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MOORE-PENROSE INVERSION IN COMPLEX CONTRACTED INVERSE SEMIGROUP ALGEBRAS

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Abstract

It is shown that every element of the complex contracted semigroup algebra of an inverse semigroup $S = S^0$ has a Moore-Penrose inverse, with respect to the natural involution, if and only if S is locally finite. In particular, every element of a complex group algebra has such an inverse if and only if the group is locally finite.

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Let A be an algebra over the complex field $\mathbb{C}$ with an involution *. By a *Moore-Penrose inverse* of an element $a \in A$ (relative to $*$) we mean an element $a^\dagger \in A$ such that

$$aa^\dagger a = a, \qquad a^\dagger aa^\dagger = a^\dagger,$$
$$(aa^\dagger)^* = aa^\dagger, \quad (a^\dagger a)^* = a^\dagger a.$$

It is readily demonstrated that there is at most one such $a^\dagger$ for a given a (see [6]); and clearly $0 = 0^\dagger$. The fundamental case is that in which A is the algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices over $\mathbb{C}$ and * is hermitian conjugation. In [6, Theorem 1], Penrose proved that $a^\dagger$ exists for each $a \in M_n(\mathbb{C})$. An equivalent result, using a different definition of $a^\dagger$, had been obtained earlier by Moore [2]. The purpose of this note is to extend Penrose's theorem to a wider class of complex algebras.

The semigroup algebra of a semigroup S over $\mathbb{C}$ is designated by $\mathbb{C}[S]$. Adopting the convention in [1], we write '$S = S^0$' to indicate that a semigroup S has a zero and at least one other element. Given such a semigroup S, we denote the set of nonzero elements of S by $\hat{S}$ and the *contracted* semigroup algebra of S over $\mathbb{C}$ by $\mathbb{C}_0[S]$

[1, Section 5.2]. The elements of $\mathbb{C}_0[S]$ are regarded as the formal sums $\sum_{x \in \hat{S}} \alpha_x x$, where in each case at most finitely many of the (complex) coefficients α_x are nonzero. Multiplication in $\mathbb{C}_0[S]$ is induced by that in S in the obvious way, the zero of S being identified with the zero of the algebra. For a typical element $a = \sum_{x \in \hat{S}} \alpha_x x$ we define $\operatorname{supp}(a)$, the *support* of a, to be $\{x \in \hat{S} : \alpha_x \neq 0\}$. Thus $\operatorname{supp}(a)$ is a finite subset of $\hat{S}$ and is empty if and only if $a = 0$. (Note that every semigroup algebra can be viewed as a contracted semigroup algebra; for if T is an arbitrary semigroup then $\mathbb{C}[T] = \mathbb{C}_0[T^+]$, where T^+ is obtained from T by adjoining a zero.)

A semigroup S is said to be *locally finite* if and only if every finite nonempty subset of S generates a finite subsemigroup of S. The following result, which is a special case of [5, Theorem 2], provides a necessary condition for a complex contracted semigroup algebra to be regular. (An algebra A is regular, in the sense of von Neumann, if and only if, for all $a \in A$ there exists $x \in A$ such that $axa = a$.)

LEMMA 1 (Okniński). *Let $S = S^0$ be a semigroup. If $\mathbb{C}_0[S]$ is regular then S is locally finite.*

We now confine our discussion to inverse semigroups. A semigroup S of this type has the defining property that to each $x \in S$ there corresponds a unique $x^{-1} \in S$ (the 'inverse' of x) such that $xx^{-1}x = x$ and $x^{-1}xx^{-1} = x^{-1}$. It can be shown that the idempotents of S necessarily commute and that the inversion ($x \mapsto x^{-1}$) is an involution on S [1, Theorem 1.17 and Lemma 1.18]. Now consider an inverse semigroup $S = S^0$. For each $\alpha \in \mathbb{C}$ denote the complex conjugate of α by $\overline{\alpha}$. Then the mapping $^* : \mathbb{C}_0[S] \to \mathbb{C}_0[S]$ defined by

$$\left(\sum_{x \in \hat{S}} \alpha_x x\right)^* := \sum_{x \in \hat{S}} \overline{\alpha_x} x^{-1} \qquad (\alpha_x \in \mathbb{C})$$

is readily seen to be an involution. We call this the *natural involution* on $\mathbb{C}_0[S]$. For the case in which S is the semigroup of $n \times n$ matrix units (that is,

$$S = \{e_{ij} : 1 \leq i, j \leq n\} \cup \{0\}, \qquad \text{with } e_{ij} e_{kl} = \delta_{jk} e_{il}),$$

$\mathbb{C}[S] = M_n(\mathbb{C})$ and * coincides with the hermitian conjugation.

A version of the next lemma, for a non-contracted complex inverse semigroup algebra, was used by the author to show that the algebra has no nonzero nil ideals [4, Lemma 2.3]. With minor adjustment, the proof applies also to the contracted case. The same result was obtained independently by Shehadah [8].

LEMMA 2 (Munn-Shehadah). *Let $S = S^0$ be an inverse semigroup and let * denote the natural involution on $\mathbb{C}_0[S]$. Then*

$$(\forall a \in \mathbb{C}_0[S]) \quad aa^* = 0 \quad \Rightarrow \quad a = 0.$$

Before proceeding to the main result we observe the following. Let S be an inverse semigroup, let T be a finite nonempty subset of S and let T^{-1} denote $\{x^{-1} : x \in T\}$. Then the inverse subsemigroup of S generated by T is the subsemigroup generated by $T \cup T^{-1}$. Thus S is locally finite if and only if every finite nonempty subset of S generates a finite *inverse* subsemigroup of S.

THEOREM 1. *Let $S = S^0$ be an inverse semigroup. Then every element of $\mathbb{C}_0[S]$ has a Moore-Penrose inverse, relative to the natural involution, if and only if S is locally finite.*

PROOF. Assume first that every element of $\mathbb{C}_0[S]$ has a Moore-Penrose inverse. Then, in particular, $\mathbb{C}_0[S]$ is regular and so, by Lemma 1, S is locally finite.

For the converse part, we adapt Penrose's argument in [6, Theorem 1]. Denote the natural involution on $\mathbb{C}_0[S]$ by *. We show first that, for all $a, b, c \in \mathbb{C}_0[S]$,

$$\text{(i)} \quad ba^*a = ca^*a \Rightarrow ba^* = ca^*, \qquad \text{(ii)} \quad baa^* = caa^* \Rightarrow ba = ca.$$

To see that (i) holds, suppose that $a, b, c \in \mathbb{C}_0[S]$ are such that $ba^*a = ca^*a$. Then $(ba^*-ca^*)(ba^*-ca^*)^* = (ba^*a-ca^*a)(b-c)^* = 0$ and so, by Lemma 2, $ba^* = ca^*$. Result (ii) follows by replacing a by a^* in (i).

Now assume that S is locally finite. Consider a nonzero element a of $\mathbb{C}_0[S]$. Let T $(= T^0)$ denote the inverse subsemigroup of S generated by $\operatorname{supp}(a) \cup \{0\}$. Then $aa^* \in \mathbb{C}_0[T]$. But $\mathbb{C}_0[T]$ is finite-dimensional. Hence, for some $k \geq 2$, there exist complex numbers $\lambda_1, \lambda_2, \dots, \lambda_k$, with $\lambda_i \neq 0$ for some $i < k$, such that $\sum_{i=1}^{k} \lambda_i (aa^*)^i = 0$. Applying * to both sides, we see that also $\sum_{i=1}^{k} \overline{\lambda_i}(aa^*)^i = 0$. From these equations, it follows readily that there exist *real* numbers $\mu_1, \mu_2, \dots, \mu_k$, with $\mu_i \neq 0$ for some $i < k$, such that

$$\mu_1(aa^*) + \mu_2(aa^*)^2 + \cdots + \mu_k(aa^*)^k = 0.$$

Let r be the smallest integer i for which $\mu_i \neq 0$. Define $x \in \mathbb{C}_0[S]$ by

$$x := -\mu_r^{-1}[\mu_{r+1}a^* + \mu_{r+2}a^*(aa^*) + \cdots + \mu_k a^*(aa^*)^{k-r-1}].$$

Clearly, $(ax)^* = ax$ and $(xa)^* = xa$. Further, it is easily verified that $ax(aa^*)^r = (aa^*)^r$ and so, by repeated applications of (i) and (ii), $axaa^* = aa^*$. Therefore $(axa-a)(axa-a)^* = (axaa^*-aa^*)x^*a^* - (axaa^*-aa^*) = 0$. Hence, by Lemma 2, $axa = a$. From this, we have that $xaxa = xa$; also $a^* = a^*x^*a^*$ and so $x = ya^*$ for some $y \in \mathbb{C}_0[S]$. Thus $xaya^*a = ya^*a$. Hence, by (i), $xaya^* = ya^*$; that is, $xax = x$. Consequently, x is the Moore-Penrose inverse of a in $\mathbb{C}_0[S]$. □

REMARKS. (1) Penrose's result on $M_n(\mathbb{C})$ is included as a special case of Theorem 1: take S to be the semigroup of $n \times n$ matrix units.

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(2) Note that $\mathbb{C}_0[S]$ need not have an identity element. However, for a finite inverse semigroup $T = T^0$, $\mathbb{C}_0[T]$ does have an identity. A formula expressing this element in terms of the idempotents of T has been obtained by Penrose (see [3, p. 11]).

We conclude with a discussion of equations of the form $ax = b$ in complex contracted inverse semigroup algebras. First, it is easy to verify (as in [6]) that, if a and b are elements of an arbitrary ring R and there exists $a' \in R$ such that $aa'a = a$, then the equation $ax = b$ is soluble in R if and only if $aa'b = b$, the solutions (where they exist) being precisely the elements of the form $a'b + r - a'ar$, where r ranges over R. In particular, these observations apply when R is a complex contracted inverse semigroup algebra and $a' = a^\dagger$.

Let $S = S^0$ be a semigroup. We define the (Euclidean) norm $\|a\|$ of $a \in \mathbb{C}_0[S]$ by the rule that, if $a = \sum_{x \in \hat{S}} \alpha_x x$, then

$$\|a\| := \left(\sum_{x \in \hat{S}} |\alpha_x|^2 \right)^{1/2}.$$

Following Penrose [7], for a and b in $\mathbb{C}_0[S]$ we say that $t \in \mathbb{C}_0[S]$ is a *best approximate solution* of the equation $ax = b$ if and only if, for all $u \in \mathbb{C}_0[S]$, (1) $\|at - b\| \leq \|au - b\|$ and (2) if $\|at - b\| = \|au - b\|$ then $\|t\| \leq \|u\|$.

Now suppose that $S = S^0$ is an inverse semigroup. Denote the set of all nonzero idempotents of S by $\hat{E}$. We say that S is *primitive* if and only if

$$(\forall e, f \in \hat{E}) \quad ef \neq 0 \quad \Rightarrow \quad e = f.$$

It can be shown [1, Section 6.5, Example 6] that S is primitive if and only if it is a 0-direct union of Brandt semigroups (that is, completely 0-simple inverse semigroups). Thus complex contracted semigroup algebras of primitive inverse semigroups include, as special cases, complex group algebras and full matrix algebras over $\mathbb{C}$.

The next theorem mirrors Penrose's result on best approximate solutions of matrix equations [7].

THEOREM 2. *Let $S = S^0$ be a primitive inverse semigroup, let a and b be elements of $\mathbb{C}_0[S]$ and assume that $a^\dagger$ exists. Then $a^\dagger b$ is the unique best approximate solution of $ax = b$ in $\mathbb{C}_0[S]$.*

REMARK. By Theorem 1, a sufficient condition for $a^\dagger$ to exist is that the (inverse) subsemigroup of S generated by supp (a) is finite.

PROOF. Let $x, y \in \hat{S}$ be such that $x^{-1}y \in \hat{E}$. Then, since $x^{-1}x \in \hat{E}$ and $(x^{-1}x)(x^{-1}y) \neq 0$ we have that $x^{-1}y = x^{-1}x$ and therefore that $x^{-1}yx^{-1} = x^{-1}$.

Similarly, $yx^{-1}y = y$. Hence $y = (x^{-1})^{-1} = x$. Thus we have that

(1) $$(\forall x, y \in \hat{S}) \quad x^{-1}y \in \hat{E} \Leftrightarrow x = y.$$

Next, we define a linear functional $\tau : \mathbb{C}_0[S] \to \mathbb{C}$ by the rule that

$$\tau\left(\sum_{x\in\hat{S}} \alpha_x x\right) := \sum_{e\in\hat{E}} \alpha_e \qquad (\alpha_x \in \mathbb{C}).$$

As before, denote the natural involution on $\mathbb{C}_0[S]$ by *. It follows readily from (1) that

(2) $$(\forall u \in \mathbb{C}_0[S]) \quad \tau(u^*u) = \|u\|^2.$$

Let $u \in \mathbb{C}_0[S]$ and write $c := au - aa^\dagger b, \quad d := aa^\dagger b - b$. Then

(3) $$c^*d = (u^* - (a^\dagger b)^*)a^*(aa^\dagger b - b) = 0,$$

since $a^*(aa^\dagger) = (aa^\dagger a)^* = a^*$. Thus, by (2) and (3),

$$\begin{aligned}\|au - b\|^2 = \|c + d\|^2 = \tau((c+d)^*(c+d)) &= \tau(c^*c + c^*d + (c^*d)^* + d^*d) \\ &= \tau(c^*c + d^*d) = \tau(c^*c) + \tau(d^*d) = \|c\|^2 + \|d\|^2.\end{aligned}$$

Hence $\|d\| \le \|au - b\|$; that is, $\|aa^\dagger b - b\| \le \|au - b\|$.

Suppose now that equality holds here. Then $\|c\| = 0$ and so $au = aa^\dagger b$. Thus $a^\dagger au = a^\dagger b$. Consequently, $u = e + f$, where $e := a^\dagger b$ and $f := u - a^\dagger au$. Now $(a^\dagger)^*(u - a^\dagger au) = 0$ and so $e^*f = 0$. Hence, by (2),

$$\|u\|^2 = \tau((e+f)^*(e+f)) = \tau(e^*e + f^*f) = \|e\|^2 + \|f\|^2.$$

Therefore $\|e\| \le \|u\|$; that is, $\|a^\dagger b\| \le \|u\|$.

Finally, suppose additionally that $\|a^\dagger b\| = \|u\|$. Then $\|f\| = 0$. Hence $f = 0$ and so $u = e = a^\dagger b$. Thus we have shown that $a^\dagger b$ is the unique best approximate solution of $ax = b$ in $\mathbb{C}_0[S]$. □

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References

[1] A. H. Clifford and G. B. Preston, *The algebraic theory of semigroups*, Math. Surveys of the Amer. Math. Soc. 7 (Providence R.I., 1961 (vol. I) and 1967 (vol. II)).
[2] E. H. Moore, 'On the reciprocal of the general algebraic matrix', *Bull. Amer. Math. Soc.* **26** (1920), 394–395.
[3] W. D. Munn, 'Matrix representations of semigroups', *Proc. Cambridge Philos. Soc.* **53** (1957), 5–12.
[4] ———, 'Semiprimitivity of inverse semigroup algebras', *Proc. Roy. Soc. Edinburgh, Sect. A* **93** (1982), 83–98.
[5] J. Okniński, 'On regular semigroup rings', *Proc. Roy. Soc. Edinburgh, Sect. A* **99** (1984), 145–151.
[6] R. Penrose, 'A generalized inverse for matrices', *Proc. Cambridge Philos. Soc.* **51** (1955), 406–413.
[7] ———, 'On best approximate solutions of linear matrix equations', *Proc. Cambridge Philos. Soc.* **52** (1956), 17–19.
[8] A. A. Shehadah, *Embedding theorems for semigroups with involution* (Ph.D. Thesis, Purdue University, Indiana, 1982).

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