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Notes on Antireciprocal Points.

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Definition. If x, y, z and ξ, η, ζ be the perpendiculars on the sides BC, CA, AB of the $\triangle$ ABC from points O and O′, then O and O′ are antireciprocal points if $x\xi : y\eta : z\zeta :: \tan A : \tan B : \tan C$.

I. CONSTRUCTION TO FIND A POINT ANTIRECIPROCAL TO O (Fig. 4).

Draw through O a line MN antiparallel to BC. Draw OY perpendicular to AC, and OZ perpendicular to AB. Draw lines parallel to AB and AC, and at distances from them respectively equal to YN and MZ, and let them cut in P. Join AP. Find a similar line BQ, and let AP and BQ cut in O′. O′ is the required point. Let the perpendiculars from O be x, y, z and those from O′, ξ, η, ζ.

$$\begin{aligned}\eta : \zeta &= MZ : YN\\ &= OZ/\tan OMZ : OY/\tan ONY\\ &= \tan B/OY : \tan C/OZ\\ &= \tan B/y : \tan C/z\end{aligned}$$

$$\therefore \quad y\eta : z\zeta = \tan B : \tan C.$$

Similarly $\qquad x\xi : z\zeta = \tan A : \tan C$

$$\therefore \quad x\xi : y\eta : z\zeta = \tan A : \tan B : \tan C.$$

$\therefore$ O′ is the antireciprocal of O.

II. Construction to find a point antireciprocal to itself (Fig. 5).

Draw AD perpendicular to BC, and produce it to meet the semicircle described on BC as diameter in E. Draw lines parallel to AB and CA, and at distances from them respectively equal to BE and CE. Let them cut in P. Join AP. Find a similar line BQ. Let AP and BQ cut in O. O is the required point.

$$\begin{aligned} y^2 : z^2 = \mathrm{CE}^2 : \mathrm{BE}^2 &= \mathrm{CD} : \mathrm{BD} \\ &= \mathrm{CD}/\mathrm{AD} : \mathrm{BD}/\mathrm{AD} \\ &= \mathrm{AD}/\mathrm{BD} : \mathrm{AD}/\mathrm{CD} \\ &= \tan \mathrm{B} : \tan \mathrm{C}. \end{aligned}$$

Similarly $\qquad x^2 : z^2 = \tan \mathrm{A} : \tan \mathrm{C}$

$$\therefore \quad x^2 : y^2 : z^2 = \tan \mathrm{A} : \tan \mathrm{B} : \tan \mathrm{C}$$

or $\quad x\xi : y\eta : z\zeta = \tan \mathrm{A} : \tan \mathrm{B} : \tan \mathrm{C}$

where $\quad x = \xi,\ y = \eta,\ z = \zeta.$

$\therefore$ O is the required point.

$$x : y : z = \sqrt{\tan \mathrm{A}} : \sqrt{\tan \mathrm{B}} : \sqrt{\tan \mathrm{C}}$$

so that the point whose trilinear coordinates are

$$\sqrt{\tan \mathrm{A}}, \quad \sqrt{\tan \mathrm{B}}, \quad \sqrt{\tan \mathrm{C}}$$

is the antireciprocal of itself.

The three triangles formed by drawing through this point lines antiparallel to the sides of the $\triangle$ ABC will be equal. The intercepts cut off on the sides by these antiparallels are proportional to

$$\sqrt{\cot \mathrm{A}}, \quad \sqrt{\cot \mathrm{B}}, \quad \sqrt{\cot \mathrm{C}}.$$

There are four such points, one internal and three external. Their coordinates are given by $\sqrt{\tan \mathrm{A}}$, $\pm \sqrt{\tan \mathrm{B}}$, $\pm \sqrt{\tan \mathrm{C}}$.

Definition. The antireciprocal of a line is the locus of the antireciprocals of all points in the line.

1. The antireciprocal of a line is a conic passing through the vertices of the triangle.

Let $lx+my+nz=0$ be the equation of the line expressed in trilinear coordinates. Then since $x\xi : y\eta : z\zeta = \tan A : \tan B : \tan C$, the antireciprocal to $lx+my+nz=0$ is

$$\frac{l\tan A}{\xi}+\frac{m\tan B}{\eta}+\frac{n\tan C}{\zeta}=0,$$

or $$\eta\zeta l\tan A+\zeta\xi m\tan B+\xi\eta n\tan C=0. \qquad (1)$$

This represents a conic passing through the vertices of the triangle.

2. The antireciprocal of the circumcircle is the axis of homology of the triangle and its orthic triangle.

$\eta\zeta\sin A+\zeta\xi\sin B+\xi\eta\sin C=0$ is the equation of the circumcircle. Its antireciprocal is $x\cos A+y\cos B+z\cos C=0$, and this is the equation of the said axis of homology.

3. The antireciprocal of a line through a vertex consists of another line through that same vertex, and the opposite side of the triangle.

Let $lx+my=0$ be a line through C.

The equation of its antireciprocal is

$$\eta\zeta l\tan A+\zeta\xi m\tan B=0,$$

or $$\zeta=0(AB),\ \eta l\tan A+\xi m\tan B=0 \text{ (a line through C)}.$$

The vertex C is the antireciprocal of any point in the opposite side AB.

4. The antireciprocal of a tangent to the circumcircle is a conic touching the line $\Sigma x\cos A=0$ at the antireciprocal of the point of contact of the tangent and the circumcircle.

The condition that $\Sigma lx=0$ touch the circumcircle $\Sigma\eta\zeta\sin A=0$ is that $\Sigma\sqrt{l\sin A}=0$, and the condition that $\Sigma x\cos A=0$ touch the conic $\Sigma\eta\zeta l\tan A=0$ is that $\Sigma\sqrt{l\tan A.\cos A}=0$, the same condition.

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If x, y, z are the coordinates of the point in which $\Sigma x\cos A = 0$ touches $\Sigma\eta\zeta l\tan A = 0$,

$$\begin{aligned} x:y:z \quad = \quad & l\tan A(\ \ l\sin A - m\sin B - n\sin C) \\ & :m\tan B(-l\sin A + m\sin B - n\sin C) \\ & :\ n\tan C\,(-l\sin A - m\sin B + n\sin C)\,; \end{aligned}$$

if ξ, η, ζ are the coordinates of the point in which the line $\Sigma lx = 0$ touches the circumcircle

$$\begin{aligned} \xi:\eta:\zeta \quad = \quad & 1/(\ \ l\sin A - m\sin B - n\sin C) \\ & :1/(-l\sin A + m\sin B - n\sin C) \\ & :1/(-l\sin A - m\sin B + n\sin C)\,; \end{aligned}$$

and these two are antireciprocals since $x\xi : y\eta : z\zeta = \tan A : \tan B : \tan C$. The equation of the tangent at C to the conic $\Sigma\eta\zeta l\tan A = 0$ is $\eta l\tan A + \xi m\tan B = 0$; or $\eta l\tan A + \xi m\tan B = 0$ is a tangent to a series of conics $\eta\zeta l\tan A + \zeta\xi m\tan B + \xi\eta n\tan C = 0$, where n has different values. But $\zeta(\eta l\tan A + \xi m\tan B) = 0$ is the antireciprocal of $lx + my = 0$, and the antireciprocal of the conic is $lx + my + nz = 0$. These lines are concurrent in a point in AB, no matter what n may be. Hence, if a number of lines are concurrent in a point in a side of a triangle, their antireciprocals have a common tangent at the opposite vertex, namely that part, passing through the vertex, of the antireciprocal of the line joining the point of concurrence of the lines to the opposite vertex.

FIGURE 6.

5. The antireciprocal of the tangent at a vertex to the circumcircle consists partly of the line joining the vertex to the point of concurrence of the opposite side and the line $\Sigma x\cos A = 0$.

The tangent at the vertex C is $x\sin B + y\sin A = 0$. The antireciprocal of this line is $\xi\cos A + \eta\cos B = 0$ and $\zeta = 0$ (CH_3 and BA). Now the lines $\xi\cos A + \eta\cos B = 0$, $\zeta = 0$, $\Sigma x\cos A = 0(H_2H_3)$ are concurrent. The side DE of the orthic triangle $(x\cos A + y\cos B - z\cos C = 0)$ also passes through H_3.

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6. If three lines passing through the vertices be concurrent, then their antireciprocals must also be concurrent, for they pass through the antireciprocal of the point of concurrence of the first three.

For example, the lines joining the vertices to the opposite excentres pass through the incentre, and the lines joining the vertices to the antireciprocals of the excentres pass through the antireciprocal of the incentre. The lines joining the vertices to the opposite exsymmedian points pass through the insymmedian point, and the lines joining the vertices to the antireciprocals of the exsymmedian points pass through the antireciprocal of the insymmedian point, *i.e.*, the orthocentre. The three antireciprocals of the exsymmedian points can thus be easily found, for if the points H_1, H_2, H_3 be found, the intersection of BH_2 and AD gives L_1, the antireciprocal of K_1, the exsymmedian point opposite A. The triangles $L_1L_2L_3$ and ABC form the antireciprocal of triangle $K_1K_2K_3$. The two triangles ABC, $L_1L_2L_3$ have a common centre of homology O, and a common axis of homology H_2H_3.

7. If O, L_1, L_2, L_3 be the points $(\sqrt{\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C})$ found as in Construction II., then

$$\text{the line } OCL_3 \text{ is } \frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} = 0,$$

$$L_1CL_2 \text{ is } \frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} = 0, \text{ etc.}$$

The line H_2H_3 is $\dfrac{x}{\sqrt{\tan A}} + \dfrac{y}{\sqrt{\tan B}} + \dfrac{z}{\sqrt{\tan C}} = 0.$

Each of the lines AOL_1 AL_2L_3 is with the side opposite the vertex through which the line passes, its own antireciprocal. The antireciprocal of

$$\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$$

is

$$\zeta\eta\sqrt{\tan A} + \xi\zeta\sqrt{\tan B} + \xi\eta\sqrt{\tan C} = 0.$$

The following table shows the connection between the lines.

Triangles.	Centre of homology.	Axis of homology.	Tangents at vertices to antireciprocal of axis of homology.
ABC, $L_1L_2L_3$	O_1, $\sqrt{\tan A}$ $\sqrt{\tan B}$ $\sqrt{\tan C}$	H_2H_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$	CH_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} = 0$ BH_2, $\frac{x}{\sqrt{\tan A}} + \frac{z}{\sqrt{\tan C}} = 0$ AH_1, $\frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$
ABC, OL_1L_2	L_3, $\sqrt{\tan A}$ $\sqrt{\tan B}$ $-\sqrt{\tan C}$	DH_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} - \frac{z}{\sqrt{\tan C}} = 0$	AD, $\frac{y}{\sqrt{\tan B}} - \frac{z}{\sqrt{\tan C}} = 0$ BE, $\frac{x}{\sqrt{\tan A}} - \frac{z}{\sqrt{\tan C}} = 0$ CH_3, $\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} = 0$
ABC, OL_1L_3	L_2, $\sqrt{\tan A}$ $-\sqrt{\tan B}$ $\sqrt{\tan C}$	FH_2, $\frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$	AD, $\frac{y}{\sqrt{\tan B}} - \frac{z}{\sqrt{\tan C}} = 0$ BH_2, $\frac{x}{\sqrt{\tan A}} + \frac{z}{\sqrt{\tan C}} = 0$ CF, . $\frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} = 0$
ABC, OL_2L_3	L_1, $-\sqrt{\tan A}$ $\sqrt{\tan B}$ $\sqrt{\tan C}$	EH_1, $-\frac{x}{\sqrt{\tan A}} + \frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$	AH_1, $\frac{y}{\sqrt{\tan B}} + \frac{z}{\sqrt{\tan C}} = 0$ BE, $\frac{x}{\sqrt{\tan A}} - \frac{z}{\sqrt{\tan C}} = 0$ CF, $\frac{x}{\sqrt{\tan A}} - \frac{y}{\sqrt{\tan B}} = 0$

There are four conics, corresponding to the four points, and each of the six lines passing through the vertices is a tangent to two of the conics. Thus each conic touches the other three conics at different vertices.

The line $px+qy+rz=0$ will touch the conic

$$\eta\zeta l\tan A+\xi\zeta m\tan B+\xi\eta n\tan C=0$$

if $$\sqrt{pl\tan A}\pm\sqrt{qm\tan B}\pm\sqrt{rn\tan C}=0.$$

Hence the line $lx+my+nz=0$ will touch its own antireciprocal if $l\sqrt{\tan A}\pm m\sqrt{\tan B}\pm n\sqrt{\tan C}=0$, that is if the line $lx+my+nz=0$ pass through one of the four points

$$(\sqrt{\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C}).$$

Since $$x:y:z=\frac{\tan A}{\xi}:\frac{\tan B}{\eta}:\frac{\tan C}{\zeta}$$

$$\therefore\quad x\xi\tan B-y\eta\tan A=0,\ \text{and}\ x\xi\tan C-z\zeta\tan A=0.$$

Hence (x, y, z) is the point of intersection of the polars of (ξ, η, ζ) with respect to two degenerate conics,

$$x^2\tan B-y^2\tan A=0,\ \ x^2\tan C-z^2\tan A=0.$$

Since a line corresponds to a conic, and to a point corresponds the intersection of its polars with respect to two fixed conics, this quadric transformation is a Beltrami one, for a discussion of the difference between which and the Hirst transformation see Mr Charles Tweedie's paper read before the Royal Society of Edinburgh on 15th July 1901.

The conics $x^2\tan B-y^2\tan A=0$, etc., break up into the lines

$$\frac{x}{\sqrt{\tan A}}-\frac{y}{\sqrt{\tan B}}=0,\quad \frac{x}{\sqrt{\tan A}}+\frac{y}{\sqrt{\tan B}}=0,\ \text{etc.},$$

or the lines joining the vertices to the points

$$(\sqrt{\tan A},\ \pm\sqrt{\tan B},\ \pm\sqrt{\tan C}).$$

8. The conic $\eta\zeta l\tan A+\xi\zeta m\tan B+\xi\eta n\tan C=0$ will be a rectangular hyperbola, if

$$l\sin A+m\sin B+n\sin C=0.$$

If the line $lx+my+nz=0$ pass through the insymmedian point (sinA, sinB, sinC), then the condition for a rectangular hyperbola is fulfilled. Hence the antireciprocals of all lines passing through the insymmedian point are rectangular hyperbolas. This is otherwise seen; for if the line pass through the insymmedian point, its antireciprocal must pass through the orthocentre, and is therefore a rectangular hyperbola. In particular, the antireciprocal of the line joining the orthocentre and insymmedian point is the rectangular hyperbola passing through the vertices and these two points. Since five points on it are known, it can easily be drawn by Pascal's theorem. (Fig. 7.)

The coordinates of its centre are

$$\frac{\sin(B-C)}{\cos A},\quad \frac{\sin(C-A)}{\cos B},\quad \frac{\sin(A-B)}{\cos C}.$$

This point lies on the nine-point circle.

The equation of this rectangular hyperbola is

$$\eta\zeta\sin 2A\sin(B-C)+\xi\zeta\sin 2B\sin(C-A)+\xi\eta\sin 2C\sin(A-B)=0.$$

The line joining the orthocentre and insymmedian point is

$$x\cos^2 A\sin(B-C)+y\cos^2 B\sin(C-A)+z\cos^2 C\sin(A-B)=0.$$