RELATIVE DISTANCE AND QUASI-CONFORMAL MAPPINGS

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1. Introduction. M. A. Lavrentiev[1] made use of a relative distance function to establish some important results concerning the correspondence between the frontiers under a conformal mapping of a simply connected domain onto the unit circle. The purpose of this note is to show that some of these results are valid for the boundary correspondences induced by the more general class of quasi-conformal mappings.

2. Relative distance. Let Ω be a simply connected domain and let F be the frontier of Ω. For any two points a, b of Ω, we shall define the relative distance, $\rho(a, b)$, to be the greatest lower bound of the lengths of all polygonal paths joining a to b and lying in Ω. It is clear that $\rho(a, b)$ is a metric in Ω and that $\rho(a, b) \geq |a-b|$ with equality if a, b lie in some convex subdomain of Ω. For points $a \in \Omega$, $c \in F$ we define $\rho(a, c)$ to be the infimum of $\varliminf \rho(a, z_n)$ on all sequences $\{z_n\}$ tending to c.

3. Quasi-conformal mappings.[2] A one-to-one function, $w = f(z) = u(x, y) + iv(x, y)$, whose partial derivatives u_x, u_y, v_x, v_y exist and are continuous in a domain Ω, and whose Jacobian $J[f(z)] = u_x v_y - u_y v_x$ is greater than zero, will be called quasi-conformal in Ω, if there exists a constant $K \geq 1$ such that $E + G \leq 2K\sqrt{EG - F^2}$, where $E = u_x^2 + v_x^2$, $F = u_x u_y + v_x v_y$, and $G = u_y^2 + v_y^2$. This condition is equivalent to the uniform boundedness of the dilatation-quotient $Q[f(z)] = \dfrac{E + G + [(E+G)^2 - 4(EG - F^2)]^{1/2}}{2[EG - F^2]^{1/2}}$ throughout the domain. Whenever $K = 1$, the function $f(z)$ is analytic and since we have required it to be one-to-one, a conformal mapping. We shall consider here only quasi-conformal mappings with uniformly bounded dilatation-quotient and shall restrict the use of the letter K to be that uniform bound, $K \geq 1$.

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[1] See Lavrentiev [4] and [5]. The numbers in brackets refer to the bibliography.

[2] For an excellent expository article concerning quasi-conformal mappings and pseudo-analytic functions see Tôki and Shibata [5].

4. **Rate of approach to the boundary.** We first shall prove the following theorem concerning rate of approach to the boundary.

THEOREM 1. *Let Ω be a bounded simply-connected domain, $z_0 \in \Omega$, and let $w=f(z)$ be a quasi-conformal mapping of Ω onto $|w|<1$ such that $f(z_0)=0$. Then given any $t>0$, for every $z \in \Omega$ such that $\rho(z_0, z)>t$ it follows that $1-|f(z)|<\frac{B}{t^{1/K}}$ where B is a constant independent of t.*

Proof. Let $z=g(\xi)$ be the conformal mapping of $|\xi|<1$ onto Ω such that $g(0)=z_0$, $g'(0)>0$. For any curve γ joining $\xi=\xi^*$, $|\xi^*|<1$ to $\xi=0$

(1) $$\rho(z_0, z^*) \leq \int_{\substack{\xi=0 \\ \gamma}}^{\xi=\xi^*} |g'(\xi)| \cdot |d\xi|$$

where $z^*=g(\xi^*)$. Thus it follows that

(2) $$\rho(z_0, z^*)[1-|\xi^*|] \leq \int_{\substack{\xi=0 \\ \gamma}}^{\xi=\xi^*} [1-|\xi^*|]\,|g'(\xi)| \cdot |d\xi|.$$

Since Ω is a bounded domain, by an application of Cauchy's estimate to the derivative of $g(\xi)$ we have $|g'(\xi)| \leq \frac{M}{1-|\xi|}$, where $M>|z|$ for all $z \in \Omega$. From this it follows that

(3) $$\rho(z_0, z^*)[1-|\xi^*|] \leq \int_{\substack{\xi=0 \\ \gamma}}^{\xi=\xi^*} M\frac{1-|\xi^*|}{1-|\xi|}\,|d\xi|$$

and if γ is taken as the ray from $\xi=0$ to $\xi=\xi^*$, we obtain

(4) $$\rho(z_0, z^*)[1-|\xi^*|] < \int_{\substack{\xi=0 \\ \gamma}}^{\xi=\xi^*} M|d\xi| = M|\xi^*| < M.$$

The composition function $w=f(g(\xi))=F(\xi)$ is a quasi-conformal mapping of $|\xi|<1$ onto $|w|<1$, with $F(0)=0$. A well-known property[3)] for such quasi-conformal mappings asserts that

(5) $$|1-|w|| < C|1-|\xi||^{1/K}$$

where $w=F(\xi)$, and C is a constant. Combining this inequality with inequality (4) above, we obtain

(6) $$\rho(z_0, z^*)[1-|w^*|]^K \leq C^K[1-|\xi^*|]\rho(z_0, z^*) < C^K M = B^K$$

[3)] See Ahlfors [1; p. 11].

or

$$1 - |w^*| < \frac{B}{\rho(z_0, z^*)^{1/K}}. \tag{7}$$

Thus we have

$$1 - |f(z)| < \frac{B}{t^{1/K}} \quad \text{if} \quad \rho(z_0, z) > t > 0. \tag{8}$$

5. Two lemmas. Let $w = f(z)$ be of class C' in a domain Ω and let $f'_\theta(z)$ denote the directional derivative of f at z in the direction θ. Let $|f'_m(z)| = \max\limits_{0 \leq \theta \leq 2\pi} |f'_\theta(z)|$. For the sake of completeness we shall prove the following lemma.

LEMMA 1. *Let $w = f(z)$ be of class C' in a domain Ω, then $|f'_m(z)| = \max\limits_{0 \leq \theta \leq 2\pi} |f'_\theta(z)|$ is a continuous function of z in Ω.*

Proof. The directional derivative $f'_\theta(z) = f_x \cos\theta + f_y \sin\theta$ is a continuous function of the pair z, θ. Given any $\varepsilon > 0$, there exists a $\delta = \delta(z_0, \varepsilon) > 0$ such that for $|z - z_0| < \delta$ we have $|f'_\theta(z_0) - f'_\theta(z)| < \varepsilon$ for all θ and since $|f'_m(z_0)| \geq |f'_\theta(z_0)|$ for all θ, we have $|f'_m(z_0)| \geq |f'_\theta(z_0)| > |f'_\theta(z)| - \varepsilon$ and thus for all $|z - z_0| < \delta$,

$$|f'_m(z_0)| > |f'_m(z)| - \varepsilon. \tag{9}$$

Also $|f'_m(z)| \geq |f'_\theta(z)|$ for all θ and thus for $|z - z_0| < \delta$, $|f'_m(z)| \geq |f'_\theta(z)| > |f'_\theta(z_0)| - \varepsilon$ and so we obtain

$$|f'_m(z)| > |f'_m(z_0)| - \varepsilon \tag{10}$$

combining (9) and (10), for $|z - z_0| < \delta$ we have the desired result, $||f'_m(z)| - |f'_m(z_0)|| < \varepsilon$.

There is the following inequality[4] relating $|f'_\theta(z)|$ and $Q[f(z)]$ and $J[f(z)]$ for quasi-conformal mapping, $|f'_\theta(z)|^2 \leq Q[f(z)] \cdot J[f(z)]$ which will be used in the proof of the following lemma:

LEMMA 2. *If $w = f(z)$ is a quasi-conformal mapping of a domain Ω of area π onto the unit circle $|w| < 1$, then*

$$\iint\limits_{\Omega} |f'_m(z)|\, dxdy \leq (K+1)\pi. \tag{11}$$

[4] See Tôki and Shibata [7; p. 147].

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Proof. Decompose Ω into two sets Ω_1 and Ω_2, by letting $\Omega_1 = \{z \mid z \in \Omega, |f'_m(z)| > 1\}$ and $\Omega_2 = \Omega - \Omega_1$. It follows then that

$$B = \iint_{\Omega} |f'_m(z)|\, dx\, dy = \iint_{\Omega_1} |f'_m(z)|\, dx dy + \iint_{\Omega_2} |f'_m(z)|\, dx\, dy \tag{12}$$

$$\leq \iint_{\Omega_1} |f'_m(z)|^2 dx\, dy + \iint_{\Omega_2} dx dy \tag{13}$$

$$\leqq \iint_{\Omega_1} Q[f(z)] \cdot J[f(z)] dx dy + \pi \tag{14}$$

$$\leq K \iint_{\Omega_1} J[f(z)] dx dy + \pi = (K+1)\pi. \tag{15}$$

6. The boundary correspondence.

Let Ω be a simply connected domain with at least two boundary points. A quasi-conformal mapping $w = f(z)$ of Ω onto $|w| < 1$ can be factored (as was done in the proof of theorem one above), into a conformal mapping $\xi = h(z)$ of Ω onto $|\xi| < 1$ followed by a quasi-conformal mapping $w = F(\xi)$ of $|\xi| < 1$ onto $|w| < 1$ which sends $\xi = 0$ into $w = 0$, i.e. $f(z) = F(h(z))$. The function $w = F(\xi)$ can be extended[5] to give a homeomorphism between $|\xi| \leq 1$ and $|w| \leq 1$. It is well known[6] that under the conformal mapping $\xi = h(z)$ of Ω onto $|\xi| < 1$, one can make correspond to each accessible boundary point z_1 of Ω, a definite point ξ_1 on $|\xi| = 1$, with the following properties. As z converges to the point z_1, along an arbitrary defining curve l_1, then the corresponding point ξ tends to a point ξ_1, and two distinct accessible boundary points of Ω correspond to distinct points on $|\xi| = 1$. The set of those points on $|\xi| = 1$ which corresponds to the set of all accessible boundary points is dense on $|\xi| = 1$.

Let Ω be a simply connected domain and let $z_0 \in \Omega$. The sets of points $\{z \mid \rho(z_0, z) = t\}$ we shall refer to as *relative circles*. It is clear that for small t they are ordinary circles; but for values of t between $\underset{\xi \in F}{\text{g.l.b}}\, \rho(z_0, \xi)$ and $\underset{\xi \in F}{\text{l.u.b.}}\, \rho(z_0, \xi)$, the relative circle $\rho(z_0, z) = t$ consists of a countable[7] collection of Jordan arcs with end points on F. Because each relative circle possesses a continuously turning tangent, we can consider the orthogonal trajectories to the family of relative circles, which we shall call relative radii. It is convenient to introduce the element of arc length $d\sigma$ along relative circles and the element of

[5] See Tôki and Shibata [7; p. 150–151].
[6] See Koebe [3].
[7] i.e. finite or denumerably infinite.

arc length $d\tau$ along relative radii and the element of area $d\sigma d\tau$.

Let us consider now a simply connected domain Ω of area π and a fixed point $z_0 \in \Omega$. Let $w = f(z)$ be a quasi-conformal mapping of Ω onto $|w| < 1$ such that $f(z_0) = 0$. Letting $\sup_{\xi \in F} \rho(z_0, \xi) = T \leq +\infty$ and $C(t)$ equal the length of the relative circle $\rho(z_0, z) = t$, we consider the integral

(16) $$I = \int_0^T \left(\int_0^{C(t)} |f_\theta'(z)|\, d\sigma \right) d\tau$$

where the directional derivative $f_\theta'(z)$ is taken along the tangent to the relative circle $\rho(z_0, z) = t$. Denoting by $L(t)$ the length of the image of the relative circle $\rho(z_0, z) = t$ under $w = f(z)$, i.e. $L(t) = \int_0^{C(t)} |f_\theta'(z)|\, d\sigma$, since $I \leq (K+1)\pi$, we obtain

(17) $$\int_0^T L(t)\, d\tau \leq (K+1)\pi.$$

For any number t_1, $1 < t_1 < T$, there exists by the integral law of the mean a number t^*, $1 < t^* < t_1$, such that

(18) $$\int_1^{t_1} L(t)\, d\tau = L(t^*)[t_1 - 1], \text{ and thus}$$

(19) $$L(t^*) \leq \frac{(K+1)\pi}{t_1 - 1}.$$

Let us denote by $\{l_j^*\}$ the countable set of image curves of the curves comprising the relative circle $\rho(z_0, z) = t^*$. Each l_j^* is a Jordan arc in $|w| \leq 1$ with end points α^*, β^*, on $|w| = 1$. If we denote by d_j^* the length of the[8] arc of $|w| = 1$ with end points α^* and β^*, it follows then that

(20) $$\sum_1^\infty d_n^* \leq \pi L(t^*) \leq \frac{(K+1)\pi^2}{t_1 - 1}.$$

If we denote by $\{l_n'\}$ the set of arcs on $|w| = 1$ determined by the image of $\rho(z_0, z) = t_1$, we find that since $1 < t^* < t_1$, we have

(21) $$\sum_1^\infty d_n' \leq \sum_1^\infty d_n^* \leq \frac{(K+1)\pi^2}{t_1 - 1}.$$

If we denote by F_{t_1} the set of all points of F which belong to the boundary of the set $\{z \mid z \in \Omega,\ \rho(z_0, z) < t_1\}$, then it follows from the above inequality that

[8] The domain bounded by d_j^* and l_j^* does not contain $w = 0$.

under a quasi-conformal mapping of Ω onto $|w|<1$ the measure of the set of points on $|w|=1$, which correspond to the set F_{t_1} is greater than $2\pi-\dfrac{(K+1)\pi^2}{t_1-1}$. By passing to limit as $t_1 \to T = \underset{\xi\in F}{\text{l.u.b.}}\ \rho(z_0,\ \xi)$ we obtain the following theorem:

THEOREM 2. *Under a quasi-conformal mapping of a simply connected domain Ω of area π onto the unit circle $|w|<1$, there corresponds to the set of those boundary points of Ω accessible by a finite path, a set of measure 2π on the unit circle $|w|=1$.*

7. Remark.[9] It has been brought to the attention of the author that Theorem 2 can be easily improved by the application of a theorem of Beurling so that in Theorem 2 the exceptional set of measure zero can be replaced by an exceptional set of outer logarithmic capacity zero. To be precise, we have

THEOREM 3. *Under a quasi-conformal mapping $w=f(z)$ of a simply connected domain Ω of finite area onto the unit circle $|w|<1$, every point of $|w|=1$, except for at most a set of outer logarithmic capacity zero on $|w|=1$, corresponds to a boundary point of Ω accessible by a path of finite length.*

Proof. Map Ω by $\xi=h(z)$ conformally onto $|\xi|<1$ and denote by $z=\psi(\xi)$ the inverse of $\xi=h(z)$. Then, we have $w=f(z)=F[h(z)]$, where $w=F(\xi)$ is a quasi-conformal mapping of $|\xi|<1$ onto $|w|<1$ such that $F(0)=0$. By Beurling's theorem,[10] we have $\int_0^1 |\psi'(\rho e^{i\theta})|\,d\rho<\infty$ for every point $\xi=e^{i\theta}$, except for a set E_ξ of outer logarithmic capacity zero on $|\xi|=1$. It is known that $w=F(\xi)$ can be extended to a topological mapping of $|\xi|\leq 1$ onto $|w|\leq 1$ and that the image E_w of E_ξ by $w=F(\xi)$ is also of outer logarithmic capacity zero by Mori's theorem.[11] Hence every $w=e^{i\varphi}$ not belonging to E_w corresponds to a boundary point of Ω accessible by a path of finite length under $w=f(z)$.

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9) The author is grateful to Professor Kiyoshi Noshiro who suggested the following extension of Theorem 2.

10) See Beurling [2].

11) See Mori [6].

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