])** *Let $I = (r_1, \ldots, r_s)$ and J be ideals of a commutative ring R. If $(\overline{r_1}, \ldots, \overline{r_s})$ form a regular sequence in R/J, then $IJ = I \cap J$.*

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Proposition 3.3 *Let $\mathbb{X}$ be a reduced variety of $\mathbb{P}^n$ with Hilbert function $\{e_i\}_{i\geq 0}$ and let $\{c_i\}_{i\geq 0}$ be the Hilbert function of a hypersurface $\mathcal{S}$ of degree $d \geq 1$ in $\mathbb{P}^n$.*

If $\mathcal{S}$ contains no irreducible component of $\mathbb{X}$, then $\mathbb{X} \cup \mathcal{S}$ has Hilbert function $\{b_i\}_{i\geq 0}$, where

$$b_i = c_i + e_{i-d} \quad (e_{-1} = \cdots = e_{-d} = 0).$$

Proof (See [5] for the case when $d = 1$.) Let Q be a form of degree d defining $\mathcal{S}$, *i.e.*, $I_{\mathcal{S}} = (Q)$ and let $J = I_{\mathbb{X}}$. The assumption that $\mathcal{S}$ contains no irreducible component of $\mathbb{X}$ means that Q is not a zero-divisor in R/J ($R = k[x_0, \ldots, x_n]$). By Lemma 3.2, we have $(Q) \cap J = QJ$. Thus

$$QJ = JQ = J \cap (Q) = I_{\mathbb{X}} \cap I_{\mathcal{S}} = I_{\mathbb{X}\cup\mathcal{S}}.$$

Moreover, we have an exact sequence

$$0 \longrightarrow (Q)/QJ \longrightarrow R/QJ \longrightarrow R/Q \longrightarrow 0. \tag{3.3}$$

Since $(Q)/QJ \simeq R/J$ as an R-module (with a shift of d in degree), we can rewrite equation (3.3) as

$$\begin{array}{ccccccccc} 0 & \longrightarrow & (R/J)(-d) & \longrightarrow & R/QJ & \longrightarrow & R/Q & \longrightarrow & 0, \\ & & & & \Big\| & & & & \\ & & & & R/I_{\mathbb{X}\cup\mathcal{S}} & & & & \end{array} \tag{3.4}$$

and so the proposition follows immediately. ∎

Proposition 3.4 *Let $\mathbb{X}$ be a subvariety of $\mathbb{P}^n$ with Hilbert function $\{e_i\}_{i\geq 0}$ and let $\mathcal{S}$ be a hypersurface of $\mathbb{P}^n$ of degree d not containing any irreducible component of $\mathbb{X}$. Let $\mathcal{V}$ be a subvariety of $\mathcal{S}$ with Hilbert function $\{c_i\}_{i\geq 0}$ where $c_i = \mathbf{H}(\mathcal{S}, i)$ for $i \leq s$. Then $\mathbb{X} \cup \mathcal{V}$ has Hilbert function $\{b_i\}_{i\geq 0}$ where $b_i = c_i + e_{i-d}$ for $i \leq s$.*

Proof (See also Corollary 2.8 in [5] for the case $d = 1$.) Let G be a form of degree $i \leq s$ vanishing on $\mathbb{X} \cup \mathcal{V}$. Then, $G \in I_{\mathbb{X}\cup\mathcal{V}} = I_{\mathbb{X}} \cap I_{\mathcal{V}} \subset I_{\mathcal{V}}$. Since $(I_{\mathcal{V}})_i = (I_{\mathcal{S}})_i$ for $i \leq s$, we have that $G \in (I_{\mathcal{S}})_i \subset I_{\mathcal{S}}$. This means that G vanishes on $\mathcal{S}$ and so contains Q as a factor where $I_{\mathcal{S}} = (Q)$ since Q does not vanish on $\mathbb{X}$. Thus $(I_{\mathbb{X}\cup\mathcal{V}})_i = (QI_{\mathbb{X}})_i = (QJ)_i$ for $i \leq s$ (where $J = I_{\mathbb{X}}$) and the result follows from the degree i portion of the exact sequence (3.4) used in the proof of Proposition 3.3. ∎

Proposition 3.5 *We maintain the notation of Proposition 3.4 and assume, moreover, that $e_{s-d} = e_{s-d+1}$ (i.e., $\mathbb{X}$ consists of e_{s-d+1} points). Then*

$$b_i = \begin{cases} c_i + e_{i-d}, & \text{if } i \leq s, \\ c_i + e_{s-d}, & \text{if } i \geq s. \end{cases}$$

Proof The first conclusion comes from Proposition 3.4. Note that

$$b_s = c_s + e_{s-d} = b_s + \mathbf{H}\big(R/(I_{\mathbb{X}} + I_{\mathcal{Y}}), s\big)$$

from equation (3.2), and so $\mathbf{H}\big(R/(I_{\mathbb{X}} + I_{\mathcal{Y}}), s\big) = 0$. In other words, $\mathbf{H}\big(R/(I_{\mathbb{X}} + I_{\mathcal{Y}}), i\big) = 0$ for all $i \geq s$. Therefore, we have

$$b_i = c_i + e_{i-d} = c_i + e_{s-d} \quad (i \geq s)$$

and that completes the proof of this proposition. ■

Remark 3.6 Let $\mathbf{H} = \{b_i\}_{i\geq 0}$ be a 0-dimensional differentiable O-sequence with $b_i = \binom{n+i}{i}$ for $i \leq d$ and let $\{c_i\}_{i\geq 0}$ be the sequence obtained from $\mathbf{H}$ by subtracting the Hilbert function $\{e_i\}_{i\geq 0}$ of the coordinate ring $k[x_0, \dots, x_n]/(Q)$ where Q is a form of degree d.

$$\begin{array}{cccccc|cc}
 & b_1 & & b_d & & & & \\
 & \| & & \| & & & & \\
1 & \binom{n+1}{1} & \cdots\cdots & \binom{n+d}{d} & \cdots\cdots & b_{h+d-1} & b_{h+d} & \cdots\cdots \\
1 & \binom{n+1}{1} & \cdots\cdots & \binom{n+d}{d} - 1 & \cdots\cdots & e_{h+d-1} & e_{h+d} & \cdots\cdots \\
 & \| & & \| & & & & \\
 & e_1 & & e_d & & & & \\
\hline
 & & & 1 & \cdots\cdots & c_{h-1} & c_h & \cdots\cdots
\end{array}$$

Then there is an integer h for which

$$1 = c_0 \leq c_1 \leq c_2 \leq \cdots \leq c_{h-1} \quad \text{and} \quad c_{h-1} > c_h.$$

Let $\mathbf{H}_1$ denote the sequence $c_0 \quad c_1 \quad c_2 \quad \cdots \quad c_{h-1} \quad \rightarrow$.
Moreover, we have

$$\begin{aligned}
e_i &= \binom{n+i}{i} - \binom{n+i-d}{i-d} \\
&= \binom{n+i-1}{i} + \binom{n+i-2}{i-1} + \cdots + \binom{n+i-d}{i-d+1},
\end{aligned}$$

for every $i \geq d$ by Lemma 2.5 in [1].

Lemma 3.7 **(Macaulay)** *Let $\{b_i\}_{i\geq 0}$ be an O-sequence where $b_0 = 1$ and $b_1 = n$. Then*

$$b_{i+1} \leq \binom{n+i}{i+1}$$

for every $i \geq 0$.

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Lemma 3.8 *Let* $\mathbf{H}$ *be as above and let* $\{\alpha_i\}_{i\geq 0}$ *be the first difference of* $\mathbf{H}$*. Then we have*

$$\begin{aligned}\binom{n+r+d-2}{r+d} + \binom{n+r+d-3}{r+d-1} + \cdots + \binom{n+r-1}{r+1}\\ \leq \alpha_{r+d} \leq \binom{n+r+d-1}{r+d}\end{aligned} \tag{3.5}$$

if $1 \leq r \leq h-1$*. Moreover,*

$$\alpha_{h+d} < \binom{n+h+d-2}{h+d} + \binom{n+h+d-3}{h+d-1} + \cdots + \binom{n+h-1}{h+1}. \tag{3.6}$$

Proof Since $c_{r-1} \leq c_r$ for $r \leq h-1$, we have

$$b_{r+d-1} - e_{r+d-1} = c_{r-1} \leq c_r = b_{r+d} - e_{r+d}$$

and so

$$e_{r+d} - e_{r+d-1} \leq b_{r+d} - b_{r+d-1} = \alpha_{r+d}.$$

Since

$$\begin{aligned} e_{r+d} - e_{r+d-1} &= \left[\binom{n+r+d}{r+d} - \binom{n+r}{r}\right] \\ &\quad - \left[\binom{n+r+d-1}{r+d-1} - \binom{n+r-1}{r-1}\right] \\ &= \left[\binom{n+r+d-1}{r+d} + \cdots + \binom{n+r}{r+1}\right] \\ &\quad - \left[\binom{n+r+d-2}{r+d-1} + \cdots + \binom{n+r-1}{r}\right] \\ &= \binom{n+r+d-1}{r+d} - \binom{n+r-1}{r} \\ &= \binom{n+r+d-2}{r+d} + \cdots + \binom{n+r-1}{r+1}, \end{aligned} \tag{3.7}$$

we obtain the left hand inequality. The other inequality is exactly Lemma 3.7.

It is straightforward to see that equation (3.6) is equivalent to the relation $c_{h-1} > c_h$. That completes the proof of this lemma. ∎

Proposition 3.9 *Let* $\mathbf{H}_1 = \{c_i\}_{i\geq 0}$ *be as above. Then* $\mathbf{H}_1$ *is a* 0*-dimensional differentiable O-sequence.*

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Proof Let $\{\beta_i\}_{i\geq 0}$ be the first difference sequence of $\mathbf{H}_1$ and let $\{\alpha_i\}_{i\geq 0}$ be as in Lemma 3.8. To prove the proposition it suffices to show that $\{\beta_i\}_{i\geq 0}$ is an O-sequence. For every $r \leq h-2$,

$$\begin{aligned}\beta_r &= \mathbf{H}_1(r) - \mathbf{H}_1(r-1)\\ &= [b_{r+d} - e_{r+d}] - [b_{r+d-1} - e_{r+d-1}]\\ &= [b_{r+d} - b_{r+d-1}] - [e_{r+d} - e_{r+d-1}]\\ &= \alpha_{r+d} - \left[\binom{n+r+d-2}{r+d} + \cdots + \binom{n+r-1}{r+1}\right]\end{aligned}$$

by equation (3.7).

If $\alpha_{r+d} = \binom{n+r+d-1}{r+d}$, then

$$\begin{aligned}\beta_r &= \alpha_{r+d} - \left[\binom{n+r+d-2}{r+d} + \cdots + \binom{n+r-1}{r+1}\right]\\ &= \binom{n+r+d-1}{r+d} - \left[\binom{n+r+d-2}{r+d} + \cdots + \binom{n+r-1}{r+1}\right]\\ &= \binom{n+r-1}{r}.\end{aligned}$$

Hence

$$\begin{aligned}\beta_r^{\langle r\rangle} &= \binom{n+r-1}{r}^{\langle r\rangle} = \binom{n+r}{r+1}\\ &\geq \beta_{r+1}.\end{aligned}$$

Thus this case is done.

Now assume $\alpha_{r+d} < \binom{n+r+d-1}{r+d}$. From the other inequality of equation (3.5), we have that the $(r+d)$-binomial expansion of α_{r+d} is:

$$\alpha_{r+d} = \left[\binom{n+r+d-2}{r+d} + \cdots + \binom{n+r-1}{r+1}\right] + \left[\binom{m_r}{r} + \cdots + \binom{m_\ell}{\ell}\right] \tag{3.8}$$

where $n+r-1 > m_r > \cdots > m_\ell \geq \ell \geq 1$. Hence the r-binomial expansion of $\alpha_{r+d} - \left[\binom{n+r+d-2}{r+d} + \cdots + \binom{n+r-1}{r+1}\right]$ is:

$$\alpha_{r+d} - \left[\binom{n+r+d-2}{r+d} + \cdots + \binom{n+r-1}{r+1}\right] = \binom{m_r}{r} + \cdots + \binom{m_\ell}{\ell}. \tag{3.9}$$

It follows from equations (3.8) and (3.9) that

$$\begin{aligned}\alpha_{r+d}^{\langle r+d\rangle} &= \left[\alpha_{r+d} - \left[\binom{n+r+d-2}{r+d} + \cdots + \binom{n+r-1}{r+1}\right]\right]^{\langle r\rangle}\\ &\quad + \left[\binom{n+r+d-1}{r+d+1} + \cdots + \binom{n+r}{r+2}\right].\end{aligned} \tag{3.10}$$

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Hence

$$\begin{aligned}\alpha_{r+d+1} &\leq \alpha_{r+d}^{\langle r+d\rangle}\\ &= \left[\alpha_{r+d} - \left[\binom{n+r+d-2}{r+d} + \cdots + \binom{n+r-1}{r+1}\right]\right]^{\langle r\rangle}\\ &\quad + \left[\binom{n+r+d-1}{r+d+1} + \cdots + \binom{n+r}{r+2}\right].\end{aligned}$$

In other words,

$$\begin{aligned}\beta_{r+1} &= \alpha_{r+d+1} - \left[\binom{n+r+d-1}{r+d+1} + \cdots + \binom{n+r}{r+2}\right]\\ &\leq \left[\alpha_{r+d} - \left[\binom{n+r+d-2}{r+d} + \cdots + \binom{n+r-1}{r+1}\right]\right]^{\langle r\rangle}\\ &= \beta_r^{\langle r\rangle}.\end{aligned}$$

That completes the proof of this proposition. ■

Proposition 3.10 *Let $\mathbf{H} = \{b_i\}_{i\geq 0}$, Q, $\{e_i\}_{i\geq 0}$, and h be as above. Define a new sequence $\mathbf{H}_1' = \{c_i'\}_{i\geq 0}$ as follows:*

$$c_i' = \begin{cases} e_i, & \text{for } i \leq h+d-1,\\ b_i - c_{h-1}, & \text{for } i \geq h+d-1.\end{cases}$$

Then $\mathbf{H}_1'$ is a 0-dimensional differentiable O-sequence.

Proof It suffices to show that the difference sequence of $\mathbf{H}_1'$ is an O-sequence.

Since the Hilbert function $\{e_i\}_{i\geq 0}$ of $k[x_0,\ldots,x_n]/(Q)$ and the sequence $\mathbf{H}$ are differentiable O-sequence, the only thing that need be shown is that

$$c_{h+d}' - c_{h+d-1}' \leq (c_{h+d-1}' - c_{h+d-2}')^{\langle h+d-1\rangle}.$$

By the construction of $\{c_i'\}_{i\geq 0}$, we have that

$$c_{h+d}' - c_{h+d-1}' = [b_{h+d} - c_{h-1}] - [b_{h+d-1} - c_{h-1}] = b_{h+d} - b_{h+d-1} = \alpha_{h+d}$$

and

$$\begin{aligned}&(c_{h+d-1}' - c_{h+d-2}')^{\langle h+d-1\rangle}\\ &\quad = (e_{h+d-1} - e_{h+d-2})^{\langle h+d-1\rangle}\\ &\quad = \left[\binom{n+h+d-3}{h+d-1} + \binom{n+h+d-4}{h+d-2} + \cdots + \binom{n+h-2}{h}\right]^{\langle h+d-1\rangle}\\ &\quad = \binom{n+h+d-2}{h+d} + \binom{n+h+d-3}{h+d-1} + \cdots + \binom{n+h-1}{h+1}\end{aligned}$$

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by equation (3.7). Hence

$$\begin{aligned}(c'_{h+d-1} - c'_{h+d-2})^{\langle h+d-1\rangle} \\ &= \binom{n+h+d-2}{h+d} + \binom{n+h+d-3}{h+d-1} + \cdots + \binom{n+h-1}{h+1} \\ &> \alpha_{h+d} = c'_{h+d} - c'_{h+d-1}\end{aligned}$$

by equation (3.6). That finishes the proof. ∎

Example 3.11 Consider the 0-dimensional differentiable O-sequence

$$\mathbf{H} \; : \; 1 \quad 4 \quad 10 \quad 20 \quad 34 \quad 50 \quad 67 \quad 84 \quad 102 \quad 122 \quad \rightarrow .$$

Let Q be a form of degree 2 in $R = k[x, y, z, w]$. Then the Hilbert function of the coordinate ring $R/(Q)$ is

$$\mathbf{H}\big(R/(Q), t\big) = t^2 + 2t + 1 \quad \text{for } t \geq 0.$$

Proceeding as in Remark 3.6 we obtain:

H	:	1	4	10	20	34	50	67	84	102	122	122	→
		1	4	9	16	25	36	49	64	81	100	121	
				1	4	9	14	18	20	21	22	1	

yielding

$$\begin{array}{lcllllllllllll} \mathbf{H}_1 & : & 1 & 4 & \mathbf{9} & 14 & 18 & 20 & 21 & 22 & \rightarrow & & \\ \mathbf{H}'_1 & : & 1 & 4 & \mathbf{9} & 16 & 25 & 36 & 49 & 64 & 81 & 100 & \rightarrow . \end{array}$$

Since $\mathbf{H}_1(2) = \mathbf{H}'_1(2) = 9 < 10$, $\mathbf{H}_1$ and $\mathbf{H}'_1$ are each the Hilbert function of a set of points on a quadric of $\mathbb{P}^3$. Using Proposition 3.4 we see that $\mathbf{H}$ is the Hilbert function of the union of: 100 points on a quadric $\mathcal{C}_1$ and 22 points on a quadric $\mathcal{C}_2$ where $\mathcal{C}_1$ and $\mathcal{C}_2$ can be chosen to be any two reduced quadrics with no common components and no point chosen on $\mathcal{C}_2$ lies on $\mathcal{C}_1$.

We now show (Proposition 3.12) that the two functions constructed from $\mathbf{H}$, namely $\mathbf{H}'_1$ and $\mathbf{H}_1$, have a subtle, but crucial, relationship.

As usual, let $R = k[x_0, \ldots, x_n] = \bigoplus_{i \geq 0} R_i$ where R_i is the set of all homogeneous polynomials in R of degree i. Let $\mathbf{H} \in \mathcal{H}_n$ and let $\mathbb{X}$ be a set of points in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$. For a closed subscheme $\mathbb{V}$ in $\mathbb{P}^n$ and a non-degenerate finite set $\mathbb{X}$ of points in $\mathbb{V}$, we put

$$\alpha(\mathbb{X}) := \min\{i \mid \mathbf{H}(\mathbb{X}, i) < \dim_k R_i\},$$

$$\sigma(\mathbb{X}) := \min\{i \mid \mathbf{H}(\mathbb{X}, i-1) = \mathbf{H}(\mathbb{X}, i)\} = \min\{i \mid \Delta\mathbf{H}(\mathbb{X}, i) = 0\},$$

$$\alpha_{\mathbb{V}}(\mathbb{X}) := \min\{i \mid \mathbf{H}(\mathbb{X}, i) < \mathbf{H}(\mathbb{V}, i)\}.$$

Notice that since all of these definitions are independent of $\mathbb{X}$, *i.e.*, depend only on $\mathbf{H}$, we can set:

$$\alpha(\mathbf{H}) = \alpha(\mathbb{X}), \quad \sigma(\mathbf{H}) = \sigma(\mathbb{X}), \quad \text{and} \quad \alpha_{\mathbb{V}}(\mathbf{H}) = \alpha_{\mathbb{V}}(\mathbb{X}).$$

Proposition 3.12 *Let $\mathbf{H} \in \mathcal{H}_n$ and let $\alpha = \alpha(\mathbf{H})$. Furthermore, let $\mathbf{H}_1$ and $\mathbf{H}'_1$ be as in Proposition 3.9 and Proposition 3.10 and let $\mathbb{V}$ be a hypersurface in $\mathbb{P}^n$ of degree $d \leq \alpha$. Then*

$$\sigma(\mathbf{H}_1) + \alpha(\mathbf{H}'_1) \leq \alpha_{\mathbb{V}}(\mathbf{H}'_1).$$

Proof Let h be as in Remark 3.6. Then

$$\sigma(\mathbf{H}_1) \leq h \quad \text{and} \quad \alpha(\mathbf{H}'_1) = d.$$

Moreover, by the construction of $\mathbf{H}'_1$,

$$\alpha_{\mathbb{V}}(\mathbf{H}'_1) \geq h + d,$$

and hence

$$\sigma(\mathbf{H}_1) + \alpha(\mathbf{H}'_1) \leq h + d \leq \alpha_{\mathbb{V}}(\mathbf{H}'_1),$$

as we wished. ∎

Theorem 3.13 *Let $\mathbf{H} \in \mathcal{H}_n$ and let $\alpha = \alpha(\mathbf{H})$ be written*

$$\alpha = d_s + d_{s-1} + \cdots + d_1.$$

Let $\mathbb{V}_i$ be a reduced hypersurface of $\mathbb{P}^n$ of degree d_i. Then there are functions $\mathbf{H}'_s, \ldots, \mathbf{H}'_1$ such that:

(1) $\mathbf{H}'_i$ is the Hilbert function of a set of points on the reduced hypersurface $\mathbb{V}_i$ and $\alpha(\mathbf{H}'_i) = d_i$;
(2) $\sigma(\mathbf{H}'_{i+1}) + \alpha(\mathbf{H}'_i) \leq \alpha_{\mathbb{V}_i}(\mathbf{H}'_i)$ for every $i = 1, \ldots, s-1$;
(3) $\mathbf{H}(t) = \mathbf{H}'_1(t) + \mathbf{H}'_2(t - d_1) + \cdots + \mathbf{H}'_s\big(t - (d_1 + \cdots + d_{s-1})\big)$.

If we choose the reduced hypersurfaces $\mathbb{V}_i$ so that $\mathbb{V}_i$ and $\mathbb{V}_j$ have no common components and let $\mathbb{X}_i$ be a subset of $\mathbb{V}_i$ with Hilbert function $\mathbf{H}'_i$—chosen so that no point of $\mathbb{X}_i$ is on $\mathbb{V}_j$ for $i \neq j$—and set $\mathbb{X} = \bigcup \mathbb{X}_i$, then

(4) $\mathbf{H}_{\mathbb{X}}(t) = \mathbf{H}(t)$ for all t.

Proof Although stated as a theorem, this is really an algorithm which decomposes the function $\mathbf{H}$ according to the data of the d_i.

We begin with $\mathbf{H}$ and write $\alpha(\mathbf{H}) = d_1 + d'$. Using Remark 3.6 and Proposition 3.10 on $\mathbf{H}$ we obtain $\mathbf{H}_1$ and $\mathbf{H}'_1$ which (by Proposition 3.9 and 3.10 respectively) imply that $\mathbf{H}_1$ and $\mathbf{H}'_1$ are the Hilbert functions of sets of points in $\mathbb{P}^n$.

From the expression for β_r in Proposition 3.9 we see that $\alpha(\mathbf{H}_1) = \alpha(\mathbf{H}) - d_1 = d'$. From the definition of $\mathbf{H}'_1$ we obtain $\alpha(\mathbf{H}'_1) = d_1$ and

$$\mathbf{H}(t) = \mathbf{H}'_1(t) + \mathbf{H}_1(t - d_1).$$

Since $\alpha(\mathbf{H}'_1) = d_1$, $\mathbf{H}'_1$ is the Hilbert function of a set of points on a reduced hypersurface $\mathbb{V}_1$ with degree d_1.

From Proposition 3.12 we get

$$\alpha_{\mathbb{Y}_1}(\mathbf{H}_1') \geq \sigma(\mathbf{H}_1) + \alpha(\mathbf{H}_1').$$

But, $\sigma(\mathbf{H}_1) \geq \sigma(\mathbf{H}_2')$, by construction, and so

$$\alpha_{\mathbb{Y}_1}(\mathbf{H}_1') \geq \sigma(\mathbf{H}_2') + \alpha(\mathbf{H}_1').$$

We now proceed by induction, using $\mathbf{H}_1$ instead of $\mathbf{H}$.

As for (4), we may assume, by induction that $\mathbb{Y} = \mathbb{X}_s \cup \cdots \cup \mathbb{X}_2$ has Hilbert function $\mathbf{H}_1$. By (3) and Proposition 3.4 we have that

$$\mathbf{H}_{\mathbb{Y}\cup\mathbb{X}_1}(t) = \mathbf{H}_1(t - d_1) + \mathbf{H}_1'(t) = \mathbf{H}(t)$$

as we wanted to show. ∎

Theorem 3.14 *Let $\mathbf{H} \in \mathcal{H}_n$ and set $\alpha(\mathbf{H}) = d_s + \cdots + d_1$. Let $\mathbb{X}$ be a finite set of points in $\mathbb{P}^n$ which has Hilbert function $\mathbf{H}$ and let $\mathbf{H}_1' = \{c_i'\}_{i\geq 0}$ be as in Theorem 3.13. Then, for every hypersurface $\mathcal{C}$ in $\mathbb{P}^n$ of degree $d_1 \geq 1$,*

$$\mathbf{H}(\mathbb{X} \cap \mathcal{C}, t) \leq \mathbf{H}_1'(t)$$

for every $t \geq 0$.

Proof Let h be as in Remark 3.6. Then, for $i \leq h + d - 1$,

$$c_i' = \begin{cases} \binom{n+i}{i}, & \text{for } i \leq d-1, \\ \binom{n+i}{i} - \binom{n+i-d}{i-d}, & \text{for } d \leq i \leq h+d-1. \end{cases}$$

This means that

$$\mathbf{H}(\mathbb{X} \cap \mathcal{C}, t) \leq \mathbf{H}(\mathcal{C}, t) = \mathbf{H}_1'(t) \tag{3.11}$$

for such t.

Now assume $t \geq h + d$. Let $\mathbf{H}_1$ be as in Remark 3.6. Note that

$$\begin{aligned} \Delta\mathbf{H}(\mathbb{X}, t) &= \Delta[\mathbf{H}_1'(t) + \mathbf{H}_1(t-d)] \\ &= \Delta\mathbf{H}_1'(t) + \Delta\mathbf{H}_1(t-d) \\ &= \Delta\mathbf{H}_1'(t) \end{aligned} \tag{3.12}$$

for $t \geq h + d$. Since $\mathbb{X} \cap \mathcal{C} \subset \mathbb{X}$, we have that

$$\Delta\mathbf{H}(\mathbb{X} \cap \mathcal{C}, t) \leq \Delta\mathbf{H}(\mathbb{X}, t) = \Delta\mathbf{H}_1'(t)$$

for $t \geq h + d$ by equation (3.12). In other words,

$$\mathbf{H}(\mathbb{X} \cap \mathcal{C}, t) \leq \mathbf{H}_1'(t) \tag{3.13}$$

for such t. Hence

$$\mathbf{H}(\mathbb{X} \cap \mathcal{C}, t) \leq \mathbf{H}_1'(t)$$

for every $t \geq 0$ by equations (3.11) and (3.13). This completes the proof. ∎

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Extremal Subsets Let $\mathbf{H} \in \mathcal{H}_n$ and let $\mathbb{X}$ be a set of points in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$. We can consider all the subsets of $\mathbb{X}$ which lie on a hypersurface of $\mathbb{P}^n$ of degree $d \geq 1$ (and, to avoid trivialities we will assume that not all of $\mathbb{X}$ is in such a hypersurface, *i.e.*, $\mathbf{H}(d) = \binom{n+d}{d}$).

We can then partially order the Hilbert functions that arise as Hilbert functions of such subsets as follows:

Suppose $\mathbb{X}_1$ and $\mathbb{X}_2$ are two subsets in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$ and $\mathbb{Y}_1 \subseteq \mathbb{X}_1$ and $\mathbb{Y}_2 \subseteq \mathbb{X}_2$ are two subsets of $\mathbb{P}^n$ which lie in hypersurfaces of $\mathbb{P}^n$ of degree d. Then we define

$$\mathbf{H}_{\mathbb{Y}_1} \leq \mathbf{H}_{\mathbb{Y}_2} := \mathbf{H}_{\mathbb{Y}_1}(i) \leq \mathbf{H}_{\mathbb{Y}_2}(i) \quad \text{for every } i.$$

Clearly, if $\mathbb{Y}_1 \subseteq \mathbb{Y}_2$ then $\mathbf{H}_{\mathbb{Y}_1} \leq \mathbf{H}_{\mathbb{Y}_2}$.

We do this for *all* sets in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$, and so obtain a finite, partially ordered set of Hilbert functions in $\mathcal{H}_n$ which we'll call $\mathrm{Sub}_d(\mathbf{H})$.

Now suppose that $\chi_n(\mathbf{H}) = \mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$. Then we have the following interesting fact about $\mathrm{Sub}_d(\mathbf{H})$.

Theorem 3.15 $\mathrm{Sub}_d(\mathbf{H})$ *contains a maximal element. It is*

$$\rho_n(\mathcal{T}_{r-(d-1)}, \ldots, \mathcal{T}_r).$$

Proof We already proved the case $d = 1$ in Theorem 3.7 in [3] and so we may assume $d \geq 2$.

Let $\rho_n(\mathcal{T}_{r-(d-1)}, \ldots, \mathcal{T}_r) = \mathbf{G}_1'$ and let $\mathbb{Z}$ be any set of points in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$ and consider $\mathcal{C}$ a hypersurface of $\mathbb{P}^n$ of degree d. We will show that

$$\Delta\mathbf{H}(\mathbb{Z} \cap \mathcal{C}, j) \leq \Delta\mathbf{G}_1'(j) \quad \text{for every } j$$

and that will be enough to prove that $\mathbf{G}_1'$ is an *upper bound* for the elements of $\mathrm{Sub}_d(\mathbf{H})$.

Now $\mathbf{G}_1'(j) = \mathbf{H}(R/I_{\mathcal{C}}, j)$ for $0 \leq j < \alpha(\mathcal{T}_{r-(d-1)}) + (d-1)$ so we obviously have

$$\Delta\mathbf{H}(\mathbb{Z} \cap \mathcal{C}, j) \leq \Delta\mathbf{G}_1'(j) \quad \text{for } 0 \leq j < \alpha(\mathcal{T}_{r-(d-1)}) + (d-1).$$

The result for $j \geq \alpha(\mathcal{T}_{r-(d-1)}) + (d-1)$ will follow easily from the following:

Claim $\Delta\mathbf{G}_1'(j) = \Delta\mathbf{H}(j)$ for all $j \geq \alpha(\mathcal{T}_{r-(d-1)}) + (d-1)$.

Proof of Claim Let $\widetilde{\mathcal{T}} = (\mathcal{T}_1, \ldots, \mathcal{T}_{r-d})$ and $\rho_n(\widetilde{\mathcal{T}}) = \mathbf{G}_1$. Then, as we have seen,

$$\mathbf{H}(j) = \mathbf{G}_1'(j) + \mathbf{G}_1(j-d) \quad \text{for all } j.$$

By definition $\sigma(\mathbf{G}_1) + (d-1) = \sigma(\mathcal{T}_{r-d}) + (d-1) < \alpha(\mathcal{T}_{r-(d-1)}) + (d-1)$. Let s be the (eventually) constant value of $\mathbf{G}_1$, *i.e.*, $\mathbf{G}_1(t) = s$ for all $t \geq \sigma(\mathbf{G}_1) - 1$. Then, for all $j \geq \alpha(\mathcal{T}_{r-(d-1)}) + (d-1) - 1$ we have

$$\begin{aligned}\mathbf{H}(j) &= \mathbf{G}_1'(j) + \mathbf{G}_1(j-d) \\ &= \mathbf{G}(j) + s\end{aligned}$$

and so

$$\Delta\mathbf{H}(j) = \Delta\mathbf{G}'_1(j)$$

for all $j \geq \alpha(\mathcal{T}_{r-(d-1)}) + (d-1)$, as we wanted to prove.

Since $\mathbb{Z} \cap \mathcal{C} \subseteq \mathbb{Z}$ we have $\Delta\mathbf{H}(\mathbb{Z} \cap \mathcal{C}, j) \leq \Delta\mathbf{H}(j)$ for all $j \geq 0$. Coupling that with the observations above finishes the proof. ■

Given $\mathbf{H} \in \mathcal{H}_n$, Theorems 3.14 and 3.15 apparently give two upper bounds for the Hilbert function of the points $\mathbb{X} \cap \mathcal{C}$ where $\mathbb{X}$ is any set of points with Hilbert function $\mathbf{H}$ and $\mathcal{C}$ is any hypersurface in $\mathbb{P}^n$ of degree d. We now show that the two upper bounds are equal.

Proposition 3.16 *Let* $\mathbf{H} \in \mathcal{H}_n$ *and* $\alpha = \alpha(\mathbf{H}) = d_s + \cdots + d_1$. *Let* $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$ $(r > 1)$ *be the n-type vector which corresponds to the Hilbert function* $\mathbf{H}$*, and let* $\mathbf{H}'_1$ *be as in Theorem 3.13. Then*

$$\mathbf{H}'_1(t) = \rho_n(\mathcal{T}_{r-(d_1-1)}, \ldots, \mathcal{T}_r)(t) = \mathbf{G}(t)$$

for every $t \geq 0$.

Proof We have $\mathbf{H}'_1 \geq \mathbf{G}$ by Theorem 3.14 and the fact that $\mathbf{G}$ is the Hilbert function of the last d_1 sets in a k-configuration in $\mathbb{P}^n$ with Hilbert function $\mathbf{H}$. We have $\mathbf{G} \geq \mathbf{H}'_1$ by using Theorem 3.15. ■

Remark 3.17 It is an immediate observation from the proof of this proposition that among all sets of points with Hilbert function $\mathbf{H}$, the maximum number of points that can lie on a reduced hypersurface of degree d_1 is exactly the same as the maximum number of points that can lie on a hypersurface of degree d_1 that is the union of d_1 distinct linear hypersurfaces.

There is one more observation we would like to make about sets of points $\mathbb{X} \subset \mathbb{P}^n$ which have Hilbert function $\mathbf{H}$ where $\mathbf{H} = \rho_n(\mathcal{T})$, $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$, an *n-type vector*.

Theorem 3.15 tells us that any subset of such an $\mathbb{X}$, which lies on a hypersurface of degree d, must have Hilbert function which is $\leq \rho_n(\mathcal{T}_{r-(d-1)}, \ldots, \mathcal{T}_r)$. The next proposition deals with the situation in which a set $\mathbb{X}$ with Hilbert function $\mathbf{H}$ actually has a subset $\mathcal{U}$ for which $\mathbf{H}_{\mathcal{U}} = \rho_n(\mathcal{T}_{r-(d-1)}, \ldots, \mathcal{T}_r)$.

Proposition 3.18 *Let* $\mathbb{X}$, $\mathbf{H}$ *and* $\mathcal{T}$ *be as above and let* $\mathcal{U} \subset \mathbb{X}$ *be such that the Hilbert function of* $\mathcal{U}$, $\mathbf{H}_{\mathcal{U}}$, *satisfies* $\mathbf{H}_{\mathcal{U}} = \rho_n(\mathcal{T}_{r-(d-1)}, \ldots, \mathcal{T}_r)$.

Then, if we let $\mathcal{T}' = (\mathcal{T}_1, \ldots, \mathcal{T}_{r-d})$ *and* $\mathbb{X}' = \mathbb{X} - \mathcal{U}$ *then* $\mathbf{H}_{\mathbb{X}'} = \rho_n(\mathcal{T}')$.

Proof Let C be the form in $R = k[x_0, \ldots, x_n]$ of degree d which describes the hyperplane containing the points of $\mathcal{U}$. We have the exact sequence

$$0 \longrightarrow I_{\mathbb{X}'}(-d) \xrightarrow{\times C} I_{\mathbb{X}} \longrightarrow \big(I_{\mathbb{X}} + (C)\big)/(C) \longrightarrow 0 \tag{3.14}$$

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since $\mathbb{X}'$ is precisely the set of points of $\mathbb{X}$ which do not lie on the hypersurface defined by C.

Let $I_{\mathcal{U}}$ be the ideal (in R) of the set of points $\mathcal{U}$. Then $J = I_{\mathbb{X}} + (C) \subseteq I_{\mathcal{U}}$. Thus,

$$\mathbf{H}_{R/J}(t) = \mathbf{H}\Big(R/\big(I_{\mathbb{X}} + (C)\big), t\Big) \geq \mathbf{H}_{R/I_{\mathcal{U}}}(t) = \mathbf{H}_{\mathcal{U}}(t). \tag{3.15}$$

From (3.14) we get that

$$\mathbf{H}_{\mathbb{X}}(t) = \mathbf{H}_{\mathbb{X}'}(t-d) + \mathbf{H}_{R/J}(t). \tag{3.16}$$

From our earlier discussion of *n-type vectors* we also have that

$$\mathbf{H}_{\mathbb{X}}(t) = \rho_n(\mathfrak{T}')(t-d) + \mathbf{H}_{\mathcal{U}}(t). \tag{3.17}$$

Since

$$\rho_{n-1}(\mathfrak{T}_{r-j})(t-j) = \binom{n-1+t-j}{t-j}$$

for all $t < \alpha(\mathfrak{T}_{r-(d-1)}) + d - 1$ where $0 \leq j \leq d-1$, it follows that

$$\begin{aligned}
\mathbf{H}_{\mathcal{U}}(t) &= \rho_n(\mathfrak{T}_{r-(d-1)}, \ldots, \mathfrak{T}_r)(t) \\
&= \rho_{n-1}(\mathfrak{T}_r)(t) + \rho_{n-1}(\mathfrak{T}_{r-1})(t-1) + \cdots + \rho_{n-1}(\mathfrak{T}_{r-(d-1)})\big(t-(d-1)\big) \\
&= \binom{n+t-1}{t} + \binom{n+t-2}{t-1} + \cdots + \binom{n+t-d}{t-d+1} \\
&= \binom{n+t}{t} - \binom{n+t-d}{t-d} \\
&= \mathbf{H}_{R/(C)}(t)
\end{aligned}$$

for all $t < \alpha(\mathfrak{T}_{r-(d-1)}) + d - 1$. Hence

$$\mathbf{H}_{\mathcal{U}}(t) = \mathbf{H}_{R/J}(t) = \mathbf{H}_{R/(C)}(t) \tag{3.18}$$

for $t < \alpha(\mathfrak{T}_{r-(d-1)}) + d - 1$. Moreover,

$$\Delta\mathbf{H}_{\mathcal{U}}(t) = \Delta\mathbf{H}_{R/J}(t) \tag{3.19}$$

for such t. Since $\sigma(\mathfrak{T}') = \sigma(\mathfrak{T}_{r-d}) < \alpha(\mathfrak{T}_{r-(d-1)})$, we see that

$$\Delta\rho_n(\mathfrak{T}')(t) = 0 \tag{3.20}$$

for every $t \geq \alpha(\mathfrak{T}_{r-(d-1)}) - 1$. From (3.16) and (3.17), we have

$$\begin{aligned}
\Delta\mathbf{H}_{\mathbb{X}}(t) &= \Delta\mathbf{H}_{\mathbb{X}'}(t-d) + \Delta\mathbf{H}_{R/J}(t) \\
&= \Delta\rho_n(\mathfrak{T}')(t-d) + \Delta\mathbf{H}_{\mathcal{U}}(t).
\end{aligned}$$

Since $\Delta\rho_n(\mathcal{T}')(t-d) = 0$ and $\Delta\mathbf{H}_{\mathbb{X}'}(t-d) \geq 0$ for every $t-d \geq \alpha(\mathcal{T}_{r-(d-1)}) - 1$, we have

$$\Delta\mathbf{H}_{\mathcal{U}}(t) \geq \Delta\mathbf{H}_{R/J}(t) \tag{3.21}$$

for every $t \geq \alpha(\mathcal{T}_{r-(d-1)}) + d - 1$. From (3.19) and (3.21), we obtain

$$\Delta\mathbf{H}_{\mathcal{U}}(t) \geq \Delta\mathbf{H}_{R/J}(t) \tag{3.22}$$

for every $t \geq 0$. Hence we have

$$\mathbf{H}_{\mathcal{U}}(t) \geq \mathbf{H}_{R/J}(t) \tag{3.23}$$

for such t. It follows from (3.15) and (3.23) that

$$\mathbf{H}_{\mathcal{U}}(t) = \mathbf{H}_{R/J}(t) \tag{3.24}$$

for every $t \geq 0$. Therefore, we obtain from (3.16), (3.17), and (3.24) that

$$\mathbf{H}_{\mathbb{X}'}(t) = \rho_n(\mathcal{T}')(t)$$

for every $t \geq 0$ and we are done. ■

Notice that, as an extra bonus, we get that $I_{\mathbb{X}} + (C) = I_{\mathcal{U}}$ in this case.

The discussion above shows that if $d < r = \alpha(\mathbf{H})$, and $\mathbf{H} \in \mathcal{H}_n$, and $\mathbf{H} = \rho_n(\mathcal{T})$, $\mathcal{T} = (\mathcal{T}_1, \ldots, \mathcal{T}_r)$, then $\rho_n(\mathcal{T}_{n-(d-1)}, \ldots, \mathcal{T}_r)$ has a nice interpretation in terms of Hilbert functions of subsets of sets of points having Hilbert function $\mathbf{H}$. We complete this paper by pointing out the significance of $\rho_n(\mathcal{T}_1, \ldots, \mathcal{T}_{r-d})$.

Proposition 3.19 *Let $\mathbf{H}_1 = \{c_i\}_{i\geq 0}$ be as defined in Remark 3.6 and $\mathcal{T}$ be as above. Then $\mathbf{H}_1 = \rho_n(\mathcal{T}_1, \ldots, \mathcal{T}_{r-d})$.*

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