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Boundedness of the q-Mean-Square Operator on Vector-Valued Analytic Martingales

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Abstract. We study boundedness properties of the q-mean-square operator $S^{(q)}$ on E-valued analytic martingales, where E is a complex quasi-Banach space and $2 \le q < \infty$. We establish that a.s. finiteness of $S^{(q)}$ for every bounded E-valued analytic martingale implies strong (p,p)-type estimates for $S^{(q)}$ and all $p \in (0,\infty)$. Our results yield new characterizations (in terms of analytic and stochastic properties of the function $S^{(q)}$) of the complex spaces E that admit an equivalent q-uniformly PL-convex quasi-norm. We also obtain a vector-valued extension (and a characterization) of part of an observation due to Bourgain and Davis concerning the L^p-boundedness of the usual square-function on scalar-valued analytic martingales.

0 Introduction

Recently complex analogues of the (real) notion of uniform convexity have been studied for complex Banach or quasi-Banach spaces, following the fundamental paper by Davis, Garling and Tomczak-Jaegermann [DGT]. These geometric notions have been related to inequalities and convergence properties of special classes of complex martingales in [DGT], as well as by Bourgain and Davis [BD], Garling [G1], Haagerup and Pisier [HP] and Xu [X1], [X2]. Other applications of complex uniform convexity are contained in [D], [G2] and [M]. There are many further results linking properties of vector-valued martingales to the geometry of Banach spaces, see for instance [DU, chapter V], [B2], [B3], [Ed1], [Ed2] and [P].

Let E be a complex locally PL-convex quasi-Banach space with a quasi-norm $\|\cdot\|$ that is uniformly continuous on bounded subsets. This note studies some boundedness properties of the q-mean-square operator $S^{(q)}$ on E-valued analytic martingales (the relevant concepts are introduced below in Section 1). Among other things we establish that a.s. finiteness of $S^{(q)}(x)$ for every uniformly bounded E-valued analytic martingale x implies strong (p,p)-type estimates for $S^{(q)}$ with arbitrary $p \in (0,\infty)$ (we refer to Theorems 3 and 5 for further equivalent reformulations). These results provide an analogue for the q-mean-square operator of the equivalent conditions of Burkholder [B2, Thm. 1.1] concerning Banach space-valued martingale transforms (see also [BD, Thm. 1.1]).

Our results are connected with some properties due to Davis *et al.* [DGT] of the uniform versions of complex convexity of quasi-Banach spaces (see Section 1 below for the definition of q-uniform PL-convexity). They established renorming results connecting the L^p-boundedness of the q-mean-square function on restricted classes of vector-valued complex martingales with the existence of an equivalent q-uniformly PL-convex (quasi-) norm.

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Their results provide analogues for complex uniform convexity of the remarkable renorming theorems due to Enflo [E] and Pisier [P] for (real) uniformly convex Banach spaces.

Our results yield, in combination with [DGT, Section 5], further characterizations of the existence of an equivalent q-uniformly PL-convex quasi-norm. For instance, it follows that E admits such an equivalent quasi-norm if and only if $S^{(q)}(x) < \infty$ almost surely on the set $\{x^* < \infty\}$ for every E-valued analytic martingale x. Theorems 3 and 5 contain additional equivalent conditions. At the same time we get a vector-valued generalization (in fact a kind of characterization) to complex quasi-Banach spaces, for the full range of p, of part of an observation due to Bourgain and Davis [BD, Prop. 4.1] concerning the L^p-boundedness of the usual square-function on scalar-valued analytic martingales. In addition, our results also yield information related to Problem 6 of [DGT, p. 135] in the case of continuously quasi-normed spaces E: it follows that the existence of an equivalent q-uniformly PL-convex norm on E is independent of the integrability exponent p.

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1 Preliminaries and Notation

Let E be a complex continuously quasi-normed (*i.e.*, the quasi-norm is uniformly continuous on bounded subsets of E) quasi-Banach space with quasi-norm $\|\cdot\|$. Thus the triangle inequality holds in the form $\|x+y\| \leq K(\|x\|+\|y\|)$ for some constant $K \geq 1$ and for all $x, y \in E$. Recall also that for a quasi-norm with constant $K \geq 1$ there is an exponent $\rho = \rho(K)$, $0 < \rho \leq 1$, and an equivalent quasi-norm $\|\cdot\|_*$ so that $\|x+y\|_*^\rho \leq \|x\|_*^\rho + \|y\|_*^\rho$ for any $x, y \in E$ (see [K, Prop. 1.c.5]). For our purposes we can renorm E by $\|\cdot\|_*$, if necessary, and hence we may and will henceforth assume that $\|\cdot\|^\rho$ already satisfies the triangle inequality. It is useful to note that this implies the more general estimate $\|\sum_k x_k\|^\rho \leq \sum_k \|x_k\|^\rho$.

Recall that a complex continuously quasi-normed quasi-Banach space E is called locally PL-convex if for any $x, y \in E$ there is $\delta = \delta(x, y) > 0$ so that

$$\|x\| \leq \frac{1}{2\pi} \int_0^{2\pi} \|x + re^{i\theta} y\| \, d\theta$$

for all $0 < r \leq \delta$. This is satisfied by complex Banach spaces as as well as by the quasi-Banach spaces $L^p(\mu)$ with $0 < p < 1$ (*cf.* [DGT, Prop. 2.3])

Uniform PL-convexity and q-uniform PL-convexity strengthen local PL-convexity. To describe these notions, let $0 < p < \infty$ and define the PL-convexity moduli of E by

$$H_p^E(\varepsilon) = \inf\left\{ \left(\frac{1}{2\pi} \int_0^{2\pi} \|x + e^{i\theta} y\|^p d\theta \right)^{1/p} - 1 : \|x\| = 1, \|y\| = \varepsilon \right\}. \tag{1.1}$$

The space E is said to be uniformly PL-convex if there is some $p \in (0, \infty)$ such that $H_p^E(\varepsilon) > 0$ for all $\varepsilon > 0$. Further refinements are obtained in [DGT] by prescribing growth of the type $H_p^E(\varepsilon) \geq c\varepsilon^q$ near 0 for some $2 \leq q < \infty$. Thus E is q-uniformly PL-convex, if there is $p \in (0, \infty)$ and a constant $d > 0$ so that

$$\left(\frac{1}{2\pi}\int_0^{2\pi} \|x + e^{i\theta}y\|^p \, d\theta\right)^{1/p} \geq (\|x\|^q + d\|y\|^q)^{1/q} \tag{1.2}$$

holds for all $x, y \in E$. These notions of complex uniform convexity are in fact independent of the exponent $p \in (0, \infty)$, see [DGT, 2.4]. As an example, $L_p(0, 1)$ is 2-uniformly PL-convex for $0 < p \leq 2$ and p-uniformly PL-convex for $2 \leq p < \infty$ (see [DGT, 4.2]).

We next introduce the class of complex E-valued martingales considered in this work. Let $\Omega = [0, 2\pi]^{\mathbf{N}}$, suppose that Σ is the product σ-field and P the product measure of the normalized Lebesgue measures. We take the standard filtration (Σ_n) on Ω, where Σ_n stand for the σ-field generated by the first n coordinates $\theta_1, \theta_2, \ldots, \theta_n$ of $\theta = (\theta_k) \in \Omega$. We will say (following [BD, Section 4] and [Ed2]) that a sequence $x = (x_n)$ is an E-valued analytic martingale if

$$x_n = x_0 + \sum_{j=1}^{n} v_j(\theta_1, \ldots, \theta_{j-1})e^{i\theta_j}, \tag{1.3}$$

where x_0 is a constant and the vector-valued functions $v_n \colon \Omega \to E$ are Σ_{n-1}-measurable (with the convention that $\Sigma_0 = \{\varnothing, \Omega\}$). Note that (x_n) is an E-valued martingale in the usual sense in the case where E is a Banach space, and that $(\|x_n\|)$ is a submartingale if E is locally PL-convex (see Lemma 1 below). The additional assumption that $v_n \in L^p(E)$ for all n leads essentially to the class of H_p-shrubs considered in [DGT, Section 5]. We refer to [DGT] for background and a careful motivation of this notion. Various other classes of vector-valued analytic martingales have been used for different purposes in (for instance) [BD], [Ed1], [Ed2], [G1], [G2] and [X2].

The q-mean-square operator $S^{(q)}$ is defined by

$$S^{(q)}(x) = \left(\|x_0\|^q + \sum_{n=1}^{\infty} \|v_n\|^q\right)^{1/q} \tag{1.4}$$

for any E-valued analytic martingale $x = (x_n)$ as in (1.3). Thus $S^{(q)}(x)$ is a random variable on Ω.

We will employ standard notation related to (quasi-)Banach spaces and probability. Let $x = (x_n)$ be an E-valued martingale. The corresponding martingale difference sequence of x is denoted by (dx_n), where $dx_n = x_n - x_{n-1}$ for $n \geq 1$ and $dx_0 = x_0$. Let $x_n^* = \sup_{j \leq n} \|x_j\|$ and $x^* = \sup_j \|x_j\|$ stand for the maximal functions. Moreover, set $\|x\|_p = \sup_j \|x_j\|_p$, where $\|x_j\|_p = (\mathbf{E}\|x_j\|^p)^{1/p}$ is the L^p-norm with $0 < p \leq \infty$.

2 Boundedness of $S^{(q)}$ on Analytic Martingales

In this section E will always stand for a complex continuously quasi-normed, locally PL-convex quasi-Banach space. We reserve K for the constant and ρ for the exponent in

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the triangle inequalities for E, respectively. In general considerable care need to be exercised (compared to the Banach case) with martingales taking their values in quasi-normed spaces. We commence by verifying the following auxiliary results in the quasi-normed case.

Lemma 1 *Let E be a complex continuously quasi-normed, locally PL-convex quasi-Banach space and let $x = (x_n)$ be an E-valued analytic martingale. Then for each $p \in (0, \infty)$ there exists $c_p > 0$ such that*

$$\|x^*\|_p \leq c_p \|x\|_p.$$

Proof By the definition of local PL-convexity, the norm $\|\cdot\|$ is plurisubharmonic and equivalently the same holds for $\|\cdot\|^\alpha$ with arbitrary $\alpha > 0$. For this fact we refer to [A, Thm. 1.1, p. 40]. Hence, taking α with $0 < \alpha < p$ we obtain

$$\|x_{n-1}\|^\alpha \leq \frac{1}{2\pi} \int_0^{2\pi} \|x_{n-1} + v_n(\theta_1, \ldots, \theta_{n-1}) e^{i\theta_n}\|^\alpha \, d\theta_n$$

for any E-valued analytic martingale $x = (x_n)$ and $n \geq 1$. In other words, $(\|x_n\|^\alpha)$ is a nonnegative submartingale. The desired inequality follows by applying Doob's inequality with index p/α. ■

Lemma 2 *Let E be a complex continuously quasi-normed, locally PL-convex quasi-Banach space and let $x = (x_n)$ be an E-valued analytic martingale with $\|x\|_\rho < \infty$. If $P\big(S^{(q)}(x) = \infty\big) = a > 0$, then there exist an E-valued analytic martingale $\tilde{x} = (\tilde{x}_n)$ such that*

$$\|\tilde{x}\|_\infty < \infty, \quad P\big(S^{(q)}(\tilde{x}) = \infty\big) \geq a/2.$$

Proof The argument is similar to that of [B3, Thm. 3.1]. According to Lemma 1 and the Chebychev inequality we may take $\lambda > 0$ large enough so that

$$P(x^* > \lambda) \leq \Big(\frac{1}{\lambda}\|x^*\|_\rho\Big)^\rho \leq \Big(\frac{c_p}{\lambda}\|x\|_\rho\Big)^\rho \leq a/2.$$

Set $u_k = \chi_{A_k}$, where $A_k = \{\|x_j\| \leq \lambda \text{ for } 0 \leq j \leq k-1, \|dx_j\| \leq 2K\lambda \text{ for } 0 \leq j \leq k\}$ (we may take $x_0 = 0$ and $A_0 = \Omega$), and consider the martingale transform

$$\tilde{x}_n = \sum_{j=0}^n u_j v_j e^{i\theta_j}.$$

Then (u_j) is a predictable sequence and $\tilde{x} = (\tilde{x}_n)$ is an analytic martingale. It follows from the definition of A_k that $\tilde{x}^* \leq 3K^2\lambda$. Consequently $\|\tilde{x}\|_\infty < \infty$ and one checks that

$$\begin{aligned} P\big(S^{(q)}(\tilde{x}) < \infty\big) &\leq P\big(S^{(q)}(x) < \infty, \tilde{x} = x\big) + P(\tilde{x} \neq x) \\ &\leq P\big(S^{(q)}(x) < \infty\big) + P(x^* > \lambda) \leq 1 - \frac{a}{2}. \end{aligned}$$ ■

We next establish the equivalence of strong L^p-boundedness properties and much weaker "local finiteness" properties of the q-mean-square function considered for all E-valued analytic martingales. It should be compared with Burkholder's equivalent characterizations [B2, Thm 1.1] of the boundedness of martingale transforms of Banach-space valued martingales. There is also a particular quasi-Banach variant for transforms in [BD, Thm. 1.1]. We discuss below the relevance of these equivalences for the observation concerning scalar-valued analytic martingales in [BD, Prop. 4.1]. The strategy of the argument below for the q-mean-square function is an appropriate modification of these earlier ones.

Theorem 3 *Suppose that E is a complex continuously quasi-normed, locally PL-convex quasi-Banach space and let $2 \le q < \infty$. Then the following conditions are equivalent.*

(i) *the condition $x^* < \infty$ implies a. s. that $S^{(q)}(x) < \infty$, i.e.,*

$$P\big(\{S^{(q)}(x) = \infty\} \cap \{x^* < \infty\}\big) = 0,$$

for all E-valued analytic martingales x.

(ii) *$S^{(q)}(x) < \infty$ a.s. for all E-valued analytic martingales x satisfying $\|x\|_\rho < \infty$,*

(iii) *there is $c > 0$ such that*

$$\lambda^\rho P\big(S^{(q)}(x) > \lambda\big) \le c\|x\|_\rho^\rho, \quad \lambda > 0, \tag{2.1}$$

for any E-valued analytic martingale x,

(iv) *for any (equivalently, for some) $p \in (0,\infty)$ there is $c_p > 0$ such that*

$$\|S^{(q)}(x)\|_p \le c_p \|x\|_p \tag{2.2}$$

for any E-valued analytic martingale x.

Proof (i) $\Rightarrow$ (ii). Let x be an E-valued analytic martingale with $\|x\|_\rho < \infty$. If $P\big(S^{(q)}(x) = \infty\big) > 0$, then Lemma 2 produces an L^∞-bounded E-valued analytic martingale $\tilde{x}$ such that $P\big(S^{(q)}(\tilde{x}) = \infty\big) > 0$. This clearly contradicts condition (i).

(ii) $\Rightarrow$ (iii). We first claim that for any $\varepsilon > 0$ there is $\delta > 0$ so that $P\big(S^{(q)}(x) > \varepsilon\big) < \varepsilon$ for every E-valued analytic martingale $x = (x_n)$ with $\|x\|_\rho < \delta$. We argue by contradiction. Suppose to the contrary that there is $\varepsilon_0 > 0$ and a sequence of E-valued analytic martingales $x^{(j)} = (x_n^{(j)})$ satisfying $\|x^{(j)}\|_\rho^\rho < 2^{-j}$ for all j, but $P\big(S_{n_j}^{(q)}(x^{(j)}) > \varepsilon_0\big) > \varepsilon_0$ for some sequence (n_j) of indices. We have here denoted

$$S_{n_j}^{(q)}(x^{(j)}) = \Big(\|x_0^{(j)}\|^q + \sum_{n=1}^{n_j} \|v_n^{(j)}\|^q\Big)^{1/q},$$

where $x_n^{(j)} = x_0^{(j)} + \sum_{k=1}^n v_k^{(j)}(\theta_1,\dots,\theta_{k-1})e^{i\theta_k}$ for all $n \ge 0$ and $j \ge 1$. We may assume that $x_0^{(j)} = 0$ for all j and that (n_j) increases.

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We construct a new E-valued analytic martingale $x = (x_n)$ as follows: for $\theta = (\theta_k) \in \Omega$ set

$$x_{n_1+n_2+\cdots+n_j}(\theta) = \sum_{k=1}^{n_1} v_k^{(1)}(\theta_1,\dots,\theta_{k-1})e^{i\theta_k} + \sum_{k=n_1+1}^{n_1+n_2} v_{k-n_1}^{(2)}(\theta_{n_1+1},\dots,\theta_{k-1})e^{i\theta_k} + \cdots$$
$$+ \sum_{k=n_1+\cdots+n_{j-1}+1}^{n_1+\cdots+n_j} v_{k-n_1-\cdots-n_{j-1}}^{(j)}(\theta_{n_1+\cdots+n_{j-1}+1},\dots,\theta_{k-1})e^{i\theta_k},$$

with the obvious extension to intermediary indices. Consequently

$$\|x\|_\rho^\rho = \sup_j \|x_{n_1+n_2+\cdots+n_j}\|_\rho^\rho \le \sum_{j=1}^\infty \|x^{(j)}\|_\rho^\rho \le \sum_{j=1}^\infty 2^{-j} = 1.$$

Let (v_j) denote the sequence of Σ_{j-1}-measurable mappings related to x above and set $A_j = \{\theta : (\sum_{k=a_{j-1}+1}^{a_j} \|v_k\|^q)^{1/q} > \varepsilon_0\}$ for $j \in \mathbf{N}$, where $a_j = \sum_{r=1}^j n_r$. Observe that $\sum_{j=1}^\infty P(A_j) = \infty$ by our assumption. The independence of the coordinates θ_j guarantees that the sets A_j are also independent, and thus the second Borel-Cantelli lemma yields that $P(A_j, i.o.) = 1$. Deduce that $S^{(q)}(x) = \infty$ a.s., which contradicts condition (ii).

We next proceed to establish condition (2.1). Suppose that $x = (x_n)$ is an arbitrary E-valued martingale with $\|x\|_\rho < \infty$. Let $k \in \mathbf{N}$, $A = \{S_k^{(q)}(x) > 2\}$, $\psi = \chi_{A^c}$ and suppose that $P(A) > 0$. Introduce $\tilde{x} = (\tilde{x}_n)$ so that for any $\theta = (\theta_k) \in \Omega$ one has

$$\tilde{x}_{nk}(\theta) = \sum_{j=1}^k v_j(\theta_1,\dots,\theta_{j-1})e^{i\theta_j} + \sum_{j=k+1}^{2k} \psi(\theta_1,\dots,\theta_k)v_{j-k}(\theta_{k+1},\dots,\theta_{j-1})e^{i\theta_j} + \cdots$$
$$+ \sum_{j=(n-1)k+1}^{nk} \psi(\theta_1,\dots,\theta_k)\cdots\psi(\theta_{(n-2)k+1},\dots,\theta_{(n-1)k})$$
$$\times\, v_{j-(n-1)k}(\theta_{(n-1)k+1},\dots,\theta_{j-1})e^{i\theta_j}$$

(with the obvious definition for intermediary indices). Then $\tilde{x}$ is an analytic martingale and the independence of the coordinates θ_j yields that

$$\begin{aligned}\|\tilde{x}_{nk}\|_\rho^\rho &\le \mathbf{E}\big(1 + \psi(\theta_1,\dots,\theta_k) + \cdots + \psi(\theta_1,\dots,\theta_k)\cdots\psi(\theta_{(n-2)k+1},\dots,\theta_{(n-1)k})\big)\|x_k\|_\rho^\rho \\ &\le \big(1 + \mathbf{E}\psi + (\mathbf{E}\psi)^2 + \cdots + (\mathbf{E}\psi)^{n-1}\big)\|x_k\|_\rho^\rho \le (1-\mathbf{E}\psi)^{-1}\|x_k\|_\rho^\rho,\end{aligned}$$

since $\mathbf{E}\psi = P(A^c) < 1$. We conclude that

$$P(A)\|\tilde{x}_{nk}\|_\rho^\rho \le \|x_k\|_\rho^\rho \tag{2.3}$$

for all n and k. Consider $A_s = \{\theta : \psi(\theta_{(s-1)k+1},\dots,\theta_{sk}) = 0\}$ for $s \in \mathbf{N}$. Thus $P(A_s) = P(A) > 0$ for all $s \ge 1$. Observe further that the sets A_s are independent by definition,

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so that the second Borel-Cantelli lemma implies that $P(A_s, i.o.) = 1$. We conclude that $S^{(q)}(\tilde{x}) > 1$ a.s. Our first claim then yields that $\|\tilde{x}\|_\rho \geq c'$ for some uniform constant $c' > 0$. Finally, by using the fact that $(\|\tilde{x}_n\|^\rho)$ is a submartingale (see Lemma 1) and replacing x by $2\lambda^{-1}x$ in (2.3), we obtain the desired estimate (2.1).

(iii) $\Rightarrow$ (iv). We commence by verifying a "good λ"-type inequality from condition (iii) for E-valued analytic martingales x. Towards this, let $\alpha > 0$, $\beta^q > 1 + \alpha^q$ and $\lambda > 0$. Set

$$\mu = \inf\{n : S^{(q)}_{n+1}(x) > \lambda\}, \nu = \inf\{n : S^{(q)}_{n+1}(x) > \beta\lambda\}, \sigma = \inf\{n : \|x_n\| \vee \|v_{n+1}\| > \alpha\lambda\}.$$

Then μ, ν, σ are stopping times and $u_k = \chi_{A_k}$, where $A_k = \{\mu < k \leq \nu \wedge \sigma\}$, defines a predictable sequence. Consider the transformed martingale $\tilde{x} = (\tilde{x}_n)$ with $d\tilde{x}_n = u_n dx_n = u_n v_n e^{i\theta_n}$. Clearly $\tilde{x}$ is an analytic martingale.

On the set $\{\nu = n, \sigma = \infty\}$ we have $x^* \leq \alpha\lambda$ and $v^* \leq \alpha\lambda$. Moreover, note that here $\mu < \nu$ in view of $\beta^q > 1 + \alpha^q$, and that $\tilde{x}_{n+1} = \sum_{j=\mu+1}^{n+1} v_j e^{i\theta_j}$. Deduce that

$$S^{(q)}_{n+1}(\tilde{x})^q > (\beta^q - \alpha^q - 1)\lambda^q$$

and hence that

$$\{S^{(q)}(x) > \beta\lambda, x^* \vee v^* \leq \alpha\lambda\} \subset \{S^{(q)}(\tilde{x}) > (\beta^q - \alpha^q - 1)^{1/q}\lambda\}. \tag{2.4}$$

For $\omega \in \{\mu < \infty\}$ such that $\tilde{x}(\omega) \neq 0$ we have $\omega \notin A_j$ for $j = 1, \ldots, \mu$ and $\omega \in A_{\mu+1}$. In this case we get that $\tilde{x}_k = x_{k-1} + dx_k - x_\mu$ and hence that $\|\tilde{x}_k\|^\rho \leq 3\alpha^\rho\lambda^\rho$ for $k \in \{\mu + 1, \ldots, \nu \wedge \sigma\}$. Deduce that

$$\|\tilde{x}\|_\rho^\rho \leq 3\alpha^\rho\lambda^\rho P(\mu < \infty). \tag{2.5}$$

By taking into account (2.4), applying the assumption (2.1) to $\tilde{x}$ and finally using (2.5) we get that

$$\begin{aligned}
P\big(S^{(q)}(x) > \beta\lambda\big) &= P(\nu < \infty) \leq P(\nu < \infty, \sigma = \infty) + P(\sigma < \infty)\\
&= P\big(S^{(q)}(x) > \beta\lambda, x^* \vee v^* \leq \alpha\lambda\big) + P(x^* \vee v^* > \alpha\lambda)\\
&\leq P\big(S^{(q)}(\tilde{x}) > (\beta^q - \alpha^q - 1)^{1/q}\lambda\big) + P(x^* \vee v^* > \alpha\lambda)\\
&\leq c(\beta^q - \alpha^q - 1)^{-\rho/q}\lambda^{-\rho}\|\tilde{x}\|_\rho^\rho + P(x^* \vee v^* > \alpha\lambda)\\
&\leq 3c\alpha^\rho(\beta^q - \alpha^q - 1)^{-\rho/q}P\big(S^{(q)}(x) > \lambda\big) + P(x^* \vee v^* > \alpha\lambda).
\end{aligned}$$

This is the "good λ-inequality" for the nonnegative functions $S^{(q)}(x)$ and $x^* \vee v^*$. Hence, by applying Burkholder's inequality [B1, Lemma 7.1] with $\Phi(t) = t^p$ for $0 < p < \infty$ and with $\alpha > 0$ small enough, we find that

$$\|S^{(q)}(x)\|_p \leq c_p\|x^* \vee v^*\|_p \leq c_p(1 + 2K)\|x^*\|_p. \tag{2.6}$$

Above we also used the obvious fact that $v^* \leq 2Kx^*$, since $\|v_n\| \leq K(\|x_n\| + \|x_{n-1}\|)$. Finally, an application of Lemma 1 yields (2.2).

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(iv) $\Rightarrow$ (i). Let $x = (x_n)$ be an E-valued analytic martingale. Given $\lambda > 0$, set

$$\tau_\lambda = \inf\{n : \|x_n\| \vee \|v_{n+1}\| > \lambda\}$$

and consider $x^{(\tau_\lambda)} = (x_{\tau_\lambda \wedge n})$. Observe that $x^{(\tau_\lambda)}$ is an analytic martingale, since τ_λ is a stopping time, and that $\|x_{\tau_\lambda \wedge n}\| \leq 2K\lambda$. Deduce from condition (iv) that $S^{(q)}(x^{(\tau_\lambda)}) < \infty$ a.e. on the set $\{\tau_\lambda = \infty\}$, where $x^{(\tau_\lambda)} = x$.

Note further that $\{x^* \leq \lambda/2K\} \subset \{\tau_\lambda = \infty\}$. We obtain (i) for x by a standard argument letting $\lambda \to \infty$. ■

Recall that a quasi-Banach space E is said to be q-uniformly PL-convexifiable if there exists an equivalent quasi-norm such that E is q-uniformly PL-convex under the new quasi-norm. The following result is a consequence of Theorem 3 combined with [DGT].

Theorem 4 *Suppose that E is a continuously quasi-normed, locally PL-convex quasi-Banach space and that $2 \leq q < \infty$. Then E is q-uniformly PL-convexifiable if and only if one (equivalently any) of the conditions of Theorem 3 hold.*

Proof Condition (iv) of Theorem 3 with $p = q$ is equivalent to the q-uniform PL-convexifiability of E by [DGT, 5.2]. Note that our class of analytic E-valued martingales is smaller than their E-valued H_p-shrubs, but the difference does not affect the renorming result. In fact, one checks that it suffices in [DGT, pp. 131–134] to consider the probability space $\Omega = [0, 2\pi]^{\mathbf{N}} = [0, 2\pi] \times [0, 2\pi]^{\mathbf{N}}$ and our analytic martingales. Given a finite sequence $(x_n^{(j)})$ of analytic martingales as on p. 132, construct a new analytic martingale $x' = (x'_n)$ by

$$x'_0 = x_0, \quad x'_n(t, \theta_1, \ldots, \theta_{n-1}) = (e^{it} - e^{i\frac{j2\pi}{k}})y + x_{n-1}^{(j)}(\theta_1, \ldots, \theta_{n-1}),$$

for $t \in (\frac{(j-1)2\pi}{k}, \frac{j2\pi}{k}]$, $j = 1, \ldots, k$ and $n \geq 1$. One may then proceed as in [DGT], since $dx'_1(t) = e^{it}y$ and $dx'_n(t, \theta_1, \ldots, \theta_{n-1}) = dx_{n-1}^{(j)}(\theta_1, \ldots, \theta_{n-1})$ for $t \in (\frac{(j-1)2\pi}{k}, \frac{j2\pi}{k}]$, $j = 1, \ldots, k$ and $n \geq 2$. ■

We also record a version of our results, where the exponent ρ related to the quasi-norm of E plays no role. Moreover, we replace the L_p-norm by a more general Orlicz type expression. For that end we recall that $\Phi\colon [0, \infty) \to [0, \infty)$ is an (non-degenerate) Orlicz function, if $\Phi(0) = 0$ and Φ is strictly increasing and convex on $[0, \infty)$. Moreover, Φ is said to be moderate, if there is $c > 0$ such that $\Phi(2t) \leq c\Phi(t)$ for all $t > 0$. One defines $\|\cdot\|_\Phi$ by $\|f\|_\Phi = \mathbf{E}\Phi(f)$ for nonnegative random variables f and $\|x\|_\Phi = \sup_{n\geq 1} \|(\|x_n\|_E)\|_\Phi$ for E-valued analytic martingales $x = (x_n)$.

Theorem 5 *Suppose that E is a continuously quasi-normed, locally PL-convex quasi-Banach space and let $2 \leq q < \infty$. Then the following conditions are equivalent.*

(i) E is q-uniformly PL-convexifiable,

(ii) $S^{(q)}(x) < \infty$ a.s. for all E-valued analytic martingales x satisfying $\|x\|_1 < \infty$,

(iii) there is $c > 0$ such that

$$\lambda P\big(S^{(q)}(x) > \lambda\big) \leq c\|x\|_1, \quad \lambda > 0,$$

for any E-valued analytic martingales x, *(iv) for any (equivalently, for some) moderate Orlicz function Φ there is a constant c_Φ so that the inequality*

$$\|S^{(q)}(x)\|_\Phi \leq c_\Phi \|x\|_\Phi$$

holds for every E-valued analytic martingale x.

Proof Lemma 2 yields that condition (ii) is equivalent to (ii) of Theorem 3 and hence Theorem 4 implies that (i) and (ii) are equivalent. Similarly, using condition (iv) of Theorem 3 with $p = 1$ we see that (ii) implies (iii) and the converse implication is straightforward. Moreover, (iv) clearly implies (ii) and towards the converse we observe that condition (ii) leads to a good λ-inequality for the functions $S^{(q)}(x)$ and $x^* \vee v^*$ as in the proof of Theorem 3. Consequently an application of [B1, Lemma 7.1] yields that

$$\|S^{(q)}(x)\|_\Phi \leq c_\Phi \|x^* \vee v^*\|_\Phi \leq c' \|x^*\|_\Phi \leq c'' \|x\|_\Phi$$

for suitable constants. The last inequality above follows from a known generalization of Doob's inequality, for which [L, Chap. 3] is a general reference. In fact, $(\|x_n\|_E^{1/2})$ is a nonnegative submartingale according to Lemma 1 and the desired inequality is seen by applying [L, Thm. 3.3.3] to the Orlicz function $\tilde{\Phi}(t) \equiv \Phi(t^2)$. It is easily checked that $\tilde{\Phi}$ satisfies the required extra condition $q_{\tilde{\Phi}} > 1$ of [L, Thm. 3.3.3] (we refer to [L, Sect. 3.3] for the definition). ∎

Bourgain and Davis [BD, Prop. 4.1] observed using different methods that

$$\|S^{(2)}(x)\|_p \approx \|x\|_p \tag{2.7}$$

for scalar-valued analytic martingales x and $1 \leq p < \infty$. The preceding theorems can be viewed as a (kind of) vector-valued characterization of part of (2.7) in complex quasi-Banach spaces for the full range of p.

Remark It may be useful to note that the analyticity of our martingales is essentially used only in Lemma 1. The other results depend only on Lemma 1 together with familiar martingale properties: closure under stopping, martingale transforms, and the "patching" procedure used in the proof of Theorem 3.

Note added We are grateful to Stefan Geiss for bringing to our attention another approach to (a part of) Theorem 3. Namely, Geiss developed in a recent interesting paper [Ge] a general machinery for comparing the L_p-norms of operators defined on vector-valued martingale sequences. He deduces such bounds by extrapolation using as a starting point a (L_∞, BMO)-estimate, which has to be verified separately in each case. We briefly sketch (assuming familiarity with the notation of [Ge]) how to apply the techniques of [Ge] to prove the implication (iii) $\Rightarrow$ (iv) of our Theorem 3, where E is a Banach space and $p \geq 1$. In order to apply [Ge, Thm. 1.7], set $Ax = (\sum_j \|dx_j\|^p)^{1/p}$ and $Bx = \sup_j(\|x_{j-1}\| + \|dx_j\|)$ for E-valued analytic martingales x. The required property (EP) is satisfied by [Ge, Prop. 7.3]. Moreover, it is enough to verify that $\|(Ax^k)_{k=0}^\infty\|_{\mathrm{BMO}_{0,1/2}} \leq C\|Bx\|_\infty$ for all such x and some $C > 0$. With a certain amount of work one may show that this is in fact equivalent to the estimate $P\big(S^{(q)}(x) \geq c\|x^*\|_\infty\big) \leq 1/2$, which clearly follows from condition (iii) of our Theorem 3.

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