ISOTOPY OF 2-DIMENSIONAL CONES

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1. Introduction. Knowing the isotopy of cones is a crucial first step in knowing the isotopy of finitely triangulable spaces, for the cones are exactly the stars of vertices. Furthermore, they are the simplest examples of contractible spaces, and the non-triviality of the contractible spaces is one of the distinguishing characteristics of isotopy theory as contrasted with homotopy theory.

The present paper is concerned with the cones over 1-dimensional finitely triangulable spaces. It is clear that homeomorphic spaces have homeomorphic cones, hence cones of the same isotopy type. The surprising result of §2 is that there are very few exceptions to the converse statement. The exceptional isotopy classes of cones all contain cones over spaces that are themselves cones. Section 3 deals with one of the principal tools of isotopy theory, the deleted product factor. It contains descriptions of the homology groups of the deleted product space of a 2-dimensional cone and of the homomorphisms induced by embeddings. The latter is presented in the following manner. After some simplifying steps, the embedding is factored into elementary embeddings. An embedding may be regarded as an addition to the domain space of that part of the range space not in the image. In the elementary embeddings, the added part is a cell. The kernels and cokernels of the homomorphisms induced by these embeddings are free abelian, and the ranks of these groups are computed. Thus the kernels and cokernels of the homomorphisms induced by the original embeddings are free abelian, and their ranks may be estimated.

2. Classification of cones. If P is a topological space and p is a point not in P, then the *cone* over P (with apex p) is the join $\mathbf{P} = P*p$ of P with p. Both P and p are regarded as being contained in $\mathbf{P}$; P is called the *base* and p the *apex*. A map $f:P \to Q$ induces a map $\mathbf{f}:\mathbf{P} \to \mathbf{Q}$. This map $\mathbf{f}$ agrees with f on P, has $f(p) = q$ (the apex of $\mathbf{Q}$), and carries each segment p_1*p linearly onto $f(p_1) * q$ for $p_1 \in P$.

If X is a finite discrete space with $n \geqslant 2$ elements and if $x_0 \notin X$, then $X * x_0$ is a *linear star* of order n. If $I = [0, 1]$ is the unit interval on the real line, then any homeomorph of $(X * x_0) \times I$ is called a *book with n leaves*. The subset $(X * x_0 - X) \times (0, 1)$ is called the *interior* of the book, $x_0 \times I$ is the *spine*, and each component of the complement of the spine is called a *leaf*. An arc may be called a book with *zero leaves*.

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2.1. LEMMA. *If A is a book with m leaves, if B is a book with n leaves, and if $A \subset B$, the $m \leqslant n$. If $m > 2$, then the spine of A is contained in the spine of B.*

Proof. It suffices to prove this when A is the interior of a book with m leaves. If $n \leqslant 2$, then B is a 2-cell or an arc, and it is clear that $m \leqslant n$. Now suppose that $m > 2$, whence $n > 2$. Since each point x of the spine A_0 of A has arbitrarily small neighbourhoods homeomorphic to A, x cannot lie in a leaf of B. Thus A_0 lies in the spine B_0 of B. If x is an interior point of A_0, then x has a neighbourhood U in B such that $B_0 \cap U = A_0 \cap U$ and such that both $A_0 \cap U$ and $V = A \cap U$ are connected. Then $A_0 \cap U$ separates V into exactly m components: there are at least m, one for each leaf, and there are no more than m, for the intersection of V with each leaf A_i of A must be connected. To prove this latter, it suffices to show that any two points $y, z \in V \cap A_i$ lie in a connected subset of $V \cap A_i$. Since V is connected, there is an arc in V from y to z. If this arc does not lie entirely in $V \cap A_i$, then it must meet $A_0 \cap V$. Moving along the arc from y to z, let y' be the first point in $A_0 \cap V$ and z' the last such point. Since V is a neighbourhood of $A_0 \cap V$ in the 2-cell $A_0 \cup A_i$, there is an embedding

$$h: (A_0 \cap V) \times I \to (A_0 \cup A_i) \cap V$$

with $h(t, 0) = t$ for each $t \in A_0 \cap V$. Then

$$C = h[(A_0 \cap V) \times (0, 1]]$$

is a connected subset of $A_i \cap V$ and it meets the subarcs from y to y' and from z' to z. The union of C with these two subarcs, less the points y' and z', is the desired subset. Suppose two components V_1 and V_2 of $V - A_0$ lie in the same leaf B_j of B. It follows from the Brouwer Invariance of Domain (**4**, p. 95) that $V_1 \cup V_2 \cup (U \cap A_0)$ is open in the 2-cell $B_j \cup B_0$, and hence cannot be separated by a subset A_0 of the boundary of the cell. This contradiction shows that each of the m components of $V - A_0$ lies in a distinct one of the n leaves of B, whence $m \leqslant n$.

The foregoing argument was suggested to the author by L. L. Larmore.

A polyhedron P of dimension $\leqslant 1$ is a finite union of vertices and 1-cells (= closed arcs) running between them. The *order* of a vertex v is the number of 1-cells meeting it (a 1-cell that both originates and terminates in v being counted twice). In a given cellular decomposition of P, the vertices of order 2 may usually be eliminated: if v is the junction of the 1-cells σ and τ, then $\sigma \cup \tau$ is taken as a single 1-cell in a new decomposition. The only exceptions occur when P has isolated simple closed curves; the last vertex in a simple closed curve cannot be eliminated. Any two decompositions of P having the minimal number of vertices of order 2 are isomorphic, and such decompositions will be called *canonical*. Let $P' \subset P$ be the set of vertices of order greater than two, and let P'' be P' united with all 1-cells both of whose ends lie in P'. In forming P'', a canonical decomposition is used.

2.2. LEMMA. *Any embedding* $f:\mathbf{P}, p \to \mathbf{Q}, q$ *is isotopic to an embedding* g *with* $g[P] \subset Q$.

Proof. We shall suppose that P has a canonical decomposition and that Q is triangulated. If α is a 1-cell of P, then a 1-simplex β of Q is called *essential* for α if $\boldsymbol{\beta} \cap f[\boldsymbol{\alpha}]$ contains a neighbourhood of q in $\boldsymbol{\beta}$. Recall that $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are the cones over the 1-dimensional cells α and β respectively. Suppose α runs between two distinct vertices v and v', both of which have orders >2 in P. If β is essential for α and if w is one of the end points of β, then one of the components of $f^{-1}[\mathbf{w}] \cap \boldsymbol{\alpha}$ contains p. This component $A = A(w)$ is an arc, and p is one of its end points. If $a = a(w)$ is the other end point of A, then $a \in \alpha$, for clearly $a \in \alpha \cup \mathbf{v} \cup \mathbf{v}'$ while $\mathbf{v}$ and $\mathbf{v}'$ must be mapped into spines. Let w' be the other end point of β and set $A' = A(w')$, $a' = a(w')$. Let T be the triangular region in $\boldsymbol{\beta}$ whose vertices are $f(a)$, $f(a')$, and q. Then $f^{-1}[T]$ lies in the 2-cell $D \subset \boldsymbol{\alpha}$ bounded by A, A', and the segment of α between a and a'. Furthermore, $f^{-1}[T]$ is a neighbourhood of $(A \cup A') - \{a, a'\}$ in D. Thus there is an isotopy h_t on $\mathbf{P}$ that carries D onto $f^{-1}[T]$ and leaves the complement of D fixed. Then fh_t is an isotopy of f that agrees with f outside of D and that carries D onto T. This process is repeated for each essential simplex β of each edge α running between two vertices of order greater than 2. The composite of these modifications is an embedding f_1 such that the following assertion is true:

(2.3) $\begin{cases} \textit{If } \beta_1, \ldots, \beta_k \textit{ are the essential simplexes for } \alpha, \textit{ then } f[\alpha] \subset \beta_1 \ldots \beta_k \\ \text{and } f_1[\boldsymbol{\alpha}] \cap \beta_1 \textit{ is a straight-line segment for each } i = 1, \ldots, k. \end{cases}$

For each edge α of P that begins and terminates in a single vertex of order >2, there is a modification similar to that described above. The result of these modifications is an embedding f_2 isotopic to f and satisfying (2.3) for each α in P''. If α is an edge of P having one vertex v of order 2 and the other, v', of order 1, then $\mathbf{v}$ must be mapped into some spine $\mathbf{w}$ in Q. There is a leaf $\boldsymbol{\beta}$ radiating from $\mathbf{w}$ such that $\boldsymbol{\beta} \cap f_2[(\alpha - v) * p]$ contains $f_2[\mathbf{v}]$ in its closure. Divide β into two 1-simplexes β' and β'' by inserting a new vertex w' of order 2 in the middle of β. Assume that $w \in \beta'$ and enlarge the set of simplexes called *essential* by including β', but not β''. There is an isotopy of f_2 that pulls the image of $\boldsymbol{\alpha}$ into $\boldsymbol{\beta}'$ in such a way that α is carried onto a segment running from some point in $\mathbf{w}$ to some point in $\mathbf{w}'$. Repeat this process for each such edge α and call the resulting embedding f_3. The remainder P_0 of P consists of isolated vertices, arcs, and simple closed curves. The next isotopy produces an embedding f_4 that agrees with f_3 outside of $\mathbf{P}_0$ and that carries $\mathbf{P}_0$ into the cone over the union of the inessential 1-simplexes, the 1-simplexes essential for edges that form isolated arcs and closed curves, and the isolated vertices of Q; this isotopy is easily constructed. The final isotopy is a radial expansion of $f_4[\mathbf{P}]$ away from q, and yields the desired embedding g.

2.4. THEOREM. *The* 1-*dimensional finitely triangulable spaces* P *and* Q *have the same isotopy type if and only if the pairs* $\mathbf{P}$, p *and* $\mathbf{Q}$, q *have the same isotopy type.*

Proof. If $f:P \to Q$ has an isotopy inverse g, then $\mathbf{g}$ is an isotopy inverse to $\mathbf{f}$. Conversely, suppose $f:\mathbf{P}, p \to \mathbf{Q}, q$ is any embedding having an isotopy inverse g. Lemma 2.2 shows that f and g are isotopic to embeddings f' and g' with $f'[P] \subset Q$ and $g'[Q] \subset P$. Let $f'':P \to Q$ and $g'':Q \to P$ be the embeddings obtained by restricting f' and g'. It follows from Lemma 2.1 that each vertex of order $m > 2$ is mapped by f' (or by g') into the spine over a vertex of order $n \geqslant m$. Also, isotopies of $g'f'$ and $f'g'$ to the respective identity maps must carry such a vertex into its own spine. Thus f'' and g'' preserve the orders of the vertices of order >2. If $r:(\mathbf{P} - \{p\}) \to P$ is the radial projection from p, and if h_t is an isotopy from $g'f'$ to the identity on P, then $k_t(x) = rh_t(x)$ $(x \in P)$ defines a homotopy from $g''f''$ to the identity on P. Similarly, $f''g''$ is homotopic to the identity on Q. The final remark in **(1)** now shows that P and Q have the same isotopy type.

2.5. Lemma. *Any isotopy h_t of the identity h_0 on* $\mathbf{P}$ *must leave the apex p fixed, provided one of the following holds:*

(a) *P has at least three vertices of orders $\neq 1, 2$;*

(b) *P is the union of two disjoint closed subsets A and B each having at least two points;*

(c) *P contains two vertices of order* 3 *and the closed star* (*in a canonical decomposition*) *of each contains a vertex of order* 1*;*

(d) *P contains a simple closed curve and* (i) *a second simple closed curve disjoint from the first, or* (ii) *a second simple closed curve meeting the first in exactly one point, or* (iii) *a vertex of order $\neq 1, 2$ not on the first curve.*

Proof. (a) If $a, b, c \in P$ are vertices of orders $\neq 1, 2$, then the triod $\{a, b, c\} * p$ must be carried into itself by h_t; hence $h_t(p) = p$ for all t.

(b) Suppose to the contrary that h_t moves p, and that $h_t(p)$ enters $A * p - \{p\}$ first. Select t and $b_1, b_2 \in B$ such that $h_t(p) \in A * p - \{p\}$ and $h_t(b_i) \in B * p - \{p\}$ $(i = 1, 2)$. Then the arcs $h_t[b_i * p]$ run from $B * p$ to $A * p$, and thus must both contain p. This violates the requirement that h_t be one-to-one.

(c) Similar to (b).

(d) The cone over a simple closed curve is a disk containing p as an interior point. An isotopy that moves p must carry other interior points onto p. Furthermore, p is constrained to move along spines over vertices of orders $\neq 1, 2$. The three cases are now easily established.

Let $\mathfrak{A}_1$ be the class of all polyhedra of dimension 0 or 1 satisfying at least one of the conditions (a)–(d) of Lemma 2.5, and let $\mathfrak{A}$ be the set of equivalence classes of cones over members of $\mathfrak{A}_1$, where two cones are called equivalent when their bases are homeomorphic. Let $\mathfrak{B}_n$ $(n = 0, 2, 3, 4, \ldots)$ be the isotopy class of cones containing a book with n leaves, and let $\mathfrak{B}$ be the set of these classes.

2.6. CLASSIFICATION THEOREM. *The set of isotopy classes of cones of dimension* $\leqslant 2$ *is exactly* $\mathfrak{A} \cup \mathfrak{B}$, *and* $\mathfrak{A} \cap \mathfrak{B}$ *is empty.*

Proof. The cones in $\mathfrak{A}$ have points that are fixed under all isotopies of the identity, and those in $\mathfrak{B}$ do not. Thus $\mathfrak{A} \cap \mathfrak{B}$ is empty.

If P is a polyhedron of dimension $\leqslant 1$, let r be the number of its vertices of order zero, s the number of vertices of order at least three, t the 1-dimensional Betti number, and u the number of components. Thus to each polyhedron there is assigned a quadruple (r, s, t, u). For the present analysis, it suffices to let r, s, t, u range over the values 0, 1, 2 and *many*. The quadruples $(r, s, t, u) = (0, 0, 0, 1)$, $(0, 0, 1, 1)$, $(0, 1, 0, 1)$, $(0, 1, 1, 1)$, $(1, 0, 0, 1)$, $(1, 0, 0, 2)$, $(1, 1, 0, 2)$, and $(2, 0, 0, 2)$ are associated with polyhedra whose cones have the isotopy types of books. The quadruples $(0, 2, t, 1)$ with $t \geqslant 2$ may be associated with elements of both $\mathfrak{A}$ and $\mathfrak{B}$, and one distinguishes the two cases as follows. Suppose P is connected and has two vertices v and w of order >2, and assume that P has its canonical decomposition. If each arc starting at v ends in w and if each arc from w ends either in v or at a vertex of order 1, then the cone over P is isotopic to a book. Otherwise P is in $\mathfrak{A}_1$. A straightforward verification shows that the rest are either inconsistent (e.g., $(2, 0, 0, 1)$ describes no space) or are associated with elements of $\mathfrak{A}$. Since every polyhedron P has a quadruple (r, s, t, u) associated with it, and since all cases have been investigated, we see that every cone is either isotopic to a book, or lies in a member of $\mathfrak{A}$.

Let $P, Q \in \mathfrak{A}_1$, and assume that $f:\mathbf{P} \to \mathbf{Q}$ is an isotopy equivalence. If g is an isotopy inverse to f, then the isotopies from gf and fg to the appropriate identity maps must leave the apexes fixed. Thus both f and g must respect apexes. It follows from Lemma 2.4 that P and Q have the same isotopy type, and hence are homeomorphic. This shows that the classes in $\mathfrak{A}$ are isotopy classes of cones.

3. Deleted products. This section is devoted to describing the integral homology groups of the deleted products $R\mathbf{P}$ of cones over 1-dimensional finitely triangulable spaces P and the homomorphisms $Rf_*:H_*(R\mathbf{P}) \to H_*(R\mathbf{Q})$ induced by embeddings $f:\mathbf{P} \to \mathbf{Q}$.

The reader will recall that if X is any space, then RX is the difference space $X \times X - DX$, where DX is the diagonal subspace of the topological product $X \times X$. If $f:X \to Y$ is an embedding into the space Y, then $Rf:RX \to RY$ is defined by restricting the product map $f \times f:X \times X \to Y \times Y$. If f and g are isotopic, then Rf and Rg are isotopic, hence homotopic.

The homology groups are easily found. If P is a single point, then $\mathbf{P}$ is an arc, and $R\mathbf{P}$ has the homotopy type of a discrete 2-point space. If P is a simple closed curve, then $\mathbf{P}$ is a disk and $R\mathbf{P}$ has the homotopy type of the 1-sphere. If P is neither a point nor a simple closed curve, then $R\mathbf{P}$ is connected, while $H_k(R\mathbf{P})$ $(k > 0)$ is a direct summand in $\tilde{H}_{k-1}(RP)$ and has a complementary

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direct summand isomorphic to the direct sum $\tilde{H}_{k-1}(P) + \tilde{H}_{k-1}(P)$ **(2)**. The tildes denote reduced homology groups. The homology of RP, hence that of $R\mathbf{P}$, may be computed from **(5** or **3)**. Note that these homology groups are free abelian, and that $H_k(R\mathbf{P})$ is trivial for $k > 3$.

We now describe the homomorphism $Rf_*:H_k(R\mathbf{P}) \to H_k(R\mathbf{Q})$ induced by an embedding f of the cone $\mathbf{P}$ into the cone $\mathbf{Q}$. First note that in case f fails to send the apex p of $\mathbf{P}$ into the apex q of $\mathbf{Q}$, then (apart from trivial cases) $f(p)$ must lie in a spine in $\mathbf{Q}$. The composite of f with an isotopy that shrinks $\mathbf{P}$ towards p will then pull the image $f[\mathbf{P}]$ into the book B associated with this spine. Any point of the spine may act as an apex of B, whence f can be factored into two embeddings, each of which respects apexes.

Thus it suffices to consider embeddings $f:\mathbf{P} \to \mathbf{Q}$ having $f(p) = q$. Let us summarize the analysis of the homomorphism Rf_* first, leaving the proofs until later. Lemma 2.2 shows that we may assume that $f[P] \subset Q$. The following result shows that only the restriction of f to P and Q plays a role in the computation of Rf_*.

3.1. THEOREM. *If $f:\mathbf{P} \to \mathbf{Q}$ is an embedding with $f(p) = q$ and $f[P] \subset Q$, and if $g:P \to Q$ is the restriction of f, then the induced homomorphisms*

$$Rg_* \text{ and } Rf_* : H_*(R\mathbf{P}) \to H_*(R\mathbf{Q})$$

are equal.

The 1-dimensional spaces P and Q may be so triangulated that g is simplicial, whence P may be regarded as a subcomplex of Q, and g may be regarded as the inclusion map. But then Q can be constructed out of P by appending vertices and 1-simplexes. In other words, g may be factored as follows:

$$P_0 = P \xrightarrow[g_1]{} P_1 \xrightarrow[g_2]{} P_2 \to \ldots \xrightarrow[g_n]{} P_n = Q,$$

where P_i is obtained from P_{i-1} by (a) adding an isolated vertex, or (b) adding a simplex that meets P_{i-1} in just one of its end point vertices, or (c) adding a simplex both of whose end point vertices lie in P_{i-1}.

The four propositions (3.2)–(3.5) below describe the induced homomorphisms, except in the routine cases in which one of the spaces is a point or a simple closed curve. In these results, K_k and C_k are the ranks of the kernel and the cokernel of $Rg_*:H_k(R\mathbf{P}) \to H_k(R\mathbf{Q})$, respectively (they are free abelian), and n is the *deleted product number* of P relative to the two end points of the added simplex (Case (c)). C. W. Patty **(5)** defines the deleted product number as follows. Let G be a set of simple closed curves in P and let v, w be the two end points of the added simplex. Let $n(G)$ be the number of elements of G that separate v from w. Then the *deleted product number* of P relative to v, w is

$$n = \min\{n(G):G \text{ is a basis for the 1-cycles of } P\}.$$

Note that $R\mathbf{P}$ and $R\mathbf{Q}$ are connected, whence Rg_* is an isomorphism in dimension zero. Also, the homology of these spaces vanishes above dimension 3.

(3.2) *If* $Q = P \cup S$ *and* S *is an isolated point, then* $K_1 = 0$, $C_1 = 2\beta_0 - 2$, $K_2 = 0$, $C_2 = 2\beta_1$, $K_3 = C_3 = 0$, *when* β_k *is the* k-*dimensional Betti number of* P.

(3.3) *Suppose that* $Q = P \cup A$, *where* A *is a* 1-*simplex and* $P \cap A = \{v\}$ *is exactly one of the two end points of* A. *Suppose that* v *is a vertex of order* m *in* P, *and that* P_1 *is the component of* P *containing* v. *Consider four cases:*

(1) P_1 *is a point,*

(2) P_1 *is an arc and* v *is an end point of* P_1,

(3) P_1 *is an arc and* v *is an interior point,*

(4) P_1 *is neither a point nor an arc.*

Then,

Case 1: $K_1 = 0$, $C_1 = 2$, $K_2 = 0$, $C_2 = 0$, $K_3 = C_3 = 0$.

Case 2: $K_1 = 0$, $C_1 = 0$, $K_2 = 0$, $C_2 = 0$, $K_3 = C_3 = 0$.

Case 3: $K_1 = 1$, $C_1 = 0$, $K_2 = 0$, $C_2 = 1$, $K_3 = C_3 = 0$.

Case 4: $K_1 = 0$, $C_1 = 0$, $K_2 = 0$, $C_2 = 2m - 4$, $K_3 = C_3 = 0$.

(3.4) *Suppose that* $Q = P \cup A$, *where* A *is a* 1-*simplex and* $P \cap A = \{v, w\}$ *is the set of both end points of* A. *Let* β'_k *be the* k-*dimensional Betti number of* $P' = P - (\text{Star } v \cup \text{Star } w)$. *Assume that* v, w *both lie in the same component* P_1 *of* P.

If P_1 *is an arc, then*

$$K_1 = 1, \quad C_1 = 0, \quad K_2 = 0, \quad C_2 = 2\beta'_0 - 3, \quad K_3 = 0, \quad C_3 = 2\beta'_1.$$

If P_1 *is not an arc and if* n *is the deleted product number of* P *relative to* v, w, *then* $K_1 = 0$, $C_1 = 0$, $K_2 = 2n$, $C_2 = 2\beta'_0 - 2$, $K_3 = 0$, $C_3 = 2\beta'_1 - 2n$.

(3.5) *Suppose that* $Q = P \cup A$, *where* A *is a* 1-*simplex and* $P \cap A = \{v, w\}$ *is the set of both end points of* A. *Let* β_0 *be the number of components in* P *and let* β'_k *be the* k-*dimensional Betti number of* $P' = P - (\text{Star } v \cup \text{Star } w)$. *Assume that* v *and* w *lie in distinct components* P_1 *and* P_2 *of* P.

Then,

$$K_1 = 2\beta_0 - 2 + a,\ C_1 = 0,\ K_2 = 2\beta'_1,\ C_2 = 2\beta'_0 - 2\beta_0 - a,\ K_3 = C_3 = 0,$$

where a *is given as follows:*

$a = -4$ *when* P_1 *and* P_2 *are both points;*

$a = -1$ *when one of* P_1, P_2 *is a point, the other is an arc, and* $P_1 \cup A \cup P_2$ *is not an arc;*

$a = -2$ *when one of* P_1, P_2 *is a point, the other is an arc, and* $P_1 \cup A \cup P_2$ *is an arc;*

$a = -2$ *when one of* P_1, P_2 *is a point and the other is neither a point nor an arc;*

$a = 0$ *when* P_1, P_2, *and* $P_1 \cup A \cup P_2$ *are all arcs;*

$a = 1$ *when* P_1 *and* P_2 *are arcs but* $P_1 \cup A \cup P_2$ *is not an arc;*

$a = 0$ *when one of* P_1, P_2 *is an arc and the other is neither an arc nor a point;*

$a = -1$ *when neither* P_1 *nor* P_2 *is an arc or a point.*

W.-T. Wu has proved **(6)** that if P is a triangulated space, then the cell complex

$$JP = \{\sigma \times \tau : \sigma, \tau \text{ are closed simplexes of } P, \sigma \cap \tau \text{ is empty}\}$$

is a deformation retract of RP. In what follows, we fix triangulations on P and Q, so that the functor R (from the category of embeddings, into itself) may be abandoned in favour of the more workable functor J (from the category of simplicial embeddings of triangulated spaces into the category of embeddings).

If P, Q are triangulated spaces, then there are homomorphisms α, β, δ, α', β', and δ' such that the rows in the diagram

$$\begin{array}{ccccccccc} \ldots \to & H_{k+1}(J\mathbf{P}) & \xrightarrow{\delta} & H_k(JP) & \xrightarrow{\alpha} & H_k(P) + H_k(P) & \xrightarrow{\beta} & H_k(J\mathbf{P}) & \to \ldots \\ (3.6) & \downarrow J\mathfrak{g}_* & & \downarrow Jg_* & & \downarrow g_* + g_* & & \downarrow J\mathfrak{g}_* & \\ \ldots \to & H_{k+1}(J\mathbf{Q}) & \xrightarrow[\delta']{} & H_k(JQ) & \xrightarrow[\alpha']{} & H_k(Q) + H_k(Q) & \xrightarrow[\beta']{} & H_k(J\mathbf{Q}) & \to \ldots \end{array}$$

are exact **(2)**. If P is 1-dimensional, but not a point or a simple closed curve, then β is trivial and the top sequence splits; a similar statement holds for Q. If $g:P \to Q$ is an embedding, then it is easily verified that the diagram commutes.

If K and L are subspaces of a space M, define $K \triangle L$ to be $(K \times L) \cup (L \times K)$ with the relative topology from $M \times M$.

Proof of (3.1). It suffices to prove the corresponding result for the functor J, and this is done by showing that Jf is homotopic to $J\mathfrak{g}$. The cell complex $J\mathbf{P}$ has subcomplexes

$$P_1 = \{\sigma \times (\tau * p) : \sigma, \tau \text{ simplexes of } P, \sigma \cap \tau \text{ is empty}\},$$
$$P_2 = \{(\sigma * p) \times \tau : \sigma, \tau \text{ simplexes of } P, \sigma \cap \tau \text{ is empty}\},$$

and $J\mathbf{P} = P_1 \cup P_2$. A similar decomposition $J\mathbf{Q} = Q_1 \cup Q_2$ is defined. In these constructions, $\emptyset * p$ is defined to be p when $\emptyset$ is the empty simplex of P. Note that Jf and $J\mathfrak{g}$ both map P_i into Q_i $(i = 1, 2)$. But P_1 is a quotient space obtained from $JP \times I$; the quotient map sends $(x, x', 1)$ into (x, p) when $(x, x') \in JP \subset P \times P$, and is defined in the obvious, one-to-one fashion on $JP \times [0, 1)$. Let $\xi : JP \times I \to JQ \times I$ be a map covering $(Jf|P_1):P_1 \to Q_1$, and write

$$\xi(x, x', t) = (\xi_1(x, x', t), \xi_2(x, z', t)) \in JQ \times I$$

when $(x, x') \in JP$ and $t \in I$. Then the homotopy $\eta : JP \times I \times I \to JQ \times I$ defined by

$$\eta(x, x', t, s) = (\xi_1(x, x', ts), \xi_2(x, x', t) + (1 - s)t)$$

for $(x, x') \in JP$ and $s, t \in I$, induces a homotopy between $(Jf|P_1)$ and $(J\mathfrak{g}|P_1)$. A similar homotopy is defined between $(Jf|P_2)$ snd $(J\mathfrak{g}|P_2)$. Since these homotopies agree on $P_1 \cap P_2$, they define the desired homotopy from Jf to $J\mathfrak{g}$.

Note that our usual restriction, that P and Q be 1-dimensional, was not needed in this proof.

Proof of (3.2). From the definition of the functor J, it follows that $JQ = JP \cup S \triangle P$. Thus in the diagram (3.6) the image of Jg_* in $H_k(JQ)$ is a direct summand with cokernel of rank $2\beta_k$. The result follows.

Proof of (3.3). Let $P_0 = P - P_1$ and $Q_1 = P_1 \cup A$. Then

$$JP = JP_1 \cup JP_0 \cup P_1 \triangle P_0 \quad \text{and} \quad JQ = JQ_1 \cup JP_0 \cup Q_1 \triangle P_0,$$

and the summands in each case are mutually disjoint closed subsets. Since $P_1 \triangle P_0$ is a deformation retract of $Q_1 \triangle P_0$, the kernels and cokernels of Jg_* are isomorphic to those induced by the restriction $g_1:P_1 \to Q_1$ of g. If P_1 is a point, then JP_1 is empty while JQ_1 is a set of exactly two points. If P_1 is an arc and v is an end point of P_1, then g_1 is an isotopy equivalence. If P_1 is an arc and v is an interior point, then JP_1 has the homotopy type of a 2-point space, while JQ_1 has the homotopy type of the circle S^1. If P_1 is neither a point nor an arc, then Patty's results **(5)** apply: Jg_{1*} is an isomorphism except in dimension 1 where it is a monomorphism with cokernel of rank $2m - 2$. In all four cases, g_* is an isomorphism in all dimensions, and the result follows from a brief contemplation of diagram (3.6).

Proof of (3.4). It follows easily from the definition of JQ that

$$JQ = JP \cup A \triangle P' \quad \text{and} \quad JP \cap A \triangle P' = B \triangle P'$$

when $B = \{v, w\}$. In the diagram

$$\begin{array}{ccccccc}
\ldots \to H_{k+1}(JQ, JP) & \xrightarrow{\partial} & H_k(JP) & \xrightarrow{Jg_*} & H_k(JQ) \to \ldots \\
\uparrow i_* & & \uparrow \mu & & \\
H_{k+1}(A \triangle P', B \triangle P') & \xrightarrow{\lambda} & H_k(P') + H_k(P') & &
\end{array}$$

the top horizontal row is the exact sequence of the pair (JQ, JP), i_* is the isomorphism induced by the excision $i:(A \triangle P', B \triangle P') \to (JQ, JP)$, λ is induced by the homomorphism that sends the cycle $z \times A + A \times z'$ into (z, z'), and μ is induced by

$$(z, z') \to z \times (v - w) + (w - v) \times z'$$

(z, z' are k-cycles on P'). The diagram is clearly commutative, and λ is an isomorphism. If $k > 1$, then $H_k(P') = 0$, whence $Jg_*:H_k(JP) \to H_k(JQ)$ is monic for $k = 2$. If z is a simple closed curve that separates v from w in P, then $z \times (v - w)$ and $(w - v) \times z$ are not bounded in JP. Thus the image of $\partial:H_2(JQ, JP) \to H_1(JP)$ has rank $2n$ and the kernel of this homomorphism has rank $2\beta_1' - 2n$. If P_1 is an arc, then JP has one more component than JQ, whence the homomorphism $Jg_*:H_0(JP) \to H_0(JQ)$ is an epimorphism with kernel of rank 1. It follows from this that the cokernel of $Jg_*:H_1(JP) \to H_1(JQ)$ has rank $2\beta_0' - 1$. If P_1 is not an arc, then $Jg_*:H_k(JP) \to H_k(JQ)$ is an isomorphism for $k = 0$, and has cokernel of rank $2\beta_0'$ when $k = 1$. Since P and Q are neither points nor simple closed curves, β and β' in (3.6) are trivial for $k > 0$. In dimension zero, $J\mathbf{g}_*$ is an isomorphism between infinite cyclic groups. For $k > 1$, $H_k(P) = H_k(Q) = 0$, whence $J\mathbf{g}_*:H_{k+1}(J\mathbf{P}) \to H_{k+1}(J\mathbf{Q})$ has its kernel and cokernel isomorphic to those of $Jg_*:H_k(JP) \to H_k(JQ)$. This yields $K_3 = 0$ and $C_3 = 2\beta'_1 - 2n$. Note that $n = 0$ when P_1 is an arc. The simplex A completes a cycle in Q, whence $g_*:H_1(P) \to H_1(Q)$ is a monomorphism with cokernel of rank 1. This yields $K_2 = 2n$ while $C_2 = 2\beta'_0 - 3$ if P_1 is an arc and $C_2 = 2\beta'_0 - 2$ otherwise. Since g_* is an isomorphism in dimension 0, $K_1 = 1$ if P_1 is an arc, $K_1 = 0$ otherwise, and $C_1 = 2\beta'_0$ in either case.

The proof of (3.5) is similar to that of (3.4) and so will be omitted.

4. Isotopy equivalences.

If $f : X \to Y$ is an isotopy equivalence, then $Rf_*:H_k(RX) \to H_k(RY)$ is an isomorphism for all k. When X and Y are finitely triangulable 1-dimensional spaces, the converse also holds **(3)**. The following example shows that the converse may fail when $X = P$ and $Y = Q$ are cones over 1-dimensional spaces.

Let P be a contractible finitely triangulable space containing two vertices, v and w, of order 1. Note that $P' = P - (\text{Star } v \cup \text{Star } w)$ is again contractible. Let $Q = P \cup A$, where A is an arc with $P \cap A = \{v, w\}$ being the two end points of A. In the terminology of (3.4), we have $n = \beta'_1 = 0$, and if P is not an arc, then the inclusion map $g:P \to Q$ induces isomorphisms $J\mathbf{g}_*:H_k(R\mathbf{P}) \to H_k(R\mathbf{Q})$ for all k. If P has, for example, more than two vertices of order greater than 2, then P is not a book, and $\mathbf{g}:\mathbf{P} \to \mathbf{Q}$ is not an isotopy equivalence.

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