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Suborbit Structure of Permutation p-Groups and an Application to Cayley Digraph Isomorphism

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Abstract. Let P be a transitive permutation group of order p^m, p an odd prime, containing a regular cyclic subgroup. The main result of this paper is a determination of the suborbits of P. The main result is used to give a simple proof of a recent result by J. Morris on Cayley digraph isomorphisms.

1 Introduction

Let G be a transitive permutation group on a finite set Ω and let $\alpha \in \Omega$. The orbits of the stabilizer G_α on Ω are called *suborbits* of G relative to α. The singleton orbit $\{\alpha\}$ is said to be *trivial*. Determining the suborbit structure of a given permutation group is one of the basic problems in the theory of permutation groups (see [2, 6, 7]). It also plays a significant role in some other mathematical areas such as graph theory, combinatorics, finite geometries and so on. During the last twenty years, following the completion of the classification of finite simple groups, many new methods and results regarding this problem have appeared. However, most have involved groups that are not p-groups. We shall use the term permutation p-group for a permutation group of order p^m, p a prime.

In this paper, we shall investigate the suborbit structure of permutation p-groups containing a regular cyclic subgroup, where p is an odd prime. Our main result is the following theorem and will be proved in the next section.

Theorem 1.1 *Let p be an odd prime, let P be a transitive permutation p-group on the set Ω, and let Q be a regular cyclic subgroup of P. Then every suborbit Δ of P must be one of the orbits of the subgroup of Q of order $|\Delta|$.*

As an application of Theorem 1.1, we examine the isomorphism problem for Cayley digraphs. Let us first recall some definitions. Let G be a finite group and S a subset of $G-\{1\}$. We define the *Cayley digraph* $X = \mathrm{Cay}(G;S)$ on G with *connection set* S to be the digraph with vertex set $V(X) = G$ and arc set $A(X) = \{(g, sg) \mid g \in G, s \in S\}$.

It is well known that $\mathrm{Aut}(X)$ contains the right regular representation $R(G)$ of G and so X is vertex-transitive. Also, X is connected if and only if $G = \langle S\rangle$. Furthermore, if $S = S^{-1}$, then $\mathrm{Cay}(G;S)$ can be considered to be a *Cayley graph* by identifying an undirected edge joining g and h with the two arcs (g,h) and (h,g).

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There have been many papers written on a variety of topics dealing with isomorphisms of Cayley graphs and digraphs. One such topic is the examination of isomorphisms between two Cayley digraphs on different groups. In 1995, A. Joseph [4] examined the case when the groups are of prime square order. Recently, J. Morris [5] extended Joseph's result and determined when a Cayley digraph on an abelian group R is isomorphic to a Cayley digraph on the cyclic group Z_{p^n}, where p is an odd prime. To do so is equivalent to determining all the regular abelian subgroups of the automorphism group of a given Cayley digraph on Z_{p^n}. Theorem 1.2 below is stated in a different form than Theorem 1.1 in [5], the main result in [5], but is equivalent to it. In this paper, as an application of Theorem 1.1, we give a new and simple proof for Theorem 1.2.

Before stating Theorem 1.2, we need some preliminary definitions. Let $X = \mathrm{Cay}(Q;S)$ be a Cayley digraph on the group $Q \cong Z_{p^n}$, where p is an odd prime and $n \geq 2$. For $0 \leq i \leq n$, let Q_i denote the unique subgroup of order p^i of Q. Let $\{i_0, i_1, i_2, \ldots, i_{k-1}\}$ be the set of all numbers i such that $S - Q_i$ is a union of right cosets of Q_i in Q, where $0 = i_0 < i_1 < i_2 < \cdots < i_{k-1} < n$. For convenience, let $i_k = n$. Then the sequence $(i_0, i_1, \ldots, i_k)$ is said to be the *wreathed sequence* of X, and is denoted by $\mathrm{WS}(X)$. Note that every Cayley digraph on Q has $i_0 = 0$ as a first term—perhaps the only term—in the wreathed sequence because S is a union of right cosets of $\{1\}$. Given the Cayley digraph $X = \mathrm{Cay}(Q;S)$ with wreathed sequence $\mathrm{WS}(X) = (i_0, i_1, \ldots, i_k)$, let $\mathbf{C}(X)$ be the set of all nonisomorphic abelian groups R such that R has a series of subgroups, say $1 = R_0 \leq R_1 \leq \cdots \leq R_k = R$, such that for any $1 \leq \ell \leq k$, $|R_\ell| = p^{i_\ell}$ and $\frac{R_\ell}{R_{\ell-1}}$ is cyclic. Clearly, $Z_{p^n} \in \mathbf{C}(X)$.

Theorem 1.2 *For an odd prime p and $n \geq 2$, let $X = \mathrm{Cay}(Q;S)$ be a Cayley digraph on the group $Q \cong Z_{p^n}$ having the wreathed sequence $\mathrm{WS}(X) = (i_0, i_1, \ldots, i_k)$. Then $\mathrm{Aut}(X)$ contains a regular abelian subgroup R if and only if $R \in \mathbf{C}(X)$. In other words, X is isomorphic to a Cayley digraph on an abelian group R if and only if $R \in \mathbf{C}(X)$.*

Using Theorem 1.2, we easily may examine whether or not a Cayley digraph $\mathrm{Cay}(R;S')$ on an abelian group R is isomorphic to a Cayley digraph $\mathrm{Cay}(Q;S)$ on the cyclic group Q (see Remark 3.2 for more details).

Unless stated otherwise, the group- and graph-theoretic terminology is standard and we refer the reader to [1, 3]. Suppose G is a permutation group on Ω. For any subgroup H of G and any subset Γ of Ω, we use $H_{(\Gamma)}$ and $H_{\{\Gamma\}}$ to denote the subgroups of H fixing Γ pointwise and setwise, respectively. In particular, if $\Gamma = \{x\}$, then we denote both $H_{(\Gamma)}$ and $H_{\{\Gamma\}}$ by H_x.

2 Proof of Theorem 1.1

Henceforth, let P be a transitive permutation group of order p^m on a set Ω of cardinality p^n, where p is an odd prime. Suppose that P contains a regular cyclic subgroup Q. As in Section 1, let Q_i, $0 \leq i \leq n$, denote the unique subgroup of order p^i of Q. Then for each i, Q has a unique system of imprimitivity with blocks of length p^i, which we always refer to as a p^i-block system. We denote it by B_i and note it is the set of orbits of Q_i. For any $x \in \Omega$ and $0 \leq j < i \leq n$, we use $B_{x,i,j}$ to denote the set of

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all the p^j-blocks of Q contained in the p^i-block which contains x, in particular, $B_{x,i,0}$ will be denoted simply by $B_{x,i}$, which is the p^i-block containing x. Before proving Theorem 1.1, we prove three lemmas.

Lemma 2.1 *For any i satisfying $0 \leq i \leq n$, each p^i-block of Q is also a block of P. Hence, B_i also forms the unique p^i-block system of P.*

Proof It is well known that the set of orbits of any normal subgroup of a transitive permutation group must be a complete block system of the group (see [7, Proposition 7.1]). Take a principal series of P: $P_0 = 1 \leq P_1 \leq \cdots P_{m-1} \leq P_m = P$, where for each ℓ satisfying $0 \leq \ell \leq m-1$, P_ℓ is normal in P and $|P_{\ell+1}/P_\ell| = p$ (see [3, Chapter III, Theorem 7.2]). Clearly, if an orbit of P_ℓ has length p^k, then any orbit of $P_{\ell+1}$ has length either p^k or p^{k+1}. Since P has both p^0-blocks and p^n-blocks, it follows that for any $0 \leq i \leq n$, P has a p^i-block. Note that any block of P must be a block of Q and Q has the unique p^i-block system B_i. Therefore, P has the unique imprimitive complete p^i-block system B_i. ■

Hereafter, by Lemma 2.1, all p^i-blocks mentioned in this paper are p^i-blocks of P, which are orbits of Q_i. In particular, every element of P induces an action on B_i. Moreover, we note the following two facts.

(1) For any $x \in \Omega$ and $0 \leq i \leq n$, we have $P_x = (P_{\{B_{x,i}\}})_x$. Since $Q_i \leq P_{\{B_{x,i}\}}$ and Q_i is transitive on $B_{x,i}$, we have from Frattini-Argument that $P_{\{B_{x,i}\}} = P_x Q_i$.

(2) Suppose that R is a regular abelian subgroup of P. For any i, with $0 \leq i \leq n$, let R_i be the subgroup of R fixing each p^i-block setwise. Then R/R_i acts regularly on $B_{x,n,i}$ because R is abelian [7, Proposition 4.4]. Hence, R_i acts regularly on each p^i-block, which forces $|R_i| = p^i$.

In what follows, we first consider a special case for $m = 3$ and $n = 2$.

Lemma 2.2 *Suppose that $m = 3$ and $n = 2$. Then P is nonabelian, each noncentral subgroup of order p of P fixes one p-block pointwise and is transitive on each of the other p-blocks.*

Proof From the hypothesis, P is a transitive permutation group of order p^3 acting on Ω which has cardinality p^2. Since P is not regular, it is nonabelian. Since P contains a cyclic subgroup Q of order p^2, we may assume that P has the following defining relations:

$$P = \langle a, b \mid a^{p^2} = b^p = 1, b^{-1}ab = a^{1+p} \rangle, \tag{2.1}$$

where $Q = \langle a \rangle$. Also, $Z(P) = \langle a^p \rangle$. It is easy to observe that P has $p^2 - 1$ elements of order p with the form $\{a^{ip}b^j\}$. Hence, P has $p+1$ subgroups of order p: one is $Z(P)$ and the others are non-central subgroups. For any $x \in \Omega$, P_x is a non-central subgroup of order p. Since $|N_P(P_x)| = p^2$, it follows that P_x fixes p points and so P_x fixes the p-block $B_{x,1}$ pointwise [7, Theorem 3.6] and is transitive on $B_{y,1}$ for any $y \notin B_{x,1}$. This completes the proof. ■

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Lemma 2.3 *Suppose $n \geq 3$. Let $x, y \in \Omega$, and for $1 \leq i \leq n-2$, let $H = P_x \cap P_{\{B_{y,i+1}\}}$. If H induces a transitive action on $B_{y,i+1,i}$, then it also must induce a transitive action on $B_{y,i+1,i-1}$.*

Proof Suppose that H induces a transitive action on $B_{y,i+1,i}$. Then $x \notin B_{y,i+1}$. Let $N = P_{\{B_{x,i+1}\}} \cap P_{\{B_{y,i+1}\}}$. Then from fact (1) before Lemma 2.2 and Dedekind's law [3, Chapter I, Theorem 2.12], we have that $N = (P_x Q_{i+1}) \cap P_{\{B_{y,i+1}\}} = (P_x \cap P_{\{B_{y,i+1}\}})Q_{i+1} = HQ_{i+1}$. Let $K = N_{(B_{y,i+1,i-1})}$ and $\bar{N} = N/K$. Then $Q_{i-1} \leq K$ and $\bar{N}$ induces a faithful action on $B_{y,i+1,i-1}$ of degree p^2. Moreover, Q_{i+1} is transitive on $B_{y,i+1}$ which implies $\bar{N}$ is transitive on $B_{y,i+1,i-1}$.

We proceed by contradiction and assume that H does not induce a transitive action on $B_{y,i+1,i-1}$. Since H is transitive on $B_{y,i+1,i}$, it must have p orbits on $B_{y,i+1,i-1}$. Each orbit then must intersect each p^i-block in $B_{y,i+1,i}$ in exactly one p^{i-1}-block. Thus, the stabilizer of a p^{i-1}-block fixes $B_{y,i+1,i}$ pointwise, thereby, fixing $B_{y,i+1,i-1}$ pointwise. This implies that H has order p in its action on $B_{y,i+1,i-1}$ so that $\bar{H} := HK/K$ has order at most p. Since H induces a transitive action on $B_{y,i+1,i}$, $|\bar{H}| \neq 1$ and so $|\bar{H}| = p$. Therefore, $|\bar{N}| \leq |\bar{H}|\,|\overline{Q_{i+1}}| \leq p^3$. First we assume that $|\bar{N}| = p^2$. Then $\bar{N} = \overline{Q_{i+1}} \cong Z_{p^2}$, which forces $\bar{H} = \overline{Q_i}$, fixing each p^i-block setwise, contradicting the assumption that H induces a transitive action on $B_{y,i+1,i}$. Next, we assume that $|\bar{N}| = p^3$. Then $\bar{N}$ is precisely isomorphic to the group defined in (2.1). The central subgroup of order p is a subgroup of $\overline{Q_{i+1}}$ so that $\bar{H}$ is a non-central subgroup of order p. By Lemma 2.2, H fixes the p-blocks of $B_{y,i+1,i-1}$ setwise, also a contradiction. This completes the proof. ∎

Proof of Theorem 1.1 Fix a point x in Ω and let y be any point of Ω. If P_x fixes y, then the conclusion is true. Hence, we assume P_x does not fix y. Since the orbit of P_x containing y is a then a proper block, there exists an i, $1 \leq i < n$, such that P_x fixes $B_{y,i}$ setwise and induces a transitive action on $B_{y,i,i-1}$. The theorem follows trivially for $i = 1$. If $i \geq 2$, then by Lemma 2.3, P_x induces a transitive action on $B_{y,i,i-2}$. By repeating this process, we obtain that P_x induces a transitive action on $B_{y,i,0} = B_{y,i}$. In other words, $B_{y,i}$ is a suborbit of P relative to x containing y. This finishes the proof.

The following result is a corollary of Theorem 1.1, and will be used in the proof of Theorem 1.2. Before stating it, we first recall a well-known result in permutation group theory. For any transitive permutation group G on Γ, the set $\mathrm{Fix}(G_v)$ of fixed points of G_v ($v \in G_v$) has the form $\{v^g \mid g \in N_G(G_v)\}$ and is a block of G (see [7, Theorem 7.4]).

Corollary 2.4 *Let $x \in \Omega$. Suppose that for some i satisfying $1 \leq i \leq n-1$, the induced action of P_x on $B_{x,i+1,i-1}$ is non-trivial. Then for any $y \notin B_{x,i}$, $P_x \cap P_{\{B_{y,i}\}}$ is transitive on $B_{y,i}$. In particular, if P contains a regular abelian subgroup R such that for any $z \in \Omega$, the induced action of $R_{\{B_{z,i+1}\}}$ on $B_{z,i+1,i-1}$ is isomorphic to $Z_p \times Z_p$, then we have the same conclusion.*

Proof Suppose that for some $x \in \Omega$ and some i with $1 \leq i \leq n-1$, the action of P_x on $B_{x,i+1,i-1}$ is non-trivial. Then we consider the induced action of P on the

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p^{i-1}-block system B_{i-1}. Let $K = P_{(B_{i-1})}$ be the kernel of the action and $\bar{P} = P/K$. Now $(\bar{P})_{B_{x,i-1}} = \overline{P_x}$. Therefore, by the arguments before the corollary, the set of fixed points of $\overline{P_x}$ on B_{i-1} is a block of $\bar{P}$. By the hypotheses, the induced action of P_x on $B_{x,i+1,i-1}$ is non-trivial. Hence, $\overline{P_x}$ fixes the p^{i-1}-blocks in $B_{x,i}$ setwise—there are p of them—and moves any other p^{i-1}-block in $B_{y,i}$ for any $y \notin B_{x,i}$. By Theorem 1.1, we know that $B_{y,i}$ is contained in the suborbit of P relative to x containing y. In particular, $P_x \cap P_{\{B_{y,i}\}}$ is transitive on $B_{y,i}$.

Suppose that P has a regular abelian subgroup R such that for any $z \in \Omega$, the induced action of $R_{\{B_{z,i+1}\}}$ on $B_{z,i+1,i-1}$ is isomorphic to $Z_p \times Z_p$. Let $H = P_{\{B_{x,i+1}\}}$. Note that $P_x \leq H$. Let $\bar{H}$ be the permutation group on $B_{x,i+1,i-1}$ induced by H. Then $\bar{H}$ contains two regular subgroups isomorphic to Z_{p^2} and $Z_p \times Z_p$, respectively. Therefore, the induced action of P_x on $B_{x,i+1,i-1}$ is non-trivial, so we have the same conclusion as the preceding paragraph. ∎

3 Proof of Theorem 1.2

Before proving Theorem 1.2, we recall some basic concepts and facts. Given a group G and a permutation group H on a set $\Delta = \{1, 2, \cdots n\}$, then the *wreath product* $G \wr H$ is the group defined by

$$G \wr H = \{(g_1, g_2, \ldots, g_n; h) \mid g_i \in G, h \in H\},$$

where

$$(g_1, g_2, \ldots, g_n; h)(g'_1, g'_2, \ldots, g'_n; h') = (g_1 g'_{1^{h^{-1}}}, g_2 g'_{2^{h^{-1}}}, \ldots, g_n g'_{n^{h^{-1}}}; hh').$$

Furthermore, if G is a permutation group on Γ, then $G \wr H$ can be defined to be a faithful permutation group on $\Gamma \times \Delta$ by $(i, j)^{(g_1,g_2,\ldots,g_n;h)} = (i^{g_j}, j^h)$. Similarly, we may define the wreath product of more than two groups. Suppose that for $1 \leq i \leq k$, G_i is a transitive permutation group on a set Δ_i. By [3, Chapter I, Theorem 15.4], we know that the permutation group $G_1 \wr G_2 \wr \cdots \wr G_k := ((\cdots (G_1 \wr G_2) \wr G_3) \wr \cdots \wr G_k)$on $((\cdots (\Delta_1 \times \Delta_2) \times \Delta_3) \times \cdots \times \Delta_k)$is associative if we identify $((\cdots (\Delta_1 \times \Delta_2) \times \cdots \times \Delta_k)$ with $\Delta_1 \times \Delta_2 \cdots \times \Delta_k$ and in this case, $G_1 \wr G_2 \wr \cdots \wr G_k$ is a transitive group on $\Delta_1 \times \Delta_2 \cdots \times \Delta_k$. Moreover, from [3, Chapter I, Theorem 15.9] we obtain the following result.

Lemma 3.1 *Let $1 = R_0 \leq R_2 \cdots \leq R_k = R$ be a subnormal series with factor groups $H_i = R_i/R_{i-1}$ $(i = 1, 2, \ldots, k)$. Suppose that for each i, H_i is a regular permutation group on Δ_i. Then R is isomorphic to a regular subgroup of $H_1 \wr H_2 \wr \cdots \wr H_k$ acting on $\Delta_1 \times \Delta_2 \cdots \times \Delta_k$.*

Given two digraphs Y and Z, the *wreath product* $Y \wr Z$ of Y and Z is defined to be the digraph with vertex set $V(Y \wr Z) = V(Y) \times V(Z)$ and arc set

$$A(Y \wr Z) = \left\{ \left((y_1, z_1), (y_2, z_2)\right) \;\middle|\; z_1 = z_2 \text{ and } (y_1, y_2) \in A(Y), \text{ or } (z_1, z_2) \in A(Z) \right\}.$$

Given digraphs $Y_1, Y_2, \ldots, Y_k$, we denote $\left((Y_1 \wr Y_2) \cdots \wr Y_k\right)$ by $Y_1 \wr Y_2 \cdots \wr Y_k$. It is easy to show that $\mathrm{Aut}(Y_1) \wr \mathrm{Aut}(Y_2) \cdots \wr \mathrm{Aut}(Y_k) \leq \mathrm{Aut}(Y_1 \wr Y_2 \cdots \wr Y_k)$.

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Proof of Theorem 1.2 The proof is divided into three steps. We prepare the foundation in (1), prove the necessity in (2), and prove the sufficiency in (3).

(1) Let Q and Q_i, $0 \le i \le n$, be as in Section 2, and let $X = \mathrm{Cay}(Q; S)$ be a Cayley digraph on Q. Suppose that $\mathrm{WS}(X) = (i_0, i_1, \ldots, i_k)$. By Sylow's Theorem, we may assume P is a Sylow p-subgroup of $\mathrm{Aut}(X)$ containing Q. Let $\Omega := V(X) = Q$ and let B_i be as in Section 2. Then by Lemma 2.1, B_i is the unique p^i-block system for P. We say the p^i-block system B_i is *wreathed* if $i = 0$, $i = n$, or $1 \le i \le n-1$ and the subdigraph induced by X on the union of any two p^i-blocks U and W has the property that if there is one arc from U to W, then there are p^{2i} arcs from U to W. It follows from the definition of the wreathed sequence $\mathrm{WS}(X)$ that $\{B_{i_0}, B_{i_1}, \ldots, B_{i_k}\}$ is the set of all the wreathed block systems of X.

For any $x \in V(X)$ and $0 \le j < i \le n$, we define the digraph $X_{x,i,j}$ to have vertex set $B_{x,i,j}$, where for any two p^j-blocks B and B' in $B_{x,i,j}$, $(B, B') \in A(X_{x,i,j})$ if and only if there exist some arcs in X from B to B'. Clearly, the digraph $X_{x,i,j}$ is a Cayley digraph on the group $Q_i/Q_j \cong Z_{p^{i-j}}$. By the analysis in the last paragraph, we know that $X \cong U_1 \wr U_2 \wr \cdots \wr U_k$, where for any $1 \le \ell \le k$, $U_\ell = X_{x,i_\ell,i_{\ell-1}}$ is a Cayley digraph on $Z_{p^{i_\ell - i_{\ell-1}}}$, and it cannot be represented as a wreath product with k terms in any other way.

(2) Suppose T is a regular abelian subgroup of $\mathrm{Aut}(X)$. Up to group isomorphism, we assume $T \le P$. For any i satisfying $0 \le i \le n$, let T_i be the subgroup of T fixing each p^i-block setwise. Then $|T_i| = p^i$ by fact (2) before Lemma 2.2. Assume that for some ℓ, $\frac{T_{i_\ell}}{T_{i_{\ell-1}}}$ is not cyclic. Then there exists an i between $i_{\ell-1}+1$ and $i_\ell - 1$ such that the induced action of T_{i+1}/T_{i-1} on $B_{x,i+1,i-1}$ is isomorphic to $Z_p \times Z_p$. By Corollary 2.4, P_x is transitive on $B_{y,i}$ for each $y \notin B_{x,i}$. This forces $B_{y,i}$ to be wreathed, a contradiction (see (1)). Hence, $\frac{T_{i_\ell}}{T_{i_{\ell-1}}}$ is cyclic for any ℓ satisfying $1 \le \ell \le k$, implying that $T \in \mathbf{C}(X)$.

(3) Suppose $R \in \mathbf{C}(X)$. Then R has a series of subgroups, say $1 = R_0 \le R_1 \cdots \le R_k = R$, where for any $1 \le \ell \le k$, $|R_\ell| = p^{i_\ell}$ and $\frac{R_\ell}{R_{\ell-1}} \cong Z_{p^{i_\ell - i_{\ell-1}}}$. Then by Lemma 3.1, R is isomorphic to a regular subgroup of $Z_{p^{i_1}} \wr Z_{p^{i_2-i_1}} \cdots \wr Z_{p^{i_k - i_{k-1}}} \le \mathrm{Aut}(U_1) \wr \mathrm{Aut}(U_2) \cdots \wr \mathrm{Aut}(U_k) \le \mathrm{Aut}(U_1 \wr U_2 \cdots \wr U_k) = \mathrm{Aut}(X)$.

Remark 3.2 Suppose that $Y = \mathrm{Cay}(R; S)$ is a Cayley digraph on an abelian group R of order p^n for an odd prime p. In order to examine whether or not it is isomorphic to a Cayley digraph on a cyclic group, by Theorem 1.2 we may carry out the following steps.

(1) First pick up all the minimal elements, under inclusion, of the set of all the nontrivial cyclic subgroups R_1 of R such that $S - R_1$ is the union of some right cosets of R_1 in R. If there exist no such subgroups R_1, then by Theorem 1.2 the digraph Y cannot be a Cayley digraph on a cyclic group.

(2) For each R_1 above, find all the minimal elements of the set of all the subgroups R_2 such that R_2/R_1 is cyclic and $S - R_2$ is the union of some right cosets of R_2 in R. If there exists no such subgroup R_2, then by Theorem 1.2 again the digraph Y cannot be a Cayley digraph on a cyclic group.

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(3) Repeating the above process, if we eventually get a series of the above subgroups of R: $1 = R_0 \leq R_2 \leq \cdots \leq R_k = R$, then Y is isomorphic to a Cayley digraph X on a cyclic group. Suppose that for $0 \leq \ell \leq n$, $|R_\ell| = p^{i_\ell}$. Then $\mathrm{WS}(X) = (i_0, i_1, \ldots, i_n)$. In this case, by Theorem 1.2 again, Y is also isomorphic to a Cayley graph on a group in $\mathbf{C}(X)$.

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