BULL. AUSTRAL. MATH. SOC. 32G15, 30C60
VOL. 36 (1987) 187-196.

CONFORMALLY NATURAL AHLFORS-WEILL SECTIONS AND BERS' REPRODUCING FORMULAS

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We differentiate certain refined Ahlfors-Weill local sections of the Bers projections. This yields reproducing formulas for holomorphic functions — which are then shown to be naturally related to Bers' important and well-known reproducing formulas.

Introduction

At the heart of Teichmüller theory lies the problem of realizing the Teichmüller space $T(G)$ of an arbitrary Fuchsian group G as an open domain in a complex Banach space B via the 'Bers embedding'. The chief problem is to establish that the Bers projection Φ from proper Beltrami differentials into the complex Banach space B of bounded holomorphic quadratic forms is a holomorphic (split-) submersion onto its image. This image, which is the Teichmüller space of G, is therefore an open domain in B -- thus providing $T(G)$ with a natural complex structure.

Bers [3] achieved the proof of the submersivity of Φ using crucially some reproducing formulas for the functions in B (Theorem B below). Recently, (in Earle and Nag [5]), we have given a very direct

Received 4 September 1986. This work owes much to the inspiration of Professor Clifford Earle. It is also a pleasant duty to thank the Mathematical Sciences Research Institute, Berkeley, for their warm hospitality during the preparation of this and the joint work [5].

proof of this submersivity by exhibiting explicit local holomorphic right inverses for Φ generalizing the Ahlfors-Weill sections (see Theorem A here). The purpose of this note is to differentiate the equation implied by Theorem A in order to obtain a reproducing formula for functions in the Banach space B. The good thing is that we can transform in a natural fashion to show that this apparently new reproducing formula is but a version of Bers' Theorem B. The two separate methods of establishing the submersivity of the Bers projection thereby turn out to be intimately related.

In the special case of the standard and classical Ahlfors-Weill sections our method becomes an exploitation of the linearity of the section map. It was brought to our attention that Gardiner [6] had already done this special case of our calculations. We remark that such utilization of linearity for the section map is possibly only in the standard case (namely when the separating quasicircle is actually a circle or straight line).

As a spin-off, we would like to draw attention to equation (8) -- which provides a derivative formula for any Bers projection Φ at a general point. There is strong reason to expect that this, and several other formulas below, will have important applications besides those shown in this article.

1. Preliminaries

We set the stage by introducing the well-known chief characters of Teichmüller theory. As usual, let C denote an oriented quasicircle on the Riemann sphere $\hat{\mathbb{C}}$ separating the two complementary Jordan domains D_1 and D_2. We make the harmless requirement that C pass through ∞. Let $\lambda_j(z)|dz|$ denote the Poincaré metric on D_j, $(j = 1,2)$. When C is $\mathbb{R} \cup \{\infty\}$ we will say that we are in the standard case and write $D_1 = U$ (upper half-plane) and $D_2 = L$. In this case, $\lambda_1(z) = \lambda_2(z) = (2\,\mathrm{Im}\,z)^{-1}$.

Recall the Banach spaces $L^\infty(D_1)$ (the essentially bounded measurable functions) and

$$B_2(D_2) = \left\{\phi \in \mathrm{Hol}(D_2) : \|\phi\| = \sup_{D_2}(\lambda_2(z)^{-2}|\phi(z)|) < \infty\right\}.$$

https://doi.org/10.1017/S0004972700026460 Published online by Cambridge University Press

In any Banach space L we agree to write L_r for the open 0-centered ball of radius r. The Bers projection $\Phi = \Phi_C\colon L^\infty(D_1)_1 \longrightarrow B_2(D_2)$ is defined as follows. For $\mu \in L^\infty(D_1)_1$ we solve the Beltrami equation on $\hat{\mathbb{C}}$ with coefficient μ on D_1 and 0 on D_2 (call the solution w^μ -- usually normalized to fix $0, 1, \infty$) and then $\Phi(\mu)$ is the Schwarzian derivative of $w^\mu\big|_{D_2}$ (w^μ restricted to D_2).

When G is a quasi-Fuchsian group, ($G = \{1\}$ is the case described above), operating on D_1 and D_2 we have the usual Banach subspaces $L^\infty(D_1,G) \subset L^\infty(D_1)$ and $B_2(D_2,G) \subset B_2(D_2)$ consisting of (respectively) the G-invariant $(-1,1)$ forms and G-invariant $(2,0)$ forms. The nice thing is that Φ_C restricted to $L^\infty(D_1,G)_1$ maps to $B_2(D_2,G)$ and we call this restriction of Φ_C also a Bers projection. As general references for this material one may consult Bers [4] or the books [1] and [7].

Notation. We often denote $\dfrac{\partial}{\partial z}$ by ∂ and $\dfrac{\partial}{\partial \bar{z}}$ by $\bar{\partial}$. The complex dilatation (Beltrami coefficient) of a map ω is denoted by $\mu(w) = \dfrac{\bar{\partial} w}{\partial w}$.

Recall from Earle and Nag [5] that associated with any Jordan domain D on $\hat{\mathbb{C}}$ there is a special conformally natural reflection $\lambda = j(D)$ across the boundary ∂D. In case ∂D is a K-quasicircle, this λ is a $C_1(K)$-quasiconformal reflection (with the further property that it is $C_2(K)$-Lipschitz in the Euclidean metric). If G is any group of Möbius transformations operating on D then λ commutes with every member of G.

Now let G be any quasi-Fuchsian group keeping the quasidisks D_1 and D_2 invariant.

THEOREM A. (Earle and Nag [5]). *Around* $\phi = 0$ *in* $B_2(D_2,G)$ *there is an* $\varepsilon > 0$ *ball on which the 'Ahlfors map'*

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(1) $$\alpha : B_2(D_2, G)_\varepsilon \longrightarrow L^\infty(D_1, G)_1$$

given by

(2) $$\alpha(\phi) = \frac{\frac{1}{2}(\phi \circ \lambda)(id - \lambda)^2\, \bar{\partial}\lambda}{1 + \frac{1}{2}(\phi \circ \lambda)(id - \lambda)^2 \partial\lambda} \quad \text{on } D_1$$

is a holomorphic local right inverse to the Bers projection

$$\Phi : L^\infty(D_1, G)_1 \longrightarrow B_2(D_2, G) \; .$$

Here λ *is the conformally natural quasiconformal reflection ('quasi-reflection')* $j(D_2)$ *associated to the quasidisk* D_2 .

Remark. Theorem A holds as stated without having to assume that ∞ is on the bounding quasicircle $C = \partial D_1 = \partial D_2$. This follows from strong Möbius-equivariance properties of formula (2). The value of ε depends only on K where C was a K-quasicircle.

In the standard case $\lambda = j(U)$ is conjugation (j) . Then ε can be chosen equal to 2 . The Beltrami coefficient simplifies to $\alpha(\phi) = -2(\operatorname{Im} z)^2 \phi(\bar{z})$ on U . The section α is thus linearly dependent on ϕ in the standard case (since $\partial j \equiv 0$) . In this connection note also the penultimate remark of this paper.

In Earle and Nag [5] we actually identified explicitly the quasi-conformal homeomorphism $w^{\alpha(\phi)}$ on $D_1 \cup D_2$. On D_2 of course this is an univalent holomorphic map with Schwarzian derivative precisely ϕ . On D_1, $\mu(w^{\alpha(\phi)}) = \alpha(\phi)$. The actual representation of $w^{\alpha(\phi)}$ is as follows.

Let v_1 and v_2 be two linearly independent solutions of $2v'' + \phi v = 0$ on D_2 (with normalization $v_1'v_2 - v_2'v_1 = 1$). Then

(2)* $$w^{\alpha(\phi)} = \begin{cases} v_1/v_2 & \text{on } D_2 \\ \dfrac{v_1 \circ \lambda + (id - \lambda)(v_1' \circ \lambda)}{v_2 \circ \lambda + (id - \lambda)(v_2' \circ \lambda)} & \text{on } D_1 \, . \end{cases}$$

(By following $w^{\alpha(\phi)}$ with a Möbius transformation we may assume that $w^{\alpha(\phi)}$ fixes any three preassigned points -- as may be required for normalization.)

THEOREM B. *Let* C, D_1, D_2 *be as above and let* h *be any uniformly Lipschitz quasiconformal reflection across* C. *Then every* $\psi \in B_2(D_2)$ *satisfies the reproducing formula*

$$\psi(\zeta) = -\frac{3}{\pi}\iint_{D_1} \frac{(z-h(z))^2\bar{\partial}h(z)\psi(h(z))}{(z-\zeta)^4}\,dxdy \,, \tag{3}$$

any $\zeta \in D_2$. *For a proof, see Bers [3 or 4], or [7].*

To connect the two theorems above we need the derivative map $d_\theta\Phi$ at any $\theta \in L^\infty(D_1)_1$. Firstly it is standard (see Bers [4]) that for any C the derivative at 0 of Φ_C is:

$$d_0\Phi_C(\nu)(z) = -\frac{6}{\pi}\iint_{D_1} \frac{\nu(\zeta)d\xi\,d\eta}{(\zeta - z)^4}\,, \qquad z \in D_2 \,. \tag{4}$$

To calculate $d_\theta\Phi_C$ at any $\theta \in L^\infty(D_1)_1$ we relate Φ_C to $\Phi_{C'}$, where $C' = w^\theta(C)$. This is easy to do using right-translation by θ, that is $R^\theta: L^\infty(w^\theta(D_1))_1 \longrightarrow L^\infty(D_1)_1$ defined by

$$R^\theta(\nu) = \text{Beltrami coefficient of } (w^\nu \circ w^\theta) \,.$$

To find the precise relation we set up the usual isometric isomorphism:

$$(w^\theta)^*: B_2(w^\theta(D_2)) \longrightarrow B_2(D_2) \,, \tag{5}$$

as the map $(w^\theta)^*(\phi) = (\phi \circ w^\theta)(\frac{dw^\theta}{dz})^2$, at $z \in D_2$. Then the rule for the Schwarzian derivative of a composite map leads to the formula

$$\Phi_C \circ R^\theta - (w^\theta)^* \circ \Phi_{w^\theta(C)} = \text{the constant map } (\nu \longmapsto \Phi_C(\theta)) \,. \tag{6}$$

Differentiating (6) at 0 leads to

$$d_\theta\Phi_C = (w^\theta)^* \circ d_0\Phi_{w^\theta(C)} \circ (d_0R^\theta)^{-1} \,. \tag{7}$$

Now one verifies

$$d_0R^\theta(\mu) = (\mu \circ w^\theta)\,\frac{\overline{(\partial w^\theta)}}{\partial w^\theta}\,(1-|\theta|^2) \,.$$

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Hence we calculate from (7), (using (4) to express $d_0 \Phi_{w^\theta(C)}$).

$$(8) \qquad d_\theta \Phi_C(\nu)(z) = -\frac{6}{\pi}\left(\frac{dw^\theta}{dz}\right)^2 \iint_{\zeta \in D_1} \frac{\nu(\zeta)(\partial w^\theta(\zeta))^2}{(w^\theta(\zeta)-w^\theta(z))^4} d\xi d\eta \ , \quad z \in D_2 \ .$$

The integral in (8) is absolutely convergent since the integral in (4) is so.

2. The main connection

Our basic program is to differentiate the equation implied by Theorem A:

$$(A) \qquad \Phi_C \circ \alpha = 1 \quad (\text{on } \|\phi\| < \varepsilon \text{ in } B_2(D_2))$$

to find a reproducing formula (equation (11) below), which we will then transform in a natural way so that it becomes a (somewhat generalized) version of Theorem B. We note at the very outset that in the standard case, $(C = \mathbb{R} \cup \{\infty\})$, α is <u>linear</u> so that differentiation of (A) becomes trivial -- and the subsequent formulas take a much simpler form -- which we will point out in passing.

A direct calculation from formula (2) for α gives:

$$(9) \qquad d_\phi \alpha(\rho) = \frac{\frac{1}{2}\rho(\lambda(z))\bar{\partial}\lambda(z)(z-\lambda(z))^2}{(1+\frac{1}{2}\phi(\lambda(z))\partial\lambda(z)(z-\lambda(z))^2)^2} ,$$

at any $z \in D_1$. Let us introduce the notation

$$(10) \qquad P(z) = \frac{1}{2}\, \bar{\partial}\lambda(z)(z-\lambda(z))^2$$

$$Q(z) = \frac{1}{2}\, \partial\lambda(z)(z-\lambda(z))^2$$

and write $\nu_\rho = d_\phi \alpha(\rho)$, for arbitrary $\rho \in B_2(D_2)$.

Let $\theta = \alpha(\phi)$. Then differentiating (A) at ϕ using formulas (8) and (9) gives the reproducing formula

$$(11) \quad \rho(\zeta) = -\frac{3}{\pi}\left(\frac{dw^\theta}{dz}(\zeta)\right)^2 \iint_{D_1} \frac{\left\{\frac{2P(z)}{(1+\phi(\lambda(z))Q(z))^2}(\partial w^\theta(z))^2\right\}\rho(\lambda(z))}{(w^\theta(z) - w(\zeta))^4} dxdy$$

for any $\zeta \in D_2$ and any $\rho \in B_2(D_2)$.

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Note that in the standard case this formula becomes

$$(11)^* \qquad \rho(\zeta) = -\frac{3}{\pi}\left(\frac{dw^\theta}{dz}(\zeta)\right)^2 \iint_U \frac{(z-\bar{z})^2(\partial w^\theta(z))^2\ \rho(\bar{z})}{(w^\theta(z) - w^\theta(\zeta))^4}\,dxdy$$

any $\zeta \in L$, any $\rho \in B_2(L)$. As remarked in the Introduction, this special case (11)* had earlier arisen in Gardiner [6, p.478].

Our concern here is to show that (11) and (11)* are actually cases of Theorem B applied with a new D_1, say $\tilde{D}_1 = w^\theta(D_1)$, a new quasicircle $\tilde{C} = w^\theta(C)$, and a quasiconformal reflection h across $\tilde{C}$ defined by

$$(12) \qquad h = w^\theta \circ \lambda \circ (w^\theta)^{-1} .$$

Although λ was a uniformly Lipschitz quasiconformal reflection across C this h need not be uniformly Lipschitz of course, nevertheless the Bers reproducing formula (3) still will be shown to hold and in fact (3) becomes equivalent to (11). (The relevant integrals continue to be absolutely convergent because the integrals in (8) and (11) are so.)

We transform variables in Bers' formula (3) using $w^\theta : D_1 \longrightarrow \tilde{D}_1$. Applying Theorem B for $\psi \in B_2(w^\theta(D_2))$, and using $(w^\theta)^*$ of equation (5) to set $\rho = (w^\theta)^*(\psi) \in B_2(D_2)$, we obtain:

$$(3)^* \qquad \rho(\zeta) = -\frac{3}{\pi}\left(\frac{dw^\theta}{dz}(\zeta)\right)^2 \iint_{D_1} \Big\{[w^\theta(z)-w^\theta(\lambda(z))]^2 \bar{\partial} h(w^\theta(z)) \cdot$$

$$\cdot \left(\frac{dw^\theta}{dz}(\lambda(z))\right)^{-2} J(w^\theta)\Big\} \frac{\rho(\lambda(z))}{(w^\theta(z)-w^\theta(\zeta))^4}\,dxdy$$

for any $\zeta \in D_2$ and any $\rho \in B_2(D_2)$ (here $J(w^\theta)$ is the Jacobian of w^θ on D_1). Our claim is that (3)* is true when h is given by (12) and in fact it is exactly the same reproducing formula as (11). So we must verify that the two expressions in curly brackets appearing in the numerators of the integrands on the right hand sides of equations (3)* and (11) are identically equal. We indicate this calculation (in which some crucial cancellations occur) below:

(13) $$\bar{\partial}h(w^{\theta}(z)) = \left(\frac{dw^{\theta}}{dz}(\lambda(z))\right) \frac{[(\bar{\partial}\lambda)(\partial w^{\theta}) - (\partial\lambda)(\bar{\partial}w^{\theta})](\text{at } z)}{J(w^{\theta})(\text{at } z)}$$

Again from the explicit formula (2)* for w^{θ} we derive, for $z \in D_1$:

(14) $$\frac{[w^{\theta}(z)-w^{\theta}(\lambda(z))]^2}{[z-\lambda(z)]^2} = \frac{1}{v_2^2(\lambda(z))[v_2(\lambda(z))+(z-\lambda(z))v_2'(\lambda(z))]^2}$$

(15) $$\partial w^{\theta}(z) = \frac{1 + \phi(\lambda(z))Q(z)}{[v_2(\lambda(z)) + (z-\lambda(z))v_2'(\lambda(z))]^2}$$

(16) $$\frac{dw^{\theta}}{dz}(\zeta) = \frac{1}{v_2^2(\zeta)}, \text{ for any } \zeta \in D_2 .$$

Now recall that $\bar{\partial}w^{\theta} = \theta\cdot\partial w^{\theta}$, and $\theta = \alpha(\phi)$ is given by formula (2) as $\theta = \frac{(\phi\circ\lambda)P}{1+(\phi\circ\lambda)Q}$. Substituting (13), (14), (15), (16) and this expression for θ (applied to (13)) in the curly brackets of the numerators of (3)* and (11) we find that the curly brackets are identically equal, as claimed, and so we are through.

In the standard case, with $\lambda = j$ (conjugation) the calculation reduces to verifying the following:

(17) $$\frac{[w^{\theta}(z) - w^{\theta}(\bar{z})]^2}{[z-\bar{z}]^2} = \left(\frac{dw^{\theta}}{dz}(\bar{z})\right)\left(\frac{\partial w^{\theta}}{\partial z}(z)\right)$$

for any $z \in U$ and any $\theta = -2(\text{Im } z)^2\phi(\bar{z})$, $\phi \in B_2(L)_2$. (This special case is, of course, much easier to verify.) Let us call $X = \left\{\mu \in L^{\infty}(U): \mu(z) = 2(\text{Im } z)^2\phi(\bar{z}), \text{ for some } \phi \in B_2(L)\right\}$. X is a Banach subspace of $L^{\infty}(U)$ and its unit ball X_1 then corresponds to $\phi \in B_2(L)_2$.

Remark. We wish to highlight (17) as the crucial relation satisfied by the quasiconformal maps w^{θ} when $\theta \in X_1$. Indeed, in retrospect, we note that assuming (17) and reversing our calculation using the Bers formula (3) (with the quasireflections h as in equation (12)) we could have deduced that $d_{\theta}(\Phi|_{X_1})$ is independent of $\theta \in X_1$. By an obvious 'linearity lemma' this would imply that $\Phi|_{X_1}$ was indeed (the restriction of) a linear map. But Bers' Theorem B actually does hold for these 'conjugated' reflections h in the classical case. This was pointed out to me by C.J. Earle. Indeed the conjugating quasiconformal homeomorphism w^{θ} given by (2)* may be verified to be uniformly Lipschitz in the standard case. Consequently $\Phi|_{X_1}$ must be linear -- and this map could then be identified with its derivative at 0 (which we know very explicitly using formula (4)). This line of argument thus gives <u>an independent proof of the standard case (at least) of Theorem A</u> above.

We remark that such an exploitation of <u>linearity</u> for general Ahlfors-Weill sections is possible <u>only</u> in the case of the standard domains. In point of fact, the quasicircle C admits an anti-holomorphic quasi-reflection λ if and only if C is a circle (or straight line). The proof of this is easy.

Remark. If we do not insist on G-equivariance for our formulas (2) and (2)* relating to Theorem A then it is clear that λ need not be chosen to be the very special quasireflection $j(D_2)$. In fact, in Ahlfors' approximation arguments in [1, pp.128-133] one needs to keep all the quasireflections λ_n for the approximating domains D_n uniformly C_1-quasiconformal and uniformly C_2-Lipschitz. By standard normality properties of quasiconformal maps one may choose a subsequence on which the λ_n converge to a quasiconformal and uniformly Lipschitz reflection λ across ∂D_2. Then formula (2)* would indeed provide a map which extends to a quasiconformal homeomorphism of $\hat{\mathbb{C}}$ and whose Beltrami coefficient on D_1 is given by (2).

Therefore formula (A) of Section 2 would still be valid with these new quasireflections λ. Now, for our purposes in the present article it was not important to have G-equivariance, so all our calculations are valid with those less canonical λ. In particular, Bers' formula (3) holds with h given by (12) and these λ's.

<u>Added in Proof.</u>

Research supported in part by N.S.F. Grant No. 8120890 (USA) and by the National Board for Higher Mathematics (India).

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