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FIXED POINT THEOREM AND NONLINEAR ERGODIC THEOREM FOR NONEXPANSIVE SEMIGROUPS WITHOUT CONVEXITY

WATARU TAKAHASHI

ABSTRACT. We first prove a nonlinear ergodic theorem for nonexpansive semigroups without convexity in a Hilbert space. Further we prove a fixed point theorem for non-expansive semigroups without convexity which generalizes simultaneously fixed point theorems for left amenable semigroups and left reversible semigroups.

1. **Introduction.** Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a nonempty subset of H. A mapping $T\colon C \longrightarrow C$ is said to be *Lipschitzian* if there exists a nonnegative number k such that

$$\|Tx - Ty\| \leqq k\|x - y\| \text{ for every } x, y \in C,$$

and *nonexpansive* in the case $k = 1$. If S is a semitopological semigroup and $\mathcal{S} = \{T_s : s \in S\}$ is a continuous representation of S as Lipschitzian mappings of C into itself, it is called a *Lipschitzian semigroup* on C. A Lipschitzian semigroup $\mathcal{S} = \{T_s : s \in S\}$ with Lipschitzian constants k_s, $s \in S$ is called a *nonexpansive semigroup* if $k_s = 1$ for every $s \in S$. When C is closed and convex, there are many fixed point theorems and nonlinear ergodic theorems for nonlinear semigroups in a Hilbert space; for example, see [1–8, 10–12, 14–17]. Recently, Mizoguchi and Takahashi [10] proved a fixed point theorem which generalizes some results of them by introducing the notion of submean. And also Ishihara [7] proved a fixed point theorem for left reversible Lipschitzian semigroups without convexity.

In this paper, we first prove a nonlinear ergodic theorem for nonexpansive semigroups without convexity in a Hilbert space. This is a generalization of Rodé's result [14]. Further by the method of [10, 15–17], we prove a fixed point theorem without convexity which generalizes simultaneously fixed point theorems for left amenable semigroups and left reversible semigroups. This is a generalization of results of Lau [8], Takahashi [15] and Ishihara [7].

2. **Nonlinear ergodic theorem.** Throughout this paper, let S be a semitopological semigroup, *i.e.*, a semigroup with a Hausdorff topology such that for each $s \in S$ the mappings $t \longrightarrow t \cdot s$ and $t \longrightarrow s \cdot t$ of S into itself are continuous. Let $B(S)$ be the Banach

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space of all bounded real valued functions on S with supremum norm and let X be a subspace of $B(S)$ containing constants. Then, an element μ of X^* (the dual space of X) is called a *mean* on X if $\|\mu\| = \mu(1) = 1$. We know that $\mu \in X^*$ is a mean on X if and only if

$$\inf\{f(s) : s \in S\} \leqq \mu(f) \leqq \sup\{f(s) : s \in S\}$$

for every $f \in X$. Let μ be a mean on X and $f \in X$. Then, according to time and circumstances, we use $\mu_t\big(f(t)\big)$ instead of $\mu(f)$. For each $s \in S$ and $f \in B(S)$, we define elements $l_s f$ and $r_s f$ in $B(S)$ given by

$$(l_s f)(t) = f(st) \text{ and } (r_s f)(t) = f(ts)$$

for all $t \in S$. Let X be a subspace of $B(S)$ containing constants which is l_S-invariant (r_S-invariant), *i.e.*, $l_s(X) \subset X$ $\big(r_s(X) \subset X\big)$ for each $s \in S$. Then a mean μ on X is said to be *left invariant* (*right invariant*) if

$$\mu(f) = \mu(l_s f) \quad \big(\mu(f) = \mu(r_s f)\big)$$

for all $f \in X$ and $s \in S$. An *invariant mean* is a left and right invariant mean.

Let C be a nonempty subset of H. Then a family $\mathcal{S} = \{T_s : s \in S\}$ of mappings of C into itself is called a *Lipschitzian semigroup* on C if it satisfies the following:

(1) $T_{st}x = T_s T_t x$ for all $s, t \in S$ and $x \in C$;

(2) for each $x \in C$, the mapping $s \longrightarrow T_s x$ is continuous on S;

(3) for each $s \in S$, T_s is a Lipschitzian mapping of C into itself, *i.e.*, there is $k_s \geqq 0$ such that

$$\|T_s x - T_s y\| \leqq k_s \|x - y\|$$

for all $x, y \in C$. A Lipschitzian semigroup $\mathcal{S} = \{T_t : t \in S\}$ on C is said to be *nonexpansive* if $k_s = 1$ for every $s \in S$. For a Lipschitzian semigroup $\mathcal{S} = \{T_s : s \in S\}$ on C, we denote by $F(\mathcal{S})$ the set of common fixed points of $T_s, s \in S$.

Let $C(S)$ be the Banach space of all bounded continuous real-valued functions on S and let RUC(S) be the space of all bounded right uniformly continuous functions on S, *i.e.*, all $f \in C(S)$ such that the mapping $s \longrightarrow r_s f$ is continuous. Then RUC(S) is a closed subalgebra of $C(S)$ containing constants and invariant under l_S and r_S; see [9] for details. When $\mathcal{S} = \{T_s : s \in S\}$ is a nonexpansive semigroup on C such that $\{T_s x : s \in S\}$ is bounded for some $x \in C$, then we know from [8] that for each $u \in C$ and $v \in H$, the functions $f(t) = \|T_t u - v\|^2$ and $g(t) = \langle T_t u, v\rangle$ are in RUC(S). Let μ be a mean on RUC(S). Then since for each y in H, the real valued function $t \longrightarrow \langle T_t x, y\rangle$ is in RUC(S), we define the value $\mu_t\langle T_t x, y\rangle$ of μ at this function. By linearity of μ and of the inner product, this is linear in y; moreover, since

$$|\mu_t\langle T_t x, y\rangle| \leqq \|\mu\| \cdot \sup_t |\langle T_t x, y\rangle| \leqq (\sup_t \|T_t x\|) \cdot \|y\|,$$

it is continuous in y. So, by the Riesz theorem, there exists an $x_0 \in H$ such that

$$\mu_t\langle T_t x, y\rangle = \langle x_0, y\rangle \text{ for every } y \in H.$$

We write such an x_0 by $T_\mu x$. Before proving a nonlinear ergodic theorem for nonexpansive semigroups without convexity, we give a definition concerning means. Let $\{\mu_\alpha : \alpha \in A\}$ be a net of means on RUC(S). Then $\{\mu_\alpha : \alpha \in A\}$ is said to be *asymptotically invariant* if for each $f \in$ RUC(S) and $s \in S$,

$$\mu_\alpha(f) - \mu_\alpha(l_s f) \longrightarrow 0 \text{ and } \mu_\alpha(f) - \mu_\alpha(r_s f) \longrightarrow 0.$$

THEOREM 1. *Let C be a nonempty subset of a Hilbert space H and let S be a semitopological semigroup such that* RUC(S) *has an invariant mean. Let $\mathcal{S} = \{T_t : t \in S\}$ be a nonexpansive semigroup on C such that $\{T_t x : t \in S\}$ is bounded and $\sqcap_{s\in S}\overline{\mathrm{co}}\{T_{st}x : t \in S\} \subset C$ for some $x \in C$. Then, $F(\mathcal{S}) \neq \emptyset$. Further, for an asymptotically invariant net $\{\mu_\alpha : \alpha \in A\}$ of means on* RUC(S), *the net $T_{\mu_\alpha}x, \alpha \in A$ converges weakly to an element $x_0 \in F(\mathcal{S})$.*

PROOF. Let μ be an invariant mean on RUC(S). Then, we know that there exists an $x_0 \in H$ such that $\mu_t\langle T_t x, y\rangle = \langle x_0, y\rangle$ for every $y \in H$. For such an x_0, we can prove $x_0 \in \sqcap_{s\in S}\overline{\mathrm{co}}\{T_{st}x : t \in S\}$. If not, we have $x_0 \notin \overline{\mathrm{co}}\{T_{st}x : t \in S\}$ for some $s \in S$. By the separation theorem, there exists a y_0 in H such that

$$\langle x_0, y_0\rangle < \inf\big\{\langle z, y_0\rangle : z \in \overline{\mathrm{co}}\{T_{st}x : t \in S\}\big\}.$$

So we have

$$\begin{aligned}\inf_{t\in S}\langle T_{st}x, y_0\rangle &\leqq \mu_t\langle T_{st}x, y_0\rangle = \mu_t\langle T_t x, y_0\rangle \\ &= \langle x_0, y_0\rangle \\ &< \inf\big\{\langle z, y_0\rangle : z \in \overline{\mathrm{co}}\{T_{st}x : t \in S\}\big\} \\ &\leqq \inf_{t\in S}\langle T_{st}x, y_0\rangle.\end{aligned}$$

This is a contradiction. Therefore we have

$$x_0 \in \sqcap_{s\in S}\overline{\mathrm{co}}\{T_{st}x : t \in S\}$$

and hence $x_0 \in C$. On the other hand, since, for each y in H, the real valued function $t \longrightarrow \|T_t x - y\|^2$ is in RUC(S), we can also define the value $\mu_t\|T_t x - y\|^2$ of μ at this function. Let

$$r = \inf\{\mu_t\|T_t x - y\|^2 : y \in H\}$$

and

$$M = \{z \in H : \mu_t\|T_t x - z\|^2 = r\}.$$

Then, since, for each $y \in H$ and $t \in S$

$$\|x_0 - y\|^2 = \|T_t x - y\|^2 - \|T_t x - x_0\|^2 - 2\langle T_t x - x_0, x_0 - y\rangle,$$

we have

$$\begin{aligned}\|x_0 - y\|^2 &= \mu_t(\|T_t x - y\|^2 - \|T_t x - x_0\|^2 - 2\langle T_t x - x_0, x_0 - y\rangle) \\ &= \mu_t\|T_t x - y\|^2 - \mu_t\|T_t x - x_0\|^2 - 2\langle x_0 - x_0, x_0 - y\rangle \\ &= \mu_t\|T_t x - y\|^2 - \mu_t\|T_t x - x_0\|^2 \geqq 0.\end{aligned}$$

This implies that the set M consists of a single point x_0. Now, we can show $x_0 \in F(\mathcal{S})$. In fact, for each $s \in S$,

$$\begin{aligned}\mu_t\|T_tx - T_sx_0\|^2 &= \mu_t\|T_{st}x - T_sx_0\|^2 \\ &= \mu_t\|T_sT_tx - T_sx_0\|^2 \\ &\leqq \mu_t\|T_tx - x_0\|^2 = r.\end{aligned}$$

Hence, $T_sx_0 = x_0$ for every $s \in S$.

Since μ is an invariant mean on RUC(S), from [16], we know

$$\mu_t\|T_tx - z\|^2 \leqq \inf_s \sup_t \|T_{ts}x - z\|^2$$

for every $z \in H$. On the other hand, since for $z \in F(\mathcal{S})$ and $a, s \in S$,

$$\begin{aligned}\inf_u \sup_t \|T_{tu}x - z\|^2 &\leqq \sup_t \|T_{tas}x - z\|^2 \\ &= \sup_t \|T_{ta}T_sx - T_{ta}z\|^2 \\ &\leqq \sup_t \|T_sx - z\|^2 = \|T_sx - z\|^2,\end{aligned}$$

it follows that

$$\inf_u \sup_t \|T_{tu}x - z\|^2 \leqq \mu_s\|T_sx - z\|^2.$$

Therefore, for each $z \in F(\mathcal{S})$, we have

$$\mu_t\|T_tx - z\|^2 = \inf_s \sup_t \|T_{ts}x - z\|^2.$$

This implies that the point x_0 is independent of μ, that is, $T_\mu x = x_0$ for each invariant mean μ. Last, we show that $T_{\mu_\alpha}x$ converges weakly to x_0.

Let μ be a cluster point of the net $\mu_\alpha, \alpha \in A$ in the weak* topology. Then μ is an invariant mean. In fact, since it is obvious that μ is a mean, we show that μ is left invariant. For each $\varepsilon > 0, f \in$ RUC(S) and $s \in S$, there exists $\alpha_0 \in A$ such that

$$|\mu_\alpha(f) - \mu_\alpha(l_sf)| \leqq \frac{\varepsilon}{3}$$

for all $\alpha \geqq \alpha_0$. Since μ is a cluster point of the net $\mu_\alpha, \alpha \in A$, we can choose α_1 ($\geqq \alpha_0$) such that

$$|\mu_{\alpha_1}(f) - \mu(f)| \leqq \frac{\varepsilon}{3}$$

and

$$|\mu_{\alpha_1}(l_sf) - \mu(l_sf)| \leqq \frac{\varepsilon}{3}.$$

Hence, we have

$$\begin{aligned}|\mu(f) - \mu(l_sf)| &\leqq |\mu(f) - \mu_{\alpha_1}(f)| + |\mu_{\alpha_1}(f) - \mu_{\alpha_1}(l_sf)| \\ &\quad + |\mu_{\alpha_1}(l_sf) - \mu(l_sf)| \\ &\leqq \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon.\end{aligned}$$

Since $\varepsilon > 0$ is arbitrary, we have

$$\mu(f) = \mu(l_s f)$$

for every $f \in \mathrm{RUC}(S)$ and $s \in S$. This implies that μ is left invariant. Similarly, μ is right invariant.

Let $\{T_{\mu_{\alpha_\beta}} x\}$ be a subnet of $\{T_{\mu_\alpha} x\}$ such that $T_{\mu_{\alpha_\beta}} x$ converges weakly to some z in H. Then since a cluster point λ of the net $\{\mu_{\alpha_\beta}\}$ is also a cluster point of the net $\{\mu_\alpha\}$, λ is an invariant mean. So, we have $z = T_\lambda x = x_0$. This implies that $T_{\mu_\alpha} x$ converges weakly to $x_0 \in F(\mathcal{S})$.

3. **Fixed point theorem.** In this section, we prove a fixed point theorem for nonlinear semigroups without convexity. Let X be a subspace of $B(S)$ containing constants. Then, according to Mizoguchi and Takahashi [10], a real valued function μ on X is called a *submean on* X if the following conditions are satisfied:

(1) $\mu(f+g) \leqq \mu(f) + \mu(g)$ for every $f, g \in X$;
(2) $\mu(\alpha f) = \alpha\mu(f)$ for every $f \in X$ and $\alpha \geqq 0$;
(3) for $f, g \in X, f \leqq g$ implies $\mu(f) \leqq \mu(g)$;
(4) $\mu(c) = c$ for every constant c.

For a submean μ on X and $f \in X$, according to time and circumstances, we also use $\mu_t\big(f(t)\big)$ instead of $\mu(f)$.

LEMMA [10]. *Let S be a semitopological semigroup, let X be a subspace of $B(S)$ containing constants and let μ be a submean on X. Let $\{x_t : t \in S\}$ be a bounded subset of a Hilbert space H and let D be a closed convex subset of H. Suppose that for each $x \in D$, the real-valued function f on S defined by*

$$f(t) = \|x_t - x\|^2 \text{ for all } t \in S$$

belongs to X. If

$$g(x) = \mu_t \|x_t - x\|^2 \text{ for all } x \in D$$

and

$$r = \inf\{g(x) : x \in D\},$$

then there exists a unique element $z \in D$ such that $g(z) = r$. Further the following inequality holds:

$$r + \|z - x\|^2 \leqq g(x) \text{ for every } x \in D.$$

Let X be a subspace of $B(S)$ containing constants which is l_S-invariant, *i.e.*, $l_s(X) \subset X$ for each $s \in S$. Then a submean μ on X is said to be *left invariant* if $\mu(f) = \mu(l_s f)$ for all $s \in S$ and $f \in X$. Now, we can prove a fixed point theorem for nonlinear semigroups without convexity in a Hilbert space.

THEOREM 2. *Let C be a nonempty subset of a Hilbert space H and let X be a l_S-invariant subspace of $B(S)$ containing constants which has a left invariant submean μ on X. Let $\mathcal{S} = \{T_S : s \in S\}$ be a Lipschitzian semigroup on C with Lipschitzian constants $k_s, s \in S$. Suppose that $\{T_s x : s \in S\}$ is bounded and $\sqcap_{s \in S} \overline{\mathrm{co}}\{T_{st}x : t \in S\} \subset C$ for some $x \in C$. If for each $u \in C$ and $v \in H$, the real valued function f on S defined by*

$$f(t) = \|T_t u - v\|^2 \text{ for all } t \in S$$

and the function h on S defined by

$$h(t) = k_t^2 \text{ for all } t \in S$$

belong to X and $\mu_t(k_t^2) \leqq 1$, then there exists an element $z \in C$ such that $T_s z = z$ for all $s \in S$.

PROOF. Define a real valued function g on H by

$$g(y) = \mu_t \|T_t x - y\|^2 \text{ for each } y \in H.$$

If $r = \inf\{g(y) : y \in H\}$, then by Lemma there exists a unique element $z \in H$ such that $g(z) = r$. Further, we know that

$$r + \|z - y\|^2 \leqq g(y) \text{ for every } y \in H.$$

For each $s \in S$, let Q_s be the metric projection of H onto $\overline{\mathrm{co}}\{T_{st}x : t \in S\}$. Then by Phelps [13], Q_s is nonexpansive and for each $t \in S$,

$$\|T_{st}x - Q_s z\|^2 = \|Q_s T_{st}x - Q_s z\|^2 \leqq \|T_{st}x - z\|^2.$$

So, we have

$$\begin{aligned} \mu_t \|T_t x - Q_s z\|^2 &= \mu_t \|T_{st}x - Q_s z\|^2 \leqq \mu_t \|T_{st}x - z\|^2 \\ &= \mu_t \|T_t x - z\|^2 \end{aligned}$$

and thus $Q_s z = z$. This implies

$$z \in \overline{\mathrm{co}}\{T_{st}x : t \in S\} \text{ for all } s \in S$$

and hence

$$z \in \sqcap_{s \in S} \overline{\mathrm{co}}\{T_{st}x : t \in S\} \subset C.$$

Since by Lemma

$$\|z - y\|^2 \leqq \mu_t \|T_t x - y\|^2 - \mu_t \|T_t x - z\|^2 \text{ for all } y \in H,$$

putting $y = T_s z$ for each $s \in S$, we have

$$\begin{aligned} \|z - T_s z\|^2 &\leqq \mu_t \|T_t x - T_s z\|^2 - \mu_t \|T_t x - z\|^2 \\ &= \mu_t \|T_{st}x - T_s z\|^2 - \mu_t \|T_t x - z\|^2 \\ &\leqq (k_s^2 - 1)\mu_t \|T_t x - z\|^2 \end{aligned}$$

and hence

$$\mu_s\|z - T_sz\|^2 \leqq \big(\mu_s(k_s^2) - 1\big)\mu_t\|T_tx - z\|^2 \leqq 0.$$

This implies $\mu_s\|z - T_sz\|^2 = 0$. Since for every $a, s \in S$,

$$\|z - T_az\|^2 \leqq 2\|z - T_sz\|^2 + 2\|T_sz - T_az\|^2,$$

we have

$$\begin{aligned}\|z - T_az\|^2 &\leqq 2\mu_s\|z - T_sz\|^2 + 2\mu_s\|T_sz - T_az\|^2 \\ &= 2\mu_s\|T_sz - T_az\|^2 = 2\mu_s\|T_{as}z - T_az\|^2 \\ &\leqq 2k_a^2\mu_s\|T_sz - z\|^2 = 0.\end{aligned}$$

Therefore, we have $T_sz = z$ for every $s \in S$.

As a direct consequence of Theorem 2, we obtain the result $F(\mathcal{S}) \neq \emptyset$ in Theorem 1. Further, we can prove the following fixed point theorem. A semitopological semigroup S is *left reversible* if any two closed right ideals of S have nonvoid intersection. In this case, $(S, \leqq)$ is a directed system when the binary relation "$\leqq$" on S is defined by $a \leqq b$ if and only if $\{a\} \sqcup \overline{aS} \supset \{b\} \sqcup \overline{bS}$.

COROLLARY [7]. *Let C be a nonempty subset of a Hilbert space H and let S be a left reversible semigroup. Let $\mathcal{S} = \{T_t : t \in S\}$ be a Lipschitzian semigroup on C such that $\{k_s : s \in S\}$ is bounded and $\lim_s \sup k_s \leqq 1$. If $\{T_tx : t \in S\}$ is bounded and $\sqcap_{s\in S}\overline{\text{co}}\{T_{st}x : t \in S\} \subset C$ for some $x \in C$. Then there exists $z \in C$ such that $T_sz = z$ for every $s \in S$.*

PROOF. Defining a real valued function μ on $B(S)$ by

$$\mu(f) = \lim_s \sup f(s) \text{ for every } f \in B(S),$$

μ is a left invariant submean on $B(S)$. Since $\lim_s \sup k_s \leqq 1$ implies $\lim_s \sup k_s^2 \leqq 1$, by using Theorem 2, the proof is complete.

We may comment on the relationship between the hypothesis of Theorem 1: "RUC(S) has an invariant mean" and Corollary [7]: "S is left reversible". As well known, they do not imply each other in general. But if RUC(S) has sufficiently many functions to separate closed sets, then "RUC(S) has an invariant mean" would imply "S is left and right reversible".

REFERENCES

1. J. B. Baillon, *Un théorème de type ergodic pour les contractions non lineáres dans un espace de Hilbert*, C.R. Acad. Sci. Paris Sér. A-B, **280**(1975), 1511–1514.

2. ———, *Quelques propriétés de convergence asymptotique pour les semigroups de contractions impaires*, C.R. Acad. Sci. Paris Sér. A-B, **283**(1976), 75–78.

3. H. Brézis and F. E. Browder, *Nonlinear ergodic theorems*, Bull Amer. Math. Soc., **82**(1976), 959–961.

4. ———, *Remarks on nonlinear ergodic theory*, Adv. in Math., **25**(1977), 165–177.

5. N. Hirano and W. Takahashi, *Nonlinear ergodic theorems for nonexpansive mappings in Hilbert space*, Kodai Math. J., **2**(1979), 11–25.

6. H. Ishihara and W. Takahashi, *Fixed point theorems for uniformly lipschitzian semigroups in Hilbert spaces*, J. Math. Anal. Appl., **127**(1987), 206–210.
7. H. Ishihara, *Fixed point theorems for lipschitzian semigroups*, Canad. Math. Bull., **32**(1989), 90–97.
8. A. T. Lau, *Semigroup of nonexpansive mappings on a Hilbert space*, J. Math. Anal. Appl., **105**(1985), 514–522.
9. T. Mitchell, *Topological semigroups and fixed points*, Illinois J. Math., **14**(1970), 630–641.
10. N. Mizoguchi and W. Takahashi, *On the existence of fixed points and ergodic retractions for Lipschitzian semigroups in Hilbert spaces*, Nonlinear Analysis, **14**(1990), 69–80.
11. A. Pazy, *On the asymptotic behavior of iterates of nonexpansive mappings in Hilbert space*, Israel J. Math., **26**(1977), 197–204.
12. ______, *On the asymptotic behavior of semigroups of nonlinear contractions in Hilbert space*, J. Funct. Anal., **27**(1978), 292–307.
13. R. P. Phelps, *Convex sets and nearest points*, Proc. Amer. Math. Soc., **8**(1957), 790–797.
14. G. Rodé, *An ergodic theorem for semigroups of nonexpansive mappings in a Hilbert space*, J. Math. Anal. Appl., **85**(1982), 172–178.
15. W. Takahashi, *A nonlinear ergodic theorem for an amenable semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc., **81**(1981), 253-256.
16. ______*Fixed point theorems for families of nonexpansive mappings on unbounded sets*, J. Math. Soc. Japan, **36**(1984), 543–553.
17. ______*A nonlinear ergodic theorem for a reversible semigroup of nonexpansive mappings in a Hilbert space*, Proc. Amer. Math. Soc., **96**(1986), 55–58.

Department of Information Science
Tokyo Institute of Technology
Oh-Okayama, Meguro-ku
Tokyo 152, Japan