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POSITIVE DEFINITE SEQUENCE OF OPERATORS AND A FIXED POINT THEOREM

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The purpose of this note is to prove the following:

THEOREM. *Let $\{A_n\}$ be a positive definite sequence of operators on a Hilbert space H with $A_0=1$. If $A_1f=f$ for some f in H, then $A_nf=f$ for all n.*

Note that a bilateral sequence of operators $\{A_n:n=0, \pm 1, \pm 2, \ldots\}$ on H is *positive definite* if

$$\sum_n \sum_m \langle A_{n-m}f_n, f_m \rangle \geq 0$$

for every finitely nonzero sequence $\{f_n\}$ of vectors in H [1].

Proof of theorem. It is well known that there exists a unitary operator U on a larger Hilbert space K containing H such that

$$A_ng = PU^ng \quad \text{for all } g \text{ in } H \text{ and for all } n,$$

where P is the orthogonal projection of K onto H [1]. If $A_1f=f$, we have $f=A_1f=PUf$. Hence

$$\|f\| = \|PUf\| \leq \|Uf\| = \|f\|.$$

Thus it readily follows that $Uf=f$ [1]; and hence $f=U^{-1}f$. This implies that

$$A_{-1}f = PU^{-1}f = Pf = f.$$

Moreover,

$$\begin{aligned} A_2f = PU^2f &= PU\cdot Uf \\ &= PUf = f, \end{aligned}$$

and

$$\begin{aligned} A_{-2}f = PU^{-2}f &= PU^{-1}\cdot U^{-1}f \\ &= PU^{-1}f = f \end{aligned}$$

Thus the proof of the theorem follows inductively so that $A_nf=f$ for all n.

REFERENCE

1. F. Riesz and B. Sz.-Nagy, *Appendix to functional analysis*, Ungar, New York.

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