ESTIMATES OF BLOW-UP TIME FOR
A NON-LOCAL REACTIVE–CONVECTIVE PROBLEM
MODELLING OHMIC HEATING OF FOODS

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Abstract   In this work, we estimate the blow-up time for the non-local hyperbolic equation of ohmic type, $u_t + u_x = \lambda f(u)/(\int_0^1 f(u)\,\mathrm{d}x)^2$, together with initial and boundary conditions. It is known that, for $f(s)$, $-f'(s)$ positive and $\int_0^\infty f(s)\,\mathrm{d}s < \infty$, there exists a critical value of the parameter $\lambda > 0$, say λ^*, such that for $\lambda > \lambda^*$ there is no stationary solution and the solution $u(x,t)$ blows up in finite time t^*, while for $\lambda \leqslant \lambda^*$ there exist stationary solutions. Moreover, the solution $u(x,t)$ also blows up for large enough initial data and $\lambda \leqslant \lambda^*$. Thus, estimates for t^* were found either for λ greater than the critical value λ^* and fixed initial data $u_0(x) \geqslant 0$, or for $u_0(x)$ greater than the greatest steady-state solution (denoted by $w_2 \geqslant w^*$) and fixed $\lambda \leqslant \lambda^*$. The estimates are obtained by comparison, by asymptotic and by numerical methods. Finally, amongst the other results, for given λ, λ^* and $0 < \lambda - \lambda^* \ll 1$, estimates of the following form were found: upper bound $\epsilon + c_1 \ln[c_2(\lambda - \lambda^*)^{-1}]$; lower bound $c_3(\lambda - \lambda^*)^{-1/2}$; asymptotic estimate $t^* \sim c_4(\lambda - \lambda^*)^{-1/2}$ for $f(s) = \mathrm{e}^{-s}$. Moreover, for $0 < \lambda \leqslant \lambda^*$ and given initial data $u_0(x)$ greater than the greatest steady-state solution $w_2(x)$, we have upper estimates: either $c_5 \ln(c_6 A_0^{-1} + 1)$ or $\epsilon + c_7 \ln(c_8 \zeta^{-1})$, where A_0, ζ measure, in some sense, the difference $u_0 - w_2$ (if $u_0 \to w_2+$, then $A_0, \zeta \to 0+$). $c_i > 0$ are some constants and $0 < \epsilon \ll 1$, $0 < A_0, \zeta$. Some numerical results are also given.

1. Introduction

We consider the non-local initial boundary-value problem

$$u_t(x,t) + u_x(x,t) = \lambda \frac{f(u(x,t))}{(\int_0^1 f(u(x,t))\,\mathrm{d}x)^2}, \quad 0 < x < 1, \quad t > 0, \qquad (1.1\,a)$$

$$u(0,t) = 0, \quad t > 0, \qquad (1.1\,b)$$

$$u(x,0) = u_0(x) \geqslant 0, \quad 0 < x < 1, \qquad (1.1\,c)$$

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where $\lambda > 0$. The function $u(x,t)$ represents the dimensionless temperature when an electric current flows through a conductor (e.g. food) with temperature dependent on electrical resistivity $f(u) > 0$, subject to a fixed potential difference $V > 0$. The (dimensionless) resistivity $f(u)$ may be either an increasing or a decreasing function of temperature depending strongly on the type of material (food). Problem (1.1) models one of the main methods for sterilizing food. The sterilization can take place by rapidly heating the food electrically. The food is passed through a conduit, part of which lies between two electrodes. A high electric current flowing between the electrodes results in ohmic heating of the food, which quickly gets hot. The problem was considered by Please *et al.* [24], who looked at the stability of models allowing for different types of flow. More background on this type of process can be found in [5], [12], [15], [26], [27] and [29].

Lacey *et al.* [22] have also studied problem (1.1): for f decreasing with $\int_0^\infty f(s)\,\mathrm{d}s < \infty$, blow-up occurs if λ is too large for a steady state to exist or if the initial condition is too big. If f is increasing with $\int_0^\infty \mathrm{d}s/f(s) < \infty$, blow-up is also possible. If f is increasing with $\int_0^\infty \mathrm{d}s/f(s) = \infty$ or decreasing with $\int_0^\infty f(s)\,\mathrm{d}s = \infty$, the solution is global in time; some special cases with particular forms of f are discussed to illustrate what the solution can do (for details see [22] and the references therein).

Here λ is a dimensionless parameter and can be equal to (amongst other things) the square of (its relationship is actually proportional to) the applied potential difference, V. In the case where $f(u)$ is a sufficiently rapidly decreasing function of temperature, there exists a critical value of the potential difference V, say V^*, such that, for $V > V^*$ (equivalently, $\lambda > \lambda^*$), a thermal runaway (blow-up of the temperature u or burning of the food) takes place (see [19, 20, 22]).

Below we assume that f satisfies

$$f(s) > 0, \quad f'(s) < 0, \quad s \geqslant 0, \tag{1.2a}$$

$$\int_0^\infty f(s)\,\mathrm{d}s < \infty; \tag{1.2b}$$

for instance, either $f(s) = \mathrm{e}^{-s}$ or $f(s) = (1+s)^{-p}$, $p > 1$, satisfy (1.2).

For the initial data it is required that $u_0(x)$, $u_0'(x)$ be bounded, $u_0(x) \geqslant 0$ in $[0,1]$ (the latter requirement is a consequence of the fact that for any initial data the solution u becomes non-negative over $(0,1]$ for some time t and so, with an appropriate redefinition of t, we can always make this assumption [19, 22]).

The solution $u(x,t)$ also blows up for large enough initial data even if $0 < \lambda \leqslant \lambda^*$ [19, 20, 22]; in this case, we give (see §5) an analogous estimate of blow-up time, as for the case of $\lambda > \lambda^*$.

The steady-state problem that corresponds to (1.1) is

$$w' = \mu f(w) = \lambda \frac{f(w)}{(\int_0^1 f(w)\,\mathrm{d}x)^2}, \quad 0 < x < 1, \quad w(0) = 0, \tag{1.3}$$

where $w = w(x) = w(x;\lambda)$ (see [6, 14, 19, 20, 22]). The parameter μ is referred to as a local parameter, while λ is referred to as a non-local one and the relation between them is $\mu = \lambda/(\int_0^1 f(w)\,\mathrm{d}x)^2$.
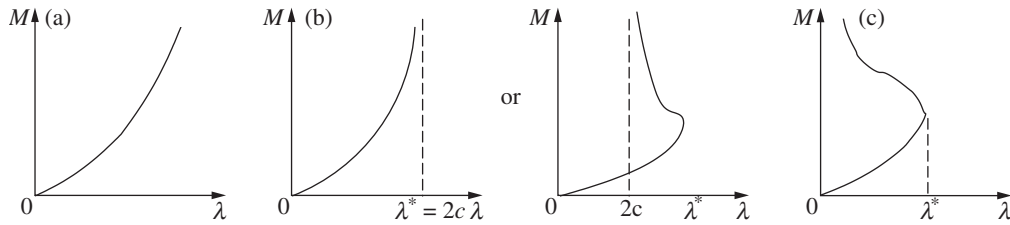
Figure 1. Possible response diagrams for equilibrium solutions: (a) $Mf(M)$ is unbounded; (b) $Mf(M) \to c$, $0 < c < \infty$, as $M \to \infty$; (c) $Mf(M) \to 0$, as $M \to \infty$. In each part of the figure it is possible for there to be more turning points than are shown (so that for some λ there are more solutions).

On integrating the ordinary differential equation (1.3) we see that $\lambda = M^2/\mu$, where $M = M(\mu) = \max_{[0,1]} w = w(1; \mu)$ and $\mu = \int_0^M \mathrm{d}s/f(s)$.

It is clear that $M(\mu) \to 0+$ as $\mu \to 0+$ and, with $f(0) > 0$, $\lambda \to 0+$ as $\mu \to 0+$.

It is also known that if $(1.2\,b)$ holds, then there exists a critical value of the parameter λ, say $\lambda^* < \infty$, such that, for $\lambda > \lambda^*$, the solution $u(x, t; \lambda)$ to problem (1.1) blows up globally in finite time t^* ($u \to \infty$ for all $x \in (0, 1]$ as $t \to t^*-$; actually, the blow-up is uniform in x on compact subsets of $(0, 1]$: for two points $0 < x_1 < x_2 \leqslant 1$, $|u(x_1, t) - u(x_2, t)| \to 0$ as $t \to t^*$ [22]) and problem (1.3) has no solutions (of any kind). For a fixed $\lambda \in (0, \lambda^*)$ there exist at least two solutions $w(x; \lambda)$ and a unique $u(x, t; \lambda)$; $u(x, t; \lambda)$ may either exist for all time or blow up globally depending on the initial data [19, 20, 22] (for the blow-up, u_0 must be greater than the greatest steady-state solution $w(x; \lambda)$ and (1.2) holds). The response (bifurcation) diagrams for problem (1.3) are as in Figure 1 (see also [22]).

Our purpose, in this work, is to find some estimates of the blow-up time t^*, either with respect to the parameter λ (more precisely, with respect to the difference $\lambda - \lambda^*$), when $\lambda > \lambda^*$ and with fixed initial data $u_0(x)$, or with respect to initial data $u_0(x)$ and with fixed $\lambda \leqslant \lambda^*$.

Work related to this model (the thermistor problem) can be found in [1, 7–10, 13].

Estimates of blow-up time are very important since they answer the question of 'when' the blow-up takes place [3, 16, 17] (for problem (1.1) this is the time when the food is burnt).

In part (b) or part (c) of Figure 1, with $2c < \lambda^*$, there may be either only one or more than one turning point (λ^*, M^*) depending on f. One can find other forms of non-local diagrams in [19, 20, 22]; their shapes depend strongly on the boundary conditions and the function f.

Under the assumptions (1.2), problem (1.3) has at least one classical (regular) steady-state solution $w^* = w(x; \lambda^*)$ (more than one w^* may exist). We assume below that w^* is unique, i.e. as in Figure 1c, and that the pair $(\underline{w}, \bar{w})$ at $\lambda < \lambda^*$ (λ close to λ^*) has the following property: $\underline{w} = w_1$ is stable while $\bar{w} = w_2$ is unstable (since, in our proofs, we require only the existence of at least one w^* at λ^* and that $\underline{w}(x) < \bar{w}(x)$ for x in $(0, 1]$, where $\bar{w}$ is the next steady-state solution greater than $\underline{w}(x)$ at $\lambda < \lambda^*$).

We also emphasize that for $\lambda > \lambda^*$ and for all $x \in (0,1]$ we have

$$F(u) = \frac{f(u)}{(\int_0^1 f(u)\,\mathrm{d}x)^2} \to \infty \quad \text{as } t \to t^* - < \infty, \qquad (1.4\,a)$$

$$u(x,t;\lambda) \to \infty \quad \text{as } t \to t^* - < \infty, \qquad (1.4\,b)$$

the latter means that $u(x,t;\lambda)$ blows up globally (see [19, 20, 22]).

Similar situations pertaining to the blow-up can be found in the study of the (local) reaction–diffusion problem [2, 18, 21]

$$u_t = \Delta u + \lambda f(u), \quad x \in \Omega, \quad t > 0, \qquad (1.5\,a)$$

$$B(u) = 0, \qquad x \in \partial\Omega, \quad t > 0, \qquad (1.5\,b)$$

$$u(x,0) = u_0(x), \qquad x \in \Omega, \qquad (1.5\,c)$$

where B represents Dirichlet or Robin boundary conditions, Ω is a bounded domain of $\mathbb{R}^n$, λ is a positive parameter, and $f(u)$ behaves like e^u:

$$f(s) > 0, \quad f'(s) > 0, \quad f''(s) \geqslant 0, \quad \text{for } s \geqslant 0 \quad \text{and} \quad \int_0^\infty \frac{\mathrm{d}s}{f(s)} < \infty. \qquad (1.6)$$

Again, under certain conditions, the solution u to (1.5) blows up:

$$\limsup_{t \to t^*-} \|u(\cdot,t;\lambda)\|_\infty = \infty \quad \text{for } \lambda > \lambda^*, \ t^* < \infty.$$

It should be emphasized that the blow-up for problem (1.5) differs from that of non-local problem (1.1) in that (1.5) does not normally blow up globally. Moreover, there exists a turning point $P^* = (\lambda^*, \|w^*\|_\infty)$ with $\|w^*\|_\infty < \infty$ of the response diagram of the steady-state problem corresponding to (1.5). For f satisfying (1.6), the following upper and lower bounds for t^* hold: $t_1 \leqslant t^* \leqslant t_2$, where $t_i = c_i(\lambda - \lambda^*)^{-1/2}$ and c_i are some constants $(c_1 < c_2)$; also, for $f(s) = \mathrm{e}^s$, $t^* \sim K(\lambda - \lambda^*)^{-1/2}$ as $\lambda \to \lambda^*+$ asymptotically, where K is a constant [18].

Estimates of blow-up time (upper and lower bounds for t^*, similar to those for the local problem (1.5)), obtained by using comparison methods, for the non-local equation $u_t = u_{xx} + \lambda f(u)/(\int_{-1}^1 f(u)\,\mathrm{d}x)^2$, with initial conditions $u_0(x)$, Robin-type boundary conditions, decreasing f and $\lambda > \lambda^*$, can also be found in [17]. Also, for $f(u) = \mathrm{e}^{-u}$, the following asymptotic estimate holds: $t^* \sim t_u(\lambda - \lambda^*)^{-1/2}$ for $0 < \lambda - \lambda^* \ll 1$, where t_u is a constant. A numerical estimate is also obtained by using a Crank–Nicolson scheme.

In the present work, we find estimates for the non-local problem (1.1), for $f(s) = \mathrm{e}^{-s}$ and for general $f(s)$ satisfying (1.2).

In both problems, estimates of t^* have been found only if the spectrum of the steady problem is an interval that is closed on the right, i.e. $(0, \lambda^*]$. It is still an open question how to estimate t^* when the spectrum is an open interval $(0, \lambda^*)$ (see [17, 18, 21]). Concerning problem (1.5), there is an upper estimate of blow-up time, given by Bellout [4], which is valid even if the spectrum is open, i.e. $(0, \lambda^*)$; actually, he uses a different method to obtain this estimate and requires f/f' to be concave.

We organize this work as follows: in §2 we use comparison techniques and find upper and lower bounds for t^*, when f satisfies (1.2). In §3 we use an asymptotic expansion and again obtain an estimate of t^* but for $f(s) = \mathrm{e}^{-s}$. In §4 we compute the blow-up time t^* numerically by using an upwind scheme and verify the previous estimate. Finally, in §5 we give an estimate for t^* with respect to the initial data $u_0(x)$.

2. Comparison methods: upper and lower bounds of t^* for $\lambda > \lambda^*$

If the function f satisfies (1.2 a), one can prove (see the appendix in [**22**]) that a maximum principle holds for (1.1) (here is where we need f to be decreasing). We may then, in the usual way, define upper and lower solutions of (1.1): an upper (lower) solution $\bar{u}$ ($\underline{u}$) is defined as a function which satisfies (1.1) if we substitute '$\geqslant$' ('$\leqslant$') for '$=$' (see [**19**, **20**, **22**, **25**, **28**]). The same comparison properties also hold for the equation $u_t + u_x = 0$ together with the data (1.1 b), (1.1 c). Below we use ideas and techniques similar to those in [**17**].

2.1. An upper bound for t^*

We now wish to find an upper bound for the blow-up time t^*. For simplicity, we assume that $0 \leqslant u_0 < w^*$. First, we write (1.3) in a slightly different way,

$$w' = \mu f(w) = \frac{\lambda f(w)}{(\int_0^1 f(w)\,\mathrm{d}x)^2} = \lambda F(w), \quad 0 < x < 1, \quad w(0) = 0, \qquad (2.1)$$

where $F(\cdot) = f(\cdot)/(\int_0^1 f(\cdot)\,\mathrm{d}x)^2$ and λ is a positive parameter (eigenvalue). The related linearized eigenvalue problem of (2.1) for a function $\phi = \phi(x; \lambda) \in \mathbb{R}$ (ϕ is actually assumed to be a real-valued function) is then

$$\phi' - \lambda \delta F(w; \phi) = -\rho \phi, \quad 0 < x < 1, \quad \phi(0) = \phi_0 = 0, \qquad (2.2)$$

where $\phi = \phi(x; \lambda)$, $\rho = \rho(w, \lambda)$ (ρ, ϕ) is the eigenpair and $\delta F(w; \phi)$ is the first variation (or Gâteaux derivative) of F at w in the direction of ϕ ($F(w; \phi) := F(w + \epsilon \phi) = J(\epsilon)$ and $\delta F(w; \phi) = J'(0) = \lim_{\epsilon \to 0}((F(w + \epsilon \phi) - F(w))/\epsilon)$).

As regards the first variation $\delta F(w; \phi)$, we have

$$\delta F(w; \phi) = \frac{f'(w)\phi}{(\int_0^1 f(w)\,\mathrm{d}x)^2} - \frac{2f(w)\int_0^1 f'(w)\phi\,\mathrm{d}x}{(\int_0^1 f(w)\,\mathrm{d}x)^3}.$$

Below, in order to simplify the expressions, we use the shorthand notation

$$I_{\nu k}(w, \phi) := \int_0^1 f^{(\nu)}(w(x))\phi^k(x)\,\mathrm{d}x,$$

and $I_\nu(w) := I_{\nu 0}(w, \phi)$, $\nu, k = 0, 1, 2, 3, \ldots$, $f^{(\nu)}(w) = (\mathrm{d}^\nu/\mathrm{d}w^\nu)f(w)$, thus

$$\delta F(w; \phi) = \frac{f'(w)\phi}{I_0^2(w)} - \frac{2f(w)I_{11}(w, \phi)}{I_0^3(w)}.$$

We now have the following lemma concerning the eigenpair of problem (2.2).

Lemma 2.1. *Problem (2.2) has the eigenpair (ρ, ϕ), where $\phi(x) > 0$, $x \in (0, 1]$, and its spectrum is a continuum of eigenvalues in $\mathbb{R}$, generated by $\rho = \rho(w, \lambda)$ for each $\lambda \in (0, \lambda^*]$. The eigenvalue $\rho = \rho(w, \lambda)$ is continuous with respect to λ.*

Proof. First, if $\phi(x)$ is a real-valued function, then equation (2.2) implies that $\rho \in \mathbb{R}$. We now write equation (2.2) in a different way:

$$\phi' + (g(x) + \rho)\phi = qh(x),$$

where $q = -I_{11}(w, \phi)$ is a number,

$$h(x) = h(x; w(x)) = \lambda \frac{2f(w(x))}{I_0^3(w)} > 0$$

and

$$g(x) = g(x; w(x)) = -\lambda \frac{f'(w(x))}{I_0^2(w)}.$$

Problem (2.2) can be written under the equivalent integral formulation:

$$\phi(x) = q\left[\exp\left(-\int_0^x g(z)\,\mathrm{d}z - \rho x\right)\right] \int_0^x h(s)\left[\exp\left(\int_0^s g(z)\,\mathrm{d}z + \rho s\right)\right]\mathrm{d}s.$$

The above form of ϕ implies that if a non-trivial ϕ satisfying (2.2) exists, then it is positive (ϕ actually does not change sign in $(0, 1)$ and can be taken to be positive).

Now we can normalize ϕ so that $q = 1$. Therefore, there exists a real-valued function ϕ satisfying (2.2) if we can find $\rho \neq 0$ satisfying the following equation:

$$1 = -I_{11}(w, \phi) = \int_0^1 -f'(w(x))\phi(x)\,\mathrm{d}x, \quad \text{where } \phi(x) = \phi(x; \rho).$$

This can be written as

$$1 = \int_0^1 -f'(w(x))\left[\int_0^x h(x-s)\exp[-\rho s - G(x) + G(x-s)]\,\mathrm{d}s\right]\mathrm{d}x$$

for $G(s) = \int_0^s g(z)\,\mathrm{d}z$. Therefore,

$$1 = \int_0^1 \exp(-\rho s)\left[\int_s^1 (-f'(w(x))h(x-s)\exp[-G(x) + G(x-s)])\,\mathrm{d}x\right]\mathrm{d}s$$

or

$$1 = \int_0^1 Y(s)\exp(-\rho s)\,\mathrm{d}s, \tag{2.3}$$

where $Y(s) = Y(s; w) > 0$ for $-f'(w)h(x-s)\exp[-G(x) + G(x-s)] > 0$ and $0 < s < x < 1$ (the eigenvalue ρ is a real number). For a fixed λ, the w are also fixed. Now, for each $Y(s) > 1$ we have only one real solution $\rho > 0$, which satisfies equation (2.3). Also, for each $Y(s) < 1$ we have only one real solution $\rho < 0$ and for $Y(s) = 1$ we have only the trivial solution $\rho = 0$. The eigenvalues $\rho = \rho(w, \lambda) \in C((0, \lambda^*]; \mathbb{R})$ since $w(x; \lambda)$, $g(x) = g(x; \lambda)$ and $Y(s) = Y(s; w)$ are continuous functions with respect to $\lambda \in (0, \lambda^*]$. This proves the lemma. $\qquad\square$

It is known that the spectrum for problem (2.1) can be either an interval closed from the right or an open one. Here we consider the case where the spectrum for problem (2.1) is a closed interval from the right and that there exists a unique turning point $(\lambda^*, \|w^*\|_\infty)$ with $\|w^*\|_\infty = w^*(1) = M^* < \infty$ (see Figure 1c); at $\lambda < \lambda^*$ two steady states correspond to w_1, w_2 with $w_1 < w_2$, while at $\lambda = \lambda^*$ they correspond to $w_1 = w_2 = w^*$. We need the following lemma.

Lemma 2.2. *Let w_1, w_2 with $w_1 < w_2$ be the solutions of (2.1) at $\lambda < \lambda^*$. Then $\rho_1 = \rho(w_1, \lambda) \leqslant 0$, $\rho_2 = \rho(w_2, \lambda) \geqslant 0$ and $\rho^* = \rho(w^*, \lambda^*) = 0$, where ρ represents the eigenvalues of problem (2.2).*

Proof. We assume that $u(x,t) = w(x) + \epsilon\phi(x)e^{\rho t} + O(\epsilon^2)$, with $\phi(x) > 0$, for $x \in (0,1]$ and for some $\epsilon \in \mathbb{R}$. Then we find, to the first order in ϵ, that ϕ and ρ must satisfy problem (2.2). We also know that w_1 is asymptotically stable, w_2 is unstable, and w^* is stable from below and unstable from above (see [**22**]). These imply that $\rho_1 \leqslant 0$, $\rho_2 \geqslant 0$ and $\rho^* = 0$. $\qquad\square$

Thus, the lower branch of the response diagram (Figure 1c) is asymptotically stable with $\rho_1 = \rho(w_1, \lambda) \leqslant 0$ (ρ is the eigenvalue of (2.2) for $\lambda < \lambda^*$ at w), while the upper branch is unstable with $\rho_2 = \rho(w_2, \lambda) \geqslant 0$ (see also [**11, 19, 20, 23**]). This continues to hold (with a suitable understanding of the 'lower and upper branch') even if there are more turning points P^*. Moreover, from Lemma 2.2 we have $\rho^* = \rho(w^*, \lambda^*) = 0$, where ρ is the eigenvalue of linearized problem (2.2) (for related results, see also [**2, 11, 23**]). Because of Lemmas 2.1 and 2.2, problem (2.2) at $\lambda = \lambda^*$, with $\phi^*(x) > 0$, becomes

$$\phi^{*\prime} - \lambda^*\delta F(w^*; \phi^*) = 0, \quad 0 < x < 1, \quad \phi^*(0) = \phi_0^* = 0. \tag{2.4}$$

Now, in order to find an upper bound for t^*, we take the difference,

$$v = v(x,t) = v(x,t;\lambda) = u(x,t;\lambda) - w^*(x) = u - w^*. \tag{2.5}$$

Since w^* is bounded, v blows up at the same time as u, and in the same manner, i.e. globally. Hence $t^* = t^*(u) = t^*(v)$ ($t^*(u)$ is the blow-up time for u) and $v(x,t) \to \infty$ as $t \to t^*-$ for all $x \in (0,1]$. In what follows, we find an A-problem (see below (2.13)), where $A = A(t)$ blows up and is such that $A(t) \leqslant c\|v(\cdot,t)\|_\infty$ where $c = 1/\sup_x \phi^*(x)$ as long as $v(x,t) \geqslant \psi(x,t) = A(t)\phi^*(x)$. The latter relation and (2.5) imply that $t^*(u) = t^*(v) \leqslant T^* = T^*(A)$, for some T^*. Thus, we find an upper bound T^* for $t^*(u)$.

Therefore, we obtain

$$v_t = u_t = -u_x + \lambda F(u) = -u_x + \lambda F(u) - \lambda^* F(w^*) + w^{*\prime}$$
$$= -v_x + (\lambda - \lambda^*)F(u) + \lambda^*(F(u) - F(w^*)). \tag{2.6}$$

By writing $J(\epsilon) = F(w^* + \epsilon v)$, $0 \leqslant \epsilon \leqslant 1$, whence $J(0) = F(w^*)$ and $J(1) = F(u)$, Taylor's formula gives

$$F(u) - F(w^*) = J(1) - J(0) = J'(0) + \tfrac{1}{2}J''(\xi),$$

for some $\xi = \xi(t) \in (0, 1)$, where

$$J'(0) = \delta F(w^*; v) = \left[\frac{\mathrm{d}}{\mathrm{d}\epsilon} J(\epsilon)\right]_{\epsilon=0}.$$

Also,

$$\delta^2 F(z; v) = \frac{f''(z)v^2}{I_0^2(z)} - \frac{4vf'(z)I_{11}(z, v)}{I_0^3(z)} - \frac{2f(z)I_{22}(z, v)}{I_0^3(z)} + \frac{6f(z)I_{11}^2(z, v)}{I_0^4(z)},$$

where $z = w^* + \xi v$ and $\delta^2 F(z; v) = J''(\xi)$ is the second Gâteaux derivative. Thus, from equation (2.6) we get the problem

$$v_t + v_x = (\lambda - \lambda^*)F(u) + \lambda^* \delta F(w^*; v) + \tfrac{1}{2}\lambda^* J''(\xi), \quad 0 < x < 1, \quad t > t_1, \qquad (2.7\,a)$$
$$v(0, t) = 0, \quad t > t_1, \qquad (2.7\,b)$$
$$v(x, t_1) = u(x, t_1) - w^*(x) \geqslant 0, \quad 0 \leqslant x \leqslant 1 \qquad (2.7\,c)$$

(it is easily seen, due to (1.4), that there exists a $t_1 \in (0, t^*)$ such that $v(x, t) = u(x, t) - w^*(x) > 0$ in $(0, 1]$ for every $t > t_1$).

Now we set $v = u - w^* = \theta\hat{v}$, for $0 < \theta = \lambda - \lambda^* \ll 1$. Thus, problem (2.7) becomes

$$\theta\hat{v}_t + \theta\hat{v}_x = \theta F(u) + \lambda^*\theta\delta F(w^*; \hat{v}) + \tfrac{1}{2}\lambda^* J''(\xi), \quad 0 < x < 1, \quad t > t_1, \qquad (2.8\,a)$$
$$\hat{v}(0, t) = 0, \quad t > t_1, \qquad (2.8\,b)$$
$$\theta\hat{v}(x, t_1) = u(x, t_1) - w^*(x) \geqslant 0, \quad 0 < x < 1. \qquad (2.8\,c)$$

This is simplified to

$$\hat{v}_t + \hat{v}_x = F(u) + \lambda^*\delta F(w^*; \hat{v}) + \tfrac{1}{2}\lambda^*\theta\hat{J}''(\xi), \quad 0 < x < 1, \quad t > t_1,$$

where $J(\xi) = \theta^2\hat{J}(\xi) = \theta^2\delta^2 F(z; \hat{v})$. Now we find a lower solution ψ for $\hat{v}$-problem (2.8). Therefore, we require that $\psi = \psi(x, t)$ satisfies

$$\psi_t + \psi_x \leqslant F(u) + \lambda^*\delta F(w^*; \psi) + \tfrac{1}{2}\lambda^*\theta\delta^2 F(z; \psi). \qquad (2.9)$$

Setting $\psi(x, t) = A(t)\phi^*(x)$ and $\dot{A}(t) = (\mathrm{d}/\mathrm{d}t)A(t)$, we obtain

$$\dot{A}(t)\phi^* + A(t)\phi^{*\prime} - \lambda^* A(t)\delta F(w^*; \phi^*) \leqslant F(u) + \tfrac{1}{2}\lambda^*\theta\delta^2 F(z; \psi). \qquad (2.10)$$

Using ϕ^*-problem (2.4), equation (2.10) becomes

$$\dot{A}(t)\phi^* \leqslant F(u) + \tfrac{1}{2}\lambda^*\theta\delta^2 F(z; \psi). \qquad (2.11)$$

Owing to relation (1.4) and the fact that u blows up as $t \to t^*-$, we see that there exists a β such that $F(u) > \beta A(t)\phi^*(x) > 0$, for some $t \in [t^* - \varepsilon, T^*(A))$, where $\varepsilon > 0$ and $T^* = T^*(A) \geqslant t^*$ is the maximum time of existence (the blow-up time) of $A(t)$. Therefore, it is sufficient to consider, for $t > \tau_1 = \max\{t_1, t^* - \varepsilon\}$,

$$\dot{A}(t)\phi^*(x) \leqslant \beta A(t)\phi^*(x) + \tfrac{1}{2}\lambda^*\theta A^2(t)\delta^2 F(z; \phi^*). \qquad (2.12)$$

For $0 < \theta \ll 1$, i.e. λ close to λ^*, we can find $\beta_1 > 0$, so that we get

$$\dot{A}(t)\phi^* \leqslant \beta_1 A(t)\phi^* \leqslant \beta A(t)\phi^* + \tfrac{1}{2}\lambda^* \theta A^2(t)\delta^2 F(z; \phi^*), \quad t > \tau_1. \qquad (2.13)$$

Taking c small enough so that $\theta \leqslant c/A(t)$ (for some fixed θ we choose c, and c_1 (see below), so that $\theta \leqslant c/A(t)$, where c is approximately the time that u is smaller than order one, i.e. $u(x,t)$ is bounded, $A(\tau)\phi^*\theta + w^* \leqslant u(x,t)$ and $0 < t^* - \tau \ll 1$) we have that $A(t) \leqslant c_1 \mathrm{e}^{\beta_1 t} \leqslant c/\theta$ with $c_1 = A(\tau_1)\mathrm{e}^{-\beta_1 \tau_1}$ and this holds for time

$$t = \tau = \frac{1}{\beta_1} \ln\left(\frac{c}{\theta c_1}\right).$$

We can now obtain an upper estimate T_u^* for $t^*(u)$, $T_u^* = \tau + t_1^* > t^*(u) = t^*$, where t_1^* is the blow-up time of the problem:

$$u_t(x,t) + u_x(x,t) = \lambda \frac{f(u(x,t))}{(\int_0^1 f(u(x,t))\,\mathrm{d}x)^2}, \quad 0 < x < 1, \quad t > \tau,$$

$$u(0,t) = 0, \quad t > \tau, \qquad u(x,\tau) = w^* + c\phi^* \geqslant 0, \quad 0 < x < 1,$$

and $t_1^* \ll \tau$.

2.2. A lower bound for t^*

We take $u_0(x)$ such that $u_0(x) < w^*(x)$ for $0 < x < 1$ and $u_0(0) = w^*(0) = 0$. Let $u^* = u^*(x,t) = u(x,t;\lambda^*)$ be the solution to (1.1) with $u_0^* = u_0$.

In the following we use a similar concept to those in [**17**, **18**]. Therefore, we set $u = u^* + u_1$ and we will prove that $u \leqslant u^* + \psi_1 = w^* - \hat{u} + \psi_1 \leqslant w^* - \psi + \psi_1$, where $\hat{u}$ is given by $\hat{u} = w^* - u^* > 0$ and satisfies (2.14), u_1 solves (2.23) (see below), ψ_1 is an upper solution to the u_1-problem, and ψ is a lower solution to the $\hat{u}$-problem, i.e. $\psi_1 \geqslant u_1$ and $\psi \leqslant \hat{u}$. The $\hat{u}$-problem is defined by

$$\hat{u}_t = -u_t^* = u_x^* - \lambda^* F(u^*) - w^{*\prime} + \lambda^* F(w^*)$$
$$= -\hat{u}_x - \lambda^*(F(u^*) - F(w^*)), \quad 0 < x < 1, \quad 0 < t < T, \qquad (2.14\,a)$$

$$\hat{u}(0,t) = w^*(0) - u^*(0,t) = 0, \qquad 0 < t < T, \qquad (2.14\,b)$$

$$\hat{u}(x,0) = \hat{u}_0(x) = w^*(x) - u_0^*(x), \qquad 0 < x < 1, \qquad (2.14\,c)$$

with $\hat{u}_0 > 0$, hence $\hat{u} > 0$, for $0 < x < 1$, $0 < t < T < t^*$ and for some $T > 0$. We write $J(\epsilon) = F(w^* - \epsilon\hat{u})$, $0 \leqslant \epsilon \leqslant 1$ and examine the difference,

$$F(u^*) - F(w^*) = J(1) - J(0) = J'(0) + \tfrac{1}{2}J''(\xi), \quad 0 < \xi < 1,$$

where

$$J'(0) = \delta F(w^*; -\hat{u}) = -\delta F(w^*; \hat{u}) = \lambda^* \frac{f'(w^*)\hat{u}}{I_0^2(w^*)} - \frac{2\lambda^* f(w^*)I_{11}(w^*, \hat{u})}{I_0^3(w^*)}$$

and

$$J''(\xi) = \delta^2 F(z; -\hat{u}) = \delta^2 F(z; \hat{u})$$
$$= \frac{f''(z)\hat{u}^2}{I_0^2(z)} - \frac{4\hat{u}f'(z)I_{11}(z,\hat{u})}{I_0^3(z)} - \frac{2f(z)I_{22}(z,\hat{u})}{I_0^3(z)} + \frac{6f(z)I_{11}^2(z,\hat{u})}{I_0^4(z)}, \qquad (2.15)$$

with $0 < z = w^* - \xi\hat{u} < w^*$, for some $\xi = \xi(t) \in (0,1)$.

Thus, equations $(2.14\,a)$ and (2.15) give

$$L(\hat{u}) := \hat{u}_t + \hat{u}_x - \lambda^* \delta F(w^*; \hat{u}) + \tfrac{1}{2}\lambda^* \delta^2 F(z; \hat{u}) = 0. \qquad (2.16)$$

Now we introduce the function

$$\psi = \psi(x,t) = \frac{c\phi^*(x)}{t + t_0} + \frac{u_2(x)}{(t + t_0)^2},$$

where c, t_0 (positive constants), $u_2 = u_2(x) \geqslant 0$ are to be determined and $\phi^* = \phi^*(x)$ satisfies problem (2.4). For

$$u^* = w^* - \psi - r = w^* - \frac{c\phi^*(x)}{t + t_0} - \frac{u_2(x)}{(t + t_0)^2} - r \geqslant 0,$$

where

$$r = r(x,t) = \frac{u_3(x)}{(t + t_0)^3} + \frac{u_4(x)}{(t + t_0)^4} + \cdots \quad (r \in \mathbb{R}),$$

since $u^* \to w^* -$ or $(\psi + r) \to 0+$ as $t \to \infty$ $(u_0^*(x) < w^*(x))$, actually it holds: $u^* < w^*$ and $\psi + r > 0$ for all $t \geqslant 0$ [**22**]. Thus, we have

$$u_t^* + u_x^* = \lambda^* F(u^*)$$

or

$$\frac{c\phi^*}{(t + t_0)^2} + w^{*\prime} - \frac{c\phi^{*\prime}}{t + t_0} - \frac{u_2'}{(t + t_0)^2} + \cdots$$
$$= \lambda^* F\left(w^* - \frac{c\phi^*}{t + t_0} - \frac{u_2}{(t + t_0)^2} + \cdots\right)$$
$$= \lambda^* F(w^*) - \lambda^* \frac{c}{t + t_0}\delta F(w^*; \phi^*) - \frac{\lambda^*}{(t + t_0)^2}[\delta F(w^*; u_2) - \tfrac{1}{2}c^2\delta^2 F(w^*; \phi^*)] + \cdots.$$

Equating terms of the same order with respect to powers of $1/(t + t_0)$ and taking into account the ϕ^*-problem, we have

$$(c\phi^* - u_2') = -\tfrac{1}{2}\lambda^* c^2 \delta^2 F(w^*; \phi^*) - \lambda^* \delta F(w^*; u_2). \qquad (2.17)$$

We choose c such that

$$c\int_0^1 \phi^* \, \mathrm{d}x = c^2 \tfrac{1}{2}\lambda^* \left|\int_0^1 \delta^2 F(w^*; \phi^*) \, \mathrm{d}x\right|,$$

which implies that $c = 2/\lambda^* |\int_0^1 \delta^2 F(w^*; \phi^*) \, dx| > 0$, since $c = 0$ is rejected and provided that $\int_0^1 \phi^* \, dx = 1$ (it should be $c > 0$, otherwise we would have $u^* > w^*$ for $t \gg 1$).

Then we need to estimate u_2 by the inequality

$$u_2' \leqslant \lambda^* \delta F(w^*; u_2) + c\phi^* + \tfrac{1}{2}\lambda^* c^2 \delta^2 F(z; \phi^*), \tag{2.18}$$

where $0 < z = w^* - \xi \hat{u} < w^*$, for some $\xi = \xi(t) \in (0, 1)$.

Therefore, by taking $m_0 = \tfrac{1}{2}\lambda^* c^2 \inf_x \{\delta^2 F(z; \phi^*)\}$ we have

$$u_2' \leqslant \lambda^* \left[\frac{f'(w^*)u_2}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*, u_2)}{I_0^3(w^*)} \right] + c\phi^* + m_0. \tag{2.19}$$

Now taking

$$c_1 = \inf_x \left\{ \lambda^* \frac{f'(w^*)}{I_0^2(w^*)} \right\}, \qquad c_2 = \sup_x \left\{ -\frac{2f(w^*)}{I_0^3(w^*)} \right\}$$

and $q = \int_0^1 f'(w^*)u_2 \, dx = I_{11}(w^*, u_2) < 0$, we need only to estimate u_2 by the expression

$$u_2'(x) \leqslant c_1 u_2(x) + c_2 q + c\phi^*(x) + m_0.$$

This implies that we need

$$u_2'(x) \leqslant c_1 u_2(x) + c_2 q + m_1,$$

where $m_1 = c \inf_x \phi^*(x) + m_0 = m_0$; hence,

$$u_2 \leqslant \frac{c_2 q + m_1}{c_1}(e^{c_1 x} - 1)$$

for $u_2(0) = 0$, which is satisfied if

$$q = \frac{c_2 q + m_1}{c_1} \int_0^1 f'(w^*)(e^{c_1 x} - 1) \, dx.$$

Therefore,

$$q = \frac{m_1 m_2}{c_1 - c_2 m_2} \quad \text{for } m_2 = \int_0^1 f'(w^*)(e^{c_1 x} - 1) \, dx,$$

which can be estimated.

Now, substituting ψ, which is known, for $\hat{u}$ in (2.16) and taking into account the ϕ^*-problem and (2.18), we obtain (for the operator L (see (2.16)))

$$L(\psi) = \frac{c}{t + t_0}(\phi^{*\prime} - \lambda^* \delta F(w^*; \phi^*))$$

$$+ \frac{1}{(t + t_0)^2}(u_2' - \lambda^* \delta F(w^*; u_2) - c\phi^* + \tfrac{1}{2}\lambda^* c^2 \delta^2 F(z; \phi^*)) + O\left(\frac{1}{(t + t_0)^3}\right)$$

$$\leqslant 0.$$

The last inequality holds since the term of order $1/(t + t_0)^2$ dominates the term of order $1/(t + t_0)^3$, this due to the u^*-problem, actually since $u^* \to w^*-$ or $(\psi + r) \to 0+$ as $t \to \infty$ and that $\psi + r > 0$ for all $t > 0$.

Requiring $\psi(x,0) \leqslant \hat{u}_0$ and now knowing u_2, since $\hat{u}_0 = w^* - u_0^* = w^* - u_0$, we choose

$$t_0 = \sup_x \frac{c\phi^* + \sqrt{c^2\phi^{*2} + 4\hat{u}_0 u_2}}{2\hat{u}_0} = \sup_x \frac{E(x)}{2\hat{u}_0} < \infty,$$

in $[0,1]$. Provided that $u_0^*(x) = u_0(x) = w^*(x) - \hat{u}_0(x)$ is chosen so that $E(x) = O(\hat{u}_0(x))$ as $x \to 0+$, without loss of generality we may choose $u_0 \to 0+$ appropriately as $x \to 0+$. Then we have that ψ is a lower solution to the $\hat{u}$-problem and thus $\psi \leqslant \hat{u}$.

We now write $u = u^* + u_1 \leqslant w^*$ and find an upper solution to the u_1-problem. The equation for u_1 is

$$u_{1t} = -u_{1x} + (\lambda - \lambda^*)F(w^*) + \lambda(F(u) - F(w^*)) - \lambda^*(F(u^*) - F(w^*)),$$
$$0 < x < 1, \quad t > 0. \quad (2.20)$$

We again examine the difference $\lambda(F(u) - F(w^*))$ and write $v = u - w^*$ $(-w^* < v < 0)$, $J_1(\epsilon) = F(w^* + \epsilon v)$, $0 \leqslant \epsilon \leqslant 1$. We have

$$
\begin{aligned}
\lambda(F(u) - F(w^*)) &= \lambda(J_1(1) - J_1(0)) \\
&= \lambda\left(\frac{f'(w^*)v}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*;v)}{I_0^3(w^*)}\right) + \tfrac{1}{2}\lambda J_1''(\xi_1) \\
&= \lambda^*\delta F(w^*;v) + (\lambda - \lambda^*)\delta F(w^*;v) + \tfrac{1}{2}\lambda\delta^2 F(z;v) \\
&= \lambda^*\delta F(w^*;v) + Q(w^*,z,v), \quad (2.21)
\end{aligned}
$$

where

$$Q(w^*,z,v) = (\lambda - \lambda^*)\delta F(w^*;v) + \tfrac{1}{2}\lambda\delta^2 F(z;v)$$

and

$$
\begin{aligned}
J_1''(\xi_1) = \delta^2 F(z;v) = \frac{1}{I_0^4(z)}[&I_0^2(z)v^2 f''(z) - 4vf'(z)I_{11}(z,v)I_0(z) \\
&- 2f(z)I_0(z)I_{22}(z,v) + 6f(z)I_0(z)I_{11}^2(z,v)],
\end{aligned}
$$

$z = w^* + \xi_1 v$, $\xi_1 = \xi_1(t) \in (0,1)$.

Also, setting $u^* = w^* - \hat{u}$ and $J_2(\epsilon) = F(w^* - \epsilon\hat{u})$, $0 \leqslant \epsilon \leqslant 1$, we may have for the quantity $\lambda^*(F(u^*) - F(w^*))$:

$$
\begin{aligned}
-\lambda^*(F(u^*) - F(w^*)) &= -\lambda^*(J_2(1) - J_2(0)) \\
&= \lambda^*\left(\frac{f'(w^*)\hat{u}}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*,\hat{u})}{I_0^3(w^*)}\right) - \tfrac{1}{2}\lambda^* J_2''(\xi_2) \\
&= \lambda^*\delta F(w^*;\hat{u}) - \tfrac{1}{2}\lambda^*\delta^2 F(\zeta;\hat{u}), \quad (2.22)
\end{aligned}
$$

with

$$
\begin{aligned}
J_2''(\xi_2) = \delta^2 F(\zeta;\hat{u}) = \frac{1}{I_0^4(\zeta)}[&I_0^2(\zeta)\hat{u}^2 f''(\zeta) - 4\hat{u}f'(\zeta)I_{11}(\zeta,\hat{u})I_0(\zeta) \\
&- 2f(\zeta)I_0(\zeta)I_{22}(\zeta,\hat{u}) + 6f(\zeta)I_0(\zeta)I_{11}^2(\zeta,\hat{u})],
\end{aligned}
$$

where $\zeta = w^* - \xi_2 \hat{u}$, $\xi_2 = \xi_2(t) \in (0,1)$.

The u_1-problem (2.20), with relations (2.21), (2.22), now becomes

$$u_{1t} = -u_{1x} + (\lambda - \lambda^*)F(w^*) + \lambda^* \delta F(w^*; v) + Q(w^*, z, v)$$
$$+ \lambda^* \delta F(w^*; \hat{u}) - \tfrac{1}{2}\lambda^* \delta^2 F(\zeta; \hat{u}), \quad 0 < x < 1, \quad t > 0, \qquad (2.23\,a)$$

$$u_1(0,t) = 0, \quad t > 0, \qquad (2.23\,b)$$

$$u_1(x,0) = u_0(x) - u_0^*(x) = 0, \quad 0 < x < 1, \qquad (2.23\,c)$$

where $0 < z < w^*$, $0 < \zeta < w^*$, $0 < \hat{u} < w^*$, $u < u_1 < w^*$ as long as $u < w^*$, so that $Q(w^*, z, v)$, $J_2''(\xi_2)$ are bounded from above and below.

Hence, for a fixed $\lambda > \lambda^*$, there exist some constants B_1, B_2 and $B > 0$ such that

$$Q(w^*, z, v) < B_1 < \tfrac{1}{2}(\lambda - \lambda^*)B, \qquad -\tfrac{1}{2}\lambda^* J_2''(\xi_2) < B_2 < \tfrac{1}{2}(\lambda - \lambda^*)B$$

and $Q - \tfrac{1}{2}\lambda^* J_2'' < (\lambda - \lambda^*)B$. From (2.20)–(2.23) we obtain

$$u_{1t} \leqslant -u_{1x} + (\lambda - \lambda^*)F(w^*) + \lambda^* \left[\frac{f'(w^*)v}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*, v)}{I_0^3(w^*)} \right]$$
$$+ \lambda^* \left[\frac{f'(w^*)\hat{u}}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*, \hat{u})}{I_0^3(w^*)} \right] + (\lambda - \lambda^*)B. \qquad (2.24)$$

Owing to the fact that $u_1 = u - u^* = u - w^* + w^* - u^* = v + \hat{u}$, the previous relation becomes

$$u_{1t} \leqslant -u_{1x} + (\lambda - \lambda^*)F(w^*) + \lambda^* \left[\frac{f'(w^*)u_1}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*, u_1)}{I_0^3(w^*)} \right] + (\lambda - \lambda^*)B.$$

Now we introduce $\psi_1(x,t) = [(\lambda - \lambda^*)\Lambda(t + t_0)]\phi^*(x)$, where Λ is a constant which is determined so that ψ_1 will be an upper solution to the u_1-problem. Here ϕ^* again satisfies problem (2.4).

By substituting $-\psi_1$ for u_1 on the right-hand side of the above relation, we get

$$-\psi_{1x} + (\lambda - \lambda^*)F(w^*) + \lambda^* \left[\frac{f'(w^*)\psi_1}{I_0^2(w^*)} - \frac{2f(w^*)I_{11}(w^*, \psi_1)}{I_0^3(w^*)} \right] + (\lambda - \lambda^*)B$$

$$= (\lambda - \lambda^*)F(w^*) + (\lambda - \lambda^*)B \leqslant (\lambda - \lambda^*)\Lambda\phi^* = \frac{\partial \psi_1}{\partial t},$$

$$F(w^*(x)) + B \leqslant \Lambda\phi^*(x), \quad \text{or} \quad \sup_x F(w^*(x)) + B \leqslant \Lambda\phi^*(x).$$

Since

$$\sup_x F(w^*(x)) = \frac{f(0)}{f^2(w^*(1))},$$

it is sufficient to take

$$\frac{f(0)}{f^2(w^*(1))} + B = \Gamma \leqslant \Lambda\phi^*(x) \quad \text{or} \quad 0 < \Gamma \leqslant \Lambda \inf_{x \in [\gamma, 1]} \phi^*(x) = \Lambda\Theta, \qquad (2.25)$$

for some $\gamma \in (0, 1)$. Also $\psi_1(x, 0) = [(\lambda - \lambda^*)\Lambda t_0]\phi^*(x) > u_1(x, 0) = 0$, and

$$\psi_1(\gamma, t) = [(\lambda - \lambda^*)\Lambda(t + t_0)]\phi^*(\gamma) > u_1(\gamma, t) \geqslant 0. \tag{2.26}$$

Relations (2.25), (2.26) lead us to choose

$$\Lambda = \max\left\{\frac{w^*(\gamma)}{(\lambda - \lambda^*)t_0\phi^*(\gamma)}, \frac{\Gamma}{\Theta}\right\},$$

for some $\gamma > 0$ (such a γ exists since $\phi^*(0) = 0$ and $\phi^*(x) > 0$ in $(0, 1]$). Then, ψ_1 is a 'restricted' upper solution for the u_1-problem in the interval $[\gamma, 1]$ [21]. Here we should note that u blows up globally (see relation (1.4 b)). This means that $u(x, t)$ is bounded for $(x, t) \in [0, \gamma] \times [0, T]$ for some $T < t^*$. In other words, u is bounded in $[\gamma, 1] \times [0, T]$ (actually, here we require $u < w^*$ (see below)) and hence a lower bound for t^* can be found by working in the restricted interval $[\gamma, 1]$.

Hence, $u \leqslant w^*$ as long as $\psi - \psi_1 \geqslant 0$. Thus, we have

$$u = u_1 + u^* \leqslant w^* - \frac{c\phi^*}{t + t_0} - \frac{u_2}{(t + t_0)^2} + (\lambda - \lambda^*)\Lambda(t + t_0)\phi^*.$$

The right-hand side of the above relation is no greater than w^*, as long as $\psi \geqslant \psi_1$ or

$$\frac{c\phi^*}{(t + t_0)} + \frac{u_2}{(t + t_0)^2} \geqslant (\lambda - \lambda^*)\Lambda\phi^*(t + t_0), \quad x \in [\gamma, 1].$$

Hence, $u \leqslant w^*$ as long as $\psi - \psi_1 \geqslant 0$; it is sufficient that $u \leqslant w^*$ for any $x \in [\gamma, 1]$, for some $\gamma > 0$, due to the fact that u blows up globally.

Therefore, for simplicity, since $0 < \lambda - \lambda^* \ll 1$, it is sufficient to choose

$$\frac{c\phi^*}{t + t_0} \geqslant (\lambda - \lambda^*)\Lambda\phi^*(t + t_0), \quad x \in [\gamma, 1],$$

so that

$$c \geqslant (\lambda - \lambda^*)\Lambda(t + t_0)^2.$$

Thus, we obtain

$$t \leqslant c^{1/2}\Lambda^{-1/2}(\lambda - \lambda^*)^{-1/2} - t_0,$$

which, for λ sufficiently close to λ^* ($\lambda > \lambda^*$), gives

$$t \lesssim c^{1/2}\Lambda^{-1/2}(\lambda - \lambda^*)^{-1/2} = t_l(\lambda - \lambda^*)^{-1/2}.$$

Hence, as long as $u = u(x, t) < w^*(x)$, $x \in [\gamma, 1]$, at $t = t_l(\lambda - \lambda^*)^{-1/2}$, we deduce that $t^* > t_l(\lambda - \lambda^*)^{-1/2}$, i.e. $t_l(\lambda - \lambda^*)^{-1/2}$ is a lower bound for t^* with $t_l = c^{1/2}\Lambda^{-1/2}$.

3. Asymptotic estimate of t^* for small $\lambda - \lambda^* > 0$

We now examine the special case $f(s) = e^{-s}$. Motivated by §2, we wish to find an estimate for the blow-up time t^* in problem (1.1) as an asymptotic series of $\eta = \theta^{1/2} = (\lambda - \lambda^*)^{1/2} \ll 1$, $\eta > 0$. We again assume that $u_0(x) < w^*(x)$ for $0 < x \leqslant 1$, with $u_0(0) = w^*(0) = 0$.

Following concepts similar to those in [**17**,**18**], and prompted by numerical calculations, we consider three intervals of time, say I, II and III.

In I and III, t varies by $O(1)$ as $\theta = (\lambda - \lambda^*) \to 0$ and we expand $u \sim u^* + \theta v_1 + \theta^2 v_2 + \cdots$ as $\theta \to 0+$, where u^* satisfies problem (1.1) at $\lambda = \lambda^*$. More precisely, in I we have $u^* < w^*$ since $u_0 = u_0^* < w^*$ in $(0,1]$, while in III we have $u^* > w^*$. Moreover, in I,

$$u^*(x,t) \sim w^*(x) - \frac{2\phi^*(x)}{\lambda^* J_2 t} \quad \text{as } t \to \infty,$$

with $J_2 = \int_0^1 R_2(w^*, \phi^*)\, dx$ and $R_2(w^*, \phi^*) = \delta^2 F(w^*; \phi^*)$, where R_2 follows from the second term of Taylor's expansion (see §2 or below relation (3.1)). This can be obtained by assuming that $u^* \sim w^* + (K/t)\phi^*$ as $t \to \infty$, $u^* < w^*$, and, substituting this expansion in the equation for u^*, we find that $K = -2/\lambda^* J_2$, provided that $\int_0^1 \phi^*(x)\, dx = 1$.

In III ($u^* > w^*$), we again expand u in a similar manner as in I, but now

$$u^*(x,t) \sim w^*(x) - \frac{2\phi^*(x)}{\lambda^* J_2 \hat{t}}$$

as $\hat{t} \to -\infty$ and $t = t_* + \hat{t} > 0$ for some large t_* ($t_* \sim t^* < \infty$ for $\theta \to 0$) and u^* becomes infinite at some finite time $\hat{t}_*$ as $\theta \to 0$.

In II, we expand $u \sim w^* + \eta v_1 + \eta^2 v_2 + \cdots$ as $\eta \to 0$ and, on making a change in timescale $t = \tau/\eta$, equation (1.1) gives

$$\eta^2 v_{1\tau} + \eta^3 v_{2\tau} + \cdots + w_x^* + \eta v_{1x} + \eta^2 v_{2x} + \cdots = \lambda R(\eta) \quad \text{as } \eta \to 0, \qquad (3.1)$$

where

$$F(u) \sim \hat{R}(x,t;\eta) := R(\eta) = \frac{e^{-(w^* + \eta v_1 + \eta^2 v_2 + \cdots)}}{(\int_0^1 e^{-(w^* + \eta v_1 + \eta^2 v_2 + \cdots)})^2\, dx} \quad \text{as } \eta \to 0.$$

We require an expansion for $R(\eta)$ as follows:

$$R(\eta) = R(0) + \eta R'(0) + \tfrac{1}{2}\eta^2 R''(0) + \cdots. \qquad (3.2)$$

From (3.1), (3.2) we obtain

$$\eta^2 v_{1\tau} + \eta^3 v_{2\tau} + \cdots + w_x^* + \eta v_{1x} + \eta^2 v_{2x} + \cdots = (\lambda^* + \eta^2)(R(0) + \eta R'(0) + \tfrac{1}{2}\eta^2 R''(0) + \cdots).$$

As regards the boundary condition $u(0,t) = 0$ at $x = 0$, we have

$$w^*(0) + \eta v_1(0,\tau) + \eta^2 v_2(0,\tau) + \cdots = 0.$$

We equate the terms of zero order ($O(1)$ or $O(\eta^0)$) and obtain

$$w^{*\prime} = \lambda^* R(0), \quad 0 < x < 1, \quad w^*(0) = 0, \tag{3.3}$$

where $R(0) = \mathrm{e}^{-w^*}/(\int_0^1 \mathrm{e}^{-w^*}\,\mathrm{d}x)^2$. Problem (3.3) is actually problem (1.3). By looking now at the terms of $O(\eta)$ we have

$$v_{1x}(x,\tau) = \lambda^* R'(0), \quad 0 < x < 1, \quad \tau > 0, \qquad v_1(0,\tau) = 0, \quad \tau > 0, \tag{3.4}$$

where

$$R'(0) = \delta F(w^*; v_1) = -\frac{\mathrm{e}^{-w^*} v_1}{(\int_0^1 \mathrm{e}^{-w^*}\,\mathrm{d}x)^2} + \frac{2\mathrm{e}^{-w^*}\int_0^1 \mathrm{e}^{-w^*} v_1\,\mathrm{d}x}{(\int_0^1 \mathrm{e}^{-w^*}\,\mathrm{d}x)^3}.$$

Problem (3.4) has the form of problem (2.4), thus we can write

$$v_1(x,\tau) = a(\tau)\phi^*(x), \tag{3.5}$$

where now we normalize ϕ^* according to $\int_0^1 \phi^{*2}(x)\,\mathrm{d}x = 1$ and we denote the normalized ϕ^* again by ϕ^*. Looking next at the $O(\eta^2)$ terms we have

$$v_{1\tau} + v_{2x} = R(0) + \tfrac{1}{2}\lambda^* R''(0),$$

which becomes

$$v_{1\tau} + v_{2x} = \frac{\mathrm{e}^{-w^*}}{S_0^2(\phi^*)} + \frac{2\lambda^*\mathrm{e}^{-w^*} v_1 S_1(v_2)}{S_0^3(\phi^*)} + \frac{2\lambda^*\mathrm{e}^{-w^*} S_1(v_2)}{S_0^3(\phi^*)}$$
$$- \frac{\lambda^*\mathrm{e}^{-w^*} S_2(v_2)}{S_0^3(\phi^*)} + \frac{3\lambda^*\mathrm{e}^{-w^*} S_1^2(v_2)}{S_0^4(\phi^*)}, \tag{3.6}$$

where now we define $I_{\nu k}(w^*, v) = (-1)^\nu S_k(v)$ with $S_k(v) = \int_0^1 \mathrm{e}^{-w^*} v^k\,\mathrm{d}x$, $k = 0, 1, 2, 3$, and $S_k(\phi^*) = S_k$.

Multiplying (3.6) by ϕ^*, integrating over $[0,1]$, using (3.5) and the normalization of ϕ^*, we obtain

$$\dot{a}(\tau) = \int_0^1 \left(-\phi^{*\prime} - \frac{\lambda^*\mathrm{e}^{-w^*}\phi^*}{S_0^2} + \frac{2\lambda^*\mathrm{e}^{-w^*} S_1}{S_0^3}\right) v_2\,\mathrm{d}x + \frac{S_1}{S_0^2}$$
$$+ \lambda^* \frac{a^2(\tau)}{2S_0^4}(S_3 S_0^2 - 6S_1 S_2 S_0 + 6S_1^3). \tag{3.7}$$

Now we look at the equation for ϕ^*, which is

$$\phi^{*\prime} = -\frac{\lambda^*\mathrm{e}^{-w^*}\phi^*}{S_0^2} + \frac{2\lambda^*\mathrm{e}^{-w^*} S_1}{S_0^3}, \quad 0 < x < 1, \quad \phi^*(0) = 0, \tag{3.8}$$

and integrate over $[0,1]$, to obtain

$$\phi^*(1) = \frac{\lambda^* S_1}{S_0^2}. \tag{3.9}$$

Moreover, multiplying (3.6) by ϕ^* and then integrating over $[0, 1]$, we obtain

$$\frac{\phi^{*2}(1)S_0^3}{2\lambda^*} = -S_2 S_0 + 2S_1^2. \tag{3.10}$$

In the same way, but multiplying by ϕ^{*2}, we get

$$\frac{\phi^{*3}(1)S_0^3}{3\lambda^*} = -S_3 S_0 + 2S_2 S_1. \tag{3.11}$$

Now, for the quantity $S = S_3 S_0^2 - 6S_2 S_1 S_0 + 6S_1^3$, after using (3.9) and relations (3.10), (3.11) we obtain

$$S = S_1^3 \left(-2 + 2\frac{\lambda^*}{S_0} - \frac{\lambda^{*2}}{3S_0^2} \right). \tag{3.12}$$

The quantity S is positive as long as the quantity $S_q = -6S_0^2 + 6\lambda^* S_0 - \lambda^{*2}$ is positive. We know the relations $\lambda^* = M^2/\mu$, where $M = \max\{w^*(x)\} = w^*(1)$, for $0 \leqslant x \leqslant 1$ and $\mu = \lambda^*/S_0^2$. Using these relations we find that S_q is positive provided

$$S_q M^2 = \lambda^{*2}(-6 + 6M - M^2)$$

is positive. We have (see [**22**]) that $M \sim 1.5936$ and $\lambda^* \sim 0.6476$. The above quantity, for these values, is positive ($S_q M^2 \sim 0.4286$).

Therefore, equation (3.7) can be written as

$$\dot{a}(\tau) = \frac{S_1}{S_0^2} + \lambda^* \frac{S}{2S_0^4} a^2(\tau), \quad a(\tau) \to -\infty \text{ as } \tau \to 0, \tag{3.13}$$

which has as a solution

$$a(\tau) = \left(\frac{B}{K} \right)^{1/2} \tan[\tau(BK)^{1/2} - \tfrac{1}{2}\pi],$$

for $K = \lambda^* S/2S_0^4$, $B = S_1/S_0^2$ (this choice of initial condition as $\tau \to 0+$ gives constant of integration $-\tfrac{1}{2}\pi$). Returning to the original time variable, this expression becomes

$$A(t) = \left(\frac{B}{K} \right)^{1/2} \tan[t(\lambda - \lambda^*)^{1/2}(BK)^{1/2} - \tfrac{1}{2}\pi].$$

Because $u = w^* + \eta v_1 + \cdots$ and $v_1(x, t) = A(t)\phi^*(x)$, it is obvious that u ceases to exist at time

$$t^* \sim t_b = t_u(\lambda - \lambda^*)^{-1/2},$$

where $t_u = \pi(4BK)^{-1/2}$ and t_b is the blow-up time of $a(\tau) = a(t(\lambda - \lambda^*)^{-1/2}) = A(t)$.

4. Numerical solutions

We solve problem (1.1) by using a two-step upwind scheme. For the linear terms we apply the usual form of the scheme:

$$v_j^{n+1} = u_j^n - r(u_j^n - u_{j-1}^n) + \lambda F(u_j^n),$$

where u_j^n is the temperature at the nth time level and at the jth space grid, $r = \delta t / \delta x$, and the non-local term $F(u_j^n)$ is evaluated at the nth time step. For this term we have

$$F(u_j^n) = \frac{f(u_j^n)}{(\int_0^1 f(u_j^n)\,\mathrm{d}x)^2}.$$

The integral in the denominator is evaluated by Simpson's rule. In the next step we evaluate w:

$$w_j^{n+1} = u_j^n - r(v_j^{n+1} - v_{j-1}^{n+1}) + \lambda F(v_j^{n+1}).$$

Finally, u at the $(n+1)$th time step is approximated by

$$u_j^{n+1} = \tfrac{1}{2}(v_j^{n+1} + w_j^{n+1}).$$

In Figure 2 we use this scheme to solve the problem numerically for $f(u) = \mathrm{e}^{-u}$ and on taking $u(x, 0) = 0$. We see that for $\lambda < \lambda^*$ the solution u tends to a steady state, for $\lambda = \lambda^*$ the behaviour is similar, and for $\lambda > \lambda^*$ the solution blows up (the decay is faster for $\lambda < \lambda^*$ than it is for $\lambda = \lambda^*$). More precisely, in Figure 2 the maxima of solutions are plotted against time.

In Figure 3, we plot the numerical solution of u for $\lambda = 0.5476$.

5. Upper bounds of t^* for sufficiently large $u_0(x)$ and $\lambda \leqslant \lambda^*$

In this section we study the case of $\lambda \leqslant \lambda^*$, where the steady-state problem (1.3) has two (bounded) solutions $w_1(x)$, $w_2(x)$ with $w_2(x) > w_1(x)$ ($w_1 = w_2 = w^*$ at $\lambda = \lambda^*$) for $x \in (0, 1]$ (see Figure 1c). Following concepts similar to those in the case of the upper bound for t^* when $\lambda > \lambda^*$, we may proceed in one of two ways: either (i) as in [17] or (ii) as in §2.

5.1. First method (as in [17])

Again we set $v(x, t) = u(x, t) - w_2(x)$, $0 \leqslant x \leqslant 1$, $t > 0$, with $v_0(x) = u_0(x) - w_2(x) > 0$, $0 < x \leqslant 1$ and $v_0(0) = u_0(0) - w_2(0) = 0$. The function $w_2(x)$ is the greatest steady-state solution to problem (1.3), while $u(x, t)$ is the unique solution to problem (1.1) which blows up globally in finite time, and (1.4) holds. Therefore, $v(x, t) \to \infty$ and $F(u) = F(v + w_2) \to \infty$ as $t \to t^*-$.

Also,

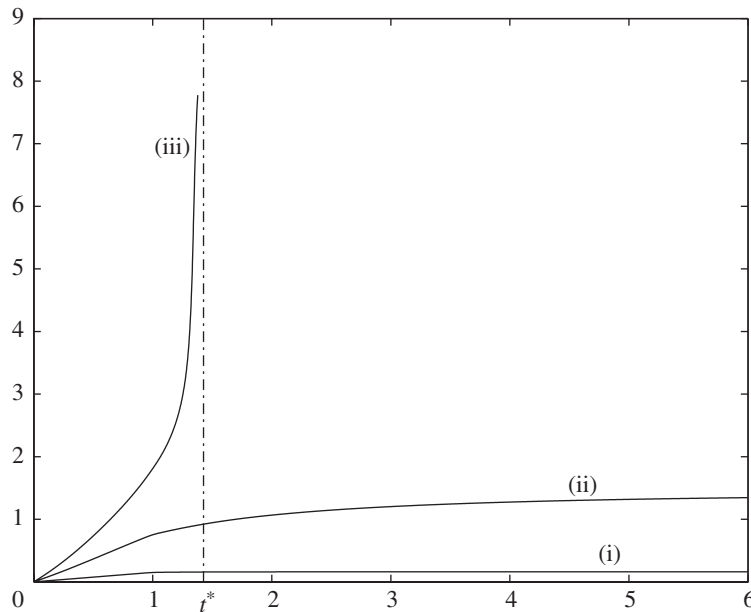$$F(u) - F(w_2) = F(v + w_2) - F(w_2) = \delta F(w_2; v) + \tfrac{1}{2}\delta^2 F(s; v).$$

Figure 2. Numerical solution to problem (1.1). We plot $\max_x(u(x,t)) = u(1,t) = M(t)$, for x in $[0,1]$, against time for $\delta x = 0.033$, $\delta t = 0.002$ (the upper curve, (iii), corresponds to $\lambda = 1.1476 > \lambda^* = 0.6476$, the intermediate curve, (ii), to $\lambda = \lambda^*$, and the lower one, (i), to $\lambda = 0.1476 < \lambda^*$). Also the dash–dotted line corresponds to the asymptotic estimate of the blow-up time $t^* \sim 1.3367$ for $\lambda = 1.1476$. To obtain this estimate we calculate w^* numerically by an iteration scheme and then we solve the equation for ϕ^* using the appropriate normalization.

Then v satisfies the problem

$$v_t + v_x = \lambda\delta F(w_2;v) + \tfrac{1}{2}\lambda\delta^2 F(s;v), \quad 0 < x < 1, \quad t > 0, \qquad (5.1\,a)$$

$$v(0,t) = 0, \quad t > 0, \qquad (5.1\,b)$$

$$v_0 = v(x,0) = u(x,0) - w_2(x) \geqslant 0, \quad 0 \leqslant x \leqslant 1, \qquad (5.1\,c)$$

where $v_0 = 0$ only at $x = 0$, $s = s(x,t) = w_2 + \xi v$, $\xi = \xi(t) \in (0,1)$, $F(w_2 + \xi v) := L(\xi)$ and $\delta^2 F(s,v) := L''(\xi)$. Moreover, $\delta^2 F(s;v) \to \infty$ as $t \to t^*-$, otherwise $v(x,t)$ would not blow up (in finite time).

Now we introduce the function Ψ, for which we need to satisfy

$$\Psi_t + \Psi_x \leqslant \lambda\delta F(w_2;\Psi) + \tfrac{1}{2}\lambda\delta^2 F(z;\Psi) = \lambda(F(w_2 + \Psi) - F(w_2)), \qquad (5.2)$$

where $\Psi(x,t) = A(t)\phi(x)$, $\phi(x) = \phi(x;\rho)$, $z = w_2 + \epsilon\Psi$, $\epsilon = \epsilon(t) \in (0,1)$, $J(\epsilon) = F(w_2 + \epsilon\Psi)$, $J''(\epsilon) = \delta^2 F(z;\Psi)$, $\phi(x)$ satisfies problem (2.2) and $A(t)$ satisfies problem (5.7) (see below). We require the function Ψ to be a lower solution to problem (5.1) which also blows up, due to the blow-up of $A(t)$ at T^*; since $\Psi \leqslant v$, we have $T^* = T^*(A) \geqslant t^*(v) = t^*(u) = t^*$, and actually this blow-up is global because of the form of Ψ.

Setting $\Psi(x,t) = A(t)\phi(x)$ in relation (5.2) we obtain

$$\dot{A}(t)\phi + A(t)\phi' - \lambda\delta F(w_2;\phi)A(t) \leqslant \tfrac{1}{2}\lambda\delta^2 F(z;\Psi). \qquad (5.3)$$
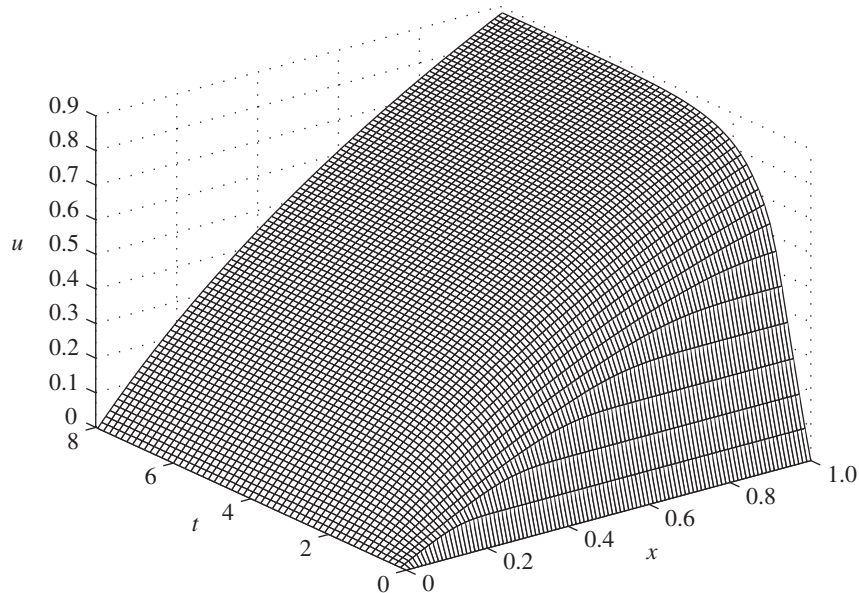
Figure 3. Numerical solution to problem (1.1) for $\lambda = 0.5476 < \lambda^*$.

Also, by the same argument as for v, we have that $J''(\epsilon) = \delta^2 F(z; \Psi) \to \infty$, $\Psi \to \infty$ as $t \to T^*-$, otherwise $\Psi(x, t)$ would not blow up. Then, for $J''(\epsilon)$, we have

$$
\begin{aligned}
J''(\epsilon) &= \Psi^2 \left[\frac{f''(z)}{I_0^2(z)} - \frac{4f'(z)I_{11}(z, \Psi)}{\Psi I_0^3(z)} - \frac{2f(z)I_{22}(z, \Psi)}{\Psi^2 I_0^3(z)} + \frac{6f(z)I_{11}^2(z, \Psi)}{\Psi^2 I_0^4(z)} \right] \\
&= \Psi^2 \left[\frac{f''(z)}{I_0^2(z)} - \frac{4f'(z)I_{11}(z, \phi)A}{A\phi I_0^3(z)} - \frac{2f(z)I_{22}(z, \phi)A^2}{A^2\phi^2 I_0^3(z)} + \frac{6f(z)I_{11}^2(z, \phi)A^2}{A^2\phi^2 I_0^4(z)} \right] \\
&= \Psi^2 \Gamma(x, t) \\
&= A^2(t)\phi(x)\Gamma_1(x, t), \quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad\quad (5.4)
\end{aligned}
$$

where $\Gamma_1(x, t) = \Gamma(x, t)\phi(x)$. Since Ψ blows up globally, we have that $F(w_2+\Psi)-F(w_2) \sim F(w_2 + \Psi) \sim F(\Psi) \to \infty$ as $t \to T^*-$ for every $x \in (0, 1]$.

Now we impose an extra condition on the function f,

$$
f(s) \leqslant \frac{c}{s^2}, \quad c > 0, \quad \text{for } s \gg 1, \quad\quad\quad\quad\quad\quad (5.5)
$$

and we have the following lemma.

Lemma 5.1. *Let f satisfy (1.2) and (5.5), then $J''(\epsilon) = \delta^2 F(z; \Psi) \geqslant \Lambda A^2(t)\phi(x) > 0$ for $\Psi \gg 1$, or $A \gg 1$, where Λ is a positive constant.*

Proof. Since Ψ blows up globally, $\delta^2 F(z; \Psi) \to \infty$ as $\Psi \to \infty$. If we assume that the conclusion of the lemma does not hold, then owing to (5.3) we have $\delta^2 F(z; \Psi) \lesssim K_0 A^p(t)\phi(x)$ as $A(t) \to \infty$ or as $t \to T^*-$, for some $1 < p < 2$ and $K_0 > 0$. Otherwise Ψ

does not blow up. The latter, the fact that $F(w_2 + \Psi) - F(w_2) \sim F(\Psi)$, for $\Psi \gg 1$, and (5.4) give

$$\begin{aligned}
F(w_2 + \Psi) - F(w_2) &= \delta F(w_2; \Psi) + \tfrac{1}{2}\delta^2 F(z; \Psi) \\
&= \delta F(w_2, \phi)A(t) + \tfrac{1}{2}\Gamma(x,t)\Psi^2 \\
&\sim F(\Psi) \\
&\lesssim K_0 A^p \phi, \quad \text{for } A \gg 1 \text{ and } 1 < p < 2.
\end{aligned}$$

Also, by using the mean-value theorem we obtain

$$F(\Psi) = \frac{f(\Psi(x,t))}{(\int_0^1 f(\Psi(x,t))\,\mathrm{d}x)^2} = \frac{f(A(t)\phi(x))}{f^2(A(t)\phi(\xi))},$$

where $\xi = \xi(t) \in (0,1)$. Then, due to the continuity of $\phi(x)$, we can always find $x_0 = x_0(t) \in (0,1)$ such that $0 < \phi(x_0(t)) < \min\{\phi(\xi(t)), \phi^2(\xi(t))\}$. This implies that

$$\frac{f(A(t)\phi(x_0(t)))}{f(A(t)\phi(\xi(t)))} > 1,$$

and, as $t \to T^*-$, we have

$$F(\Psi(x_0,t)) = \frac{f(A(t)\phi(x_0))}{f^2(A(t)\phi(\xi))} > \frac{1}{f(A(t)\phi(\xi))},$$

and

$$\frac{1}{f(A(t)\phi(\xi))} < \frac{f(A(t)\phi(x_0))}{f^2(A(t)\phi(\xi))} = F(\Psi(x_0,t)) \lesssim K_0 A^p(t)\phi(x_0).$$

Then, by using the previous relation and condition (5.5), we obtain

$$K_0 A^p(t)\phi(x_0(t)) \gtrsim \frac{1}{f(A(t)\phi(\xi(t)))} \geqslant \frac{A^2(t)\phi^2(\xi(t))}{c} \quad \text{as } t \to T^*.$$

The above inequality implies that

$$K_0 c \gtrsim A^{2-p}(t)\frac{\phi^2(\xi(t))}{\phi(x_0(t))} > A^{2-p}(t)$$

for t close to T^*, but $A^{2-p}(t) \to \infty$ as $t \to T^*$ for $1 < p < 2$, while $K_0 c < \infty$, which is a contradiction. $\qquad\square$

Owing to Lemma 5.1, relation (5.4) now becomes

$$J''(\epsilon) = \Gamma(x,t)\Psi^2 \geqslant \Lambda A^2(t)\phi(x) \quad \text{as } t \to T^*-, \quad x \in (0,1]. \tag{5.6}$$

From relation (5.6), it is sufficient to take

$$\dot{A}(t)\phi + A(t)[\phi' - \lambda\delta F(w_2; \phi)] \leqslant K_1 A^2(t)\phi, \quad t > 0, \quad A(0) = A_0,$$

where $K_1 = \frac{1}{2}\lambda\Lambda$. Now, by using problem (2.2) and Lemmas 2.2 and 5.1 we obtain

$$\dot{A}(t) - A(t)\rho_2 \leqslant K_1 A^2(t), \quad t > 0, \quad A(0) = A_0, \quad \rho_2 \geqslant 0.$$

Thus, we can take $A(t)$ in order to satisfy

$$\dot{A}(t) = \rho A(t) + K A^2(t), \quad t \in [0, T^*), \quad A(0) = A_0, \tag{5.7}$$

with $K \leqslant K_1$ and $\rho = \rho_2 \geqslant 0$.

We also take

$$\psi(x,0) = A(0)\phi(x) \leqslant v(x,0) = u_0(x) - w_2(x) \quad \text{or} \quad A_0 = A(0) \leqslant \frac{u_0(x) - w_2(x)}{\phi(x)};$$

hence, we choose

$$A_0 = \inf_x \frac{u_0(x) - w_2(x)}{\phi(x)} = \inf_x \frac{E(x)}{\phi(x)} > 0$$

(this infimum exists provided that $u_0(x)$, $u_0'(x)$ are bounded in $[0,1]$). Moreover, $\Psi(0,t) = v(0,t)$. Finally, Ψ is a lower solution to the v-problem (5.1).

Now problem (5.7) for $A(t)$ and $\rho > 0$ gives

$$A(t) = \left[-\frac{K}{\rho} + \left(A_0^{-1} + \frac{K}{\rho} \right) e^{-\rho t} \right]^{-1},$$

where K is a constant depending on c and λ, while A_0 depends on the infimum of the difference $(u_0(x) - w_2(x))/\phi(x)$.

Then the blow-up time T^* for $A(t)$ is

$$T^* = \frac{1}{\rho} \ln\left(\frac{\rho}{K} A_0^{-1} + 1 \right), \quad \text{for } \rho > 0.$$

While, for $\rho = 0$, problem (5.7) gives $T^* = (K A_0)^{-1}$. Since Ψ is a lower solution of the v-problem (5.1), we obtain

$$t^* = t^*(u) = t^*(v) \leqslant t^*(\Psi) = T^*,$$

hence T^* is an upper bound for $t^*(u)$.

5.2. Second method (as in § 2)

This method works only if $\rho = \rho_2 > 0$ or if $\rho = \rho_2 = 0$ and $\delta^2 F(z; \phi) > 0$. Now let $u(x,t) = u(x,t; u_0)$ be the solution to problem (1.1) for $\lambda < \lambda^*$ and assume, for simplicity, initial data of the form $u_0(x) = \sigma w_2(x)$, with $\sigma > 1$. Then $u_0(x) - w_2(x) = (\sigma - 1)w_2(x) = \zeta w_2(x)$, with $\zeta = \sigma - 1 > 0$. Also we have that $u \to w_2+$ as $\zeta \to 0+$. Again we set $v(x,t) = u(x,t) - w_2(x)$, $0 \leqslant x \leqslant 1$, $t > 0$, with $v_0(x) = u_0(x) - w_2(x) > 0$,

$0 < x \leqslant 1$, and $v_0(0) = u_0(0) - w_2(0) = 0$. We call $v = u - w_2 = \zeta\hat{v}$, thus problem (5.1) becomes

$$\zeta\hat{v}_t + \zeta\hat{v}_x = \lambda\zeta\delta F(w_2; \hat{v}) + \tfrac{1}{2}\lambda J''(\xi), \quad 0 < x < 1, \quad t > 0, \tag{5.8a}$$

$$\hat{v}(0, t) = 0, \quad t > 0, \tag{5.8b}$$

$$\hat{v}_0(x) = \hat{v}(x, 0) = \frac{u_0(x) - w_2(x)}{\zeta} > 0, \quad 0 < x < 1. \tag{5.8c}$$

This simplifies to

$$\hat{v}_t + \hat{v}_x = \lambda\delta F(w_2; \hat{v}) + \tfrac{1}{2}\lambda\zeta\hat{J}''(\xi), \quad 0 < x < 1, \quad t > 0,$$

where $J(\xi) = \zeta^2\hat{J}(\xi) = \zeta^2\delta^2 F(z; \hat{v})$. Now we find a lower solution ψ for the $\hat{v}$-problem. Therefore, we require $\psi = \psi(x, t)$ to satisfy

$$\psi_t + \psi_x \leqslant \lambda\delta F(w_2; \psi) + \tfrac{1}{2}\lambda\zeta\delta^2 F(z; \psi). \tag{5.9}$$

Setting $\psi = A(t)\phi(x)$ and $\dot{A}(t) = (\mathrm{d}/\mathrm{d}t)A(t)$, we obtain

$$\dot{A}(t)\phi + A(t)(\phi' - \lambda\delta F(w_2; \phi)) \leqslant \tfrac{1}{2}\lambda\zeta\delta^2 F(z; \psi). \tag{5.10}$$

Using the ϕ-problem (2.2) and Lemma 2.2, equation (5.10) becomes

$$\dot{A}(t)\phi - \rho_2\phi A(t) \leqslant \tfrac{1}{2}\lambda\zeta\delta^2 F(z; \psi), \quad \rho_2 \geqslant 0. \tag{5.11}$$

Thus, we have, on setting $\rho_2 = \rho$,

$$\dot{A}(t)\phi(x) \leqslant \rho A(t)\phi(x) + \tfrac{1}{2}\lambda\zeta A^2(t)\delta^2 F(z; \phi), \quad 0 < x < 1, \quad t > 0. \tag{5.12}$$

For $0 < \zeta \ll 1$, in both cases, i.e. either $\rho = \rho_2 > 0$, or $\rho = \rho_2 = 0$ and $\delta^2 F(z; \phi) > 0$, we can find $\beta > 0$, so that we get (for some τ_1 close to T^* and $\tau_1 < T^*$)

$$\dot{A}(t)\phi \leqslant \beta A(t)\phi \leqslant \rho A(t)\phi + \tfrac{1}{2}\lambda\zeta A^2(t)\delta^2 F(z; \phi), \quad t > \tau_1 > 0.$$

Now, taking c small enough so that $\zeta \leqslant c/A(t)$ (c is approximately the time that u becomes smaller than order one) we have that $A(t) \leqslant c_1\mathrm{e}^{\beta t} \leqslant c/\zeta$ with $c_1 = A(\tau_1)\mathrm{e}^{-\beta\tau_1}$ and this holds for time

$$t = \tau = \frac{1}{\beta}\ln\left(\frac{c}{\zeta c_1}\right).$$

Now we can obtain an upper estimate T_u^* for $t^*(u)$ which is $T_u^* = \tau + t_1^* > t^* = t^*(u)$, where t_1^* is the blow-up time of the problem

$$u_t(x, t) + u_x(x, t) = \lambda\frac{f(u(x, t))}{(\int_0^1 f(u(x, t))\,\mathrm{d}x)^2}, \quad 0 < x < 1, \quad t > \tau,$$

$$u(0, t) = 0, \quad t > \tau, \qquad u(x, \tau) = w_2(x) + c\phi(x) \geqslant 0, \quad 0 < x < 1,$$

and $t_1^* \ll \tau$.

6. Discussion

In the present work, we estimate the blow-up time t^* of the solution to problem (1.1). It is useful, from the point of view of applications, to know the time at which the temperature u becomes infinity. In our model, this is the time that the food is burnt. Similar estimates are also known for local (the reaction–diffusion problem) as well as for non-local (the ohmic heating problem) problems [**17**,**18**]. Here the results are obtained for the case where there exists a steady-state solution $w^* = w(x; \lambda^*)$ at $\lambda = \lambda^*$, either for $0 < \lambda - \lambda^* \ll 1$, with non-negative initial data, or for $0 < \lambda \leqslant \lambda^*$ with initial data greater than the greatest steady-state solution. The methods applied are comparison and asymptotic techniques, as well as numerical computations. Although the asymptotic estimate of t^* is found for $f(s) = \mathrm{e}^s$, it is possible to estimate it for a general function f satisfying (1.2).

Our main estimates, for given λ, λ^* and $0 < \lambda - \lambda^* \ll 1$, are upper bound $\epsilon + c_1 \ln[c_2(\lambda - \lambda^*)^{-1}]$, lower bound $c_3(\lambda - \lambda^*)^{-1/2}$, and asymptotic estimate $t^* \sim c_4(\lambda - \lambda^*)^{-1/2}$ as $\lambda \to \lambda^*+$. For $0 < \lambda \leqslant \lambda^*$ and given initial data $u_0(x)$ greater than the greatest steady-state solution $w_2(x)$, we have the upper estimates $c_5 \ln(c_6 A_0^{-1} + 1)$ or $\epsilon + c_7 \ln(c_8 \zeta^{-1})$, where A_0, ζ are measures of the difference $u_0 - w_2$ (if $u_0 \to w_2+$, then $A_0, \zeta \to 0+$), the c_i are constants, and $0 < \epsilon \ll 1$. Some numerical results are also represented in § 4.

It still remains an open question how to estimate t^* when there is no regular solution w^* at $\lambda = \lambda^*$.

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