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ON THE REDUCIBILITY OF APPELL'S FUNCTION F_4

BY
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1. Put

$$(1) \qquad (\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)} = \begin{cases} 1, & \text{if } n = 0, \\ \lambda(\lambda+1)\cdots(\lambda+n-1), & \text{if } n = 1, 2, 3, \ldots. \end{cases}$$

For the Appell function F_4 defined by [3, p. 224]

$$(2) \qquad F_4[\alpha, \beta; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_{m+n}}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!}, \qquad \sqrt{|x|}+\sqrt{|y|} < 1;$$

Saxena [5, p. 216] proved a reduction formula in the form

$$(3) \qquad F_4[\lambda, \mu; \nu, \nu; x, x] = {}_3F_2\left[\begin{matrix} \lambda, \mu, \nu-\frac{1}{2} & ; \\ 2\nu-1, \frac{1}{2}(\mu-\lambda); \end{matrix} 4x\right],$$

where $\sqrt{|x|}<\frac{1}{2}$.

It may be of interest to point out that formula (3) is not correct. Indeed we first establish the following general result.

THEOREM. *Let $\{c_n\}$ be a sequence of arbitrary complex numbers. Then*

$$(4) \qquad \sum_{m,n=0}^{\infty} \frac{c_{m+n}}{(\nu)_m(\sigma)_n} \frac{x^{m+n}}{m!\,n!} = \sum_{n=0}^{\infty} \frac{c_n(\nu+\sigma+n-1)_n}{(\nu)_n(\sigma)_n} \frac{x^n}{n!}$$

and

$$(5) \qquad \sum_{m,n=0}^{\infty} (-1)^n \frac{c_{m+n}}{(\nu)_m(\nu)_n} \frac{x^{m+n}}{m!\,n!} = \sum_{n=0}^{\infty} (-1)^n \frac{c_{2n}}{(\nu)_n(\nu)_{2n}} \frac{x^{2n}}{n!},$$

provided that the series involved are absolutely convergent.

In order to prove the above theorem, we consider the double series

$$I(\nu, \sigma, x, y) = \sum_{m,n=0}^{\infty} \frac{c_{m+n}}{(\nu)_m(\sigma)_n} \frac{x^m}{m!} \frac{y^n}{n!} = \sum_{N=0}^{\infty} \frac{c_N}{(\nu)_N} \frac{x^N}{N!} {}_2F_1\left[\begin{matrix} -N, 1-\nu-N; \\ \sigma \quad ; \end{matrix} \frac{y}{x}\right],$$

by setting $m+n=N$, $N\geq 0$.

The inner Gauss series can be summed when $x=y$ by using Vandermonde's

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theorem (Cf. e.g., [3, p. 61])

(6) $${}_2F_1\left[\begin{matrix}-n, b; \\ c \;\; ;\end{matrix} 1\right] = \frac{(c-b)_n}{(c)_n}, \qquad n = 0, 1, 2, \ldots; c \neq 0, -1, -2, \ldots,$$

or else when $x=-y$ and $\nu=\sigma$ by using Kummer's summation theorem [3, p. 104]

(7) $${}_2F_1\left[\begin{matrix}a, b \;\; ; \\ 1+a-b;\end{matrix} -1\right] = \frac{\Gamma(1+a-b)\Gamma(1+\frac{1}{2}a)}{\Gamma(1+\frac{1}{2}a-b)\Gamma(1+a)},$$
$$\mathrm{Re}(b) < 1; 1+a-b \neq 0, -1, -2, \ldots.$$

On simplifying the resulting series for $I(\nu, \sigma, x, x)$ and $I(\nu, \nu, x, -x)$, we shall arrive at formulas (4) and (5) respectively.

By appropriately specializing the c_n, the above theorem can readily be rewritten in terms of hypergeometric functions. In particular, for Appell's function F_4, we are thus led to the following special cases of formulas (4) and (5):

(8) $$F_4[\lambda, \mu; \nu, \sigma; x, x] = {}_4F_3\left[\begin{matrix}\lambda, \mu, \frac{1}{2}(\nu+\sigma-1), \frac{1}{2}(\nu+\sigma); \\ \nu, \sigma, \nu+\sigma-1 \;\; ;\end{matrix} 4x\right]$$

and

(9) $$F_4[\lambda, \mu; \nu, \nu; x, -x] = {}_4F_3\left[\begin{matrix}\frac{1}{2}\lambda, \frac{1}{2}(\lambda+1), \frac{1}{2}\mu, \frac{1}{2}(\mu+1); \\ \nu, \frac{1}{2}\nu, \frac{1}{2}(\nu+1) \;\; ;\end{matrix} -4x^2\right],$$

where, for convergence, $|x|<\frac{1}{4}$.

The reduction formula (8) is fairly well-known. Burchnall [1, p. 101] obtained it while investigating solutions of equivalent systems of hypergeometric differential equations. On the other hand, formula (9) does not seem to have been recorded earlier.

Note that in the special case when $\nu=\sigma$, Burchnall's formula (8) gives us

(10) $$F_4[\lambda, \mu; \nu, \nu; x, x] = {}_3F_2\left[\begin{matrix}\lambda, \mu, \nu-\frac{1}{2}; \\ \nu, 2\nu-1 \;;\end{matrix} 4x\right], \qquad |x| < \tfrac{1}{4},$$

which evidently differs from the reduction formula (3) derived by Saxena [5].

Incidentally, a special form of Burchnall's formula (8) happens also to be the main result of Gupta's paper [4]. Of course, in terms of the generalized Appell functions (Cf. e.g., [2, p. 112]), formulas (4) and (5) have been known to the present author for a long time.

2. Some more special cases of (4) and (5) are worthy of note. Indeed, in terms of Humbert's function [3, p. 225]

(11) $$\Psi_2[\alpha; \gamma, \gamma'; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}}{(\gamma)_m(\gamma')_n} \frac{x^m}{m!} \frac{y^n}{n!},$$

formulas (4) and (5) yield

(12) $$\Psi_2[\mu; \nu, \sigma; x, x] = {}_3F_3\left[\begin{matrix}\mu, \frac{1}{2}(\nu+\sigma-1), \frac{1}{2}(\nu+\sigma); \\ \nu, \sigma, \nu+\sigma-1 \;\; ;\end{matrix} 4x\right]$$

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and

$$(13)\qquad \Psi_2[\mu; \nu, \nu; x, -x] = {}_2F_3\left[\begin{matrix}\frac{1}{2}\mu, \frac{1}{2}(\mu+1)\ ; \\ \nu, \frac{1}{2}\nu, \frac{1}{2}(\nu+1);\end{matrix} -x^2\right]$$

respectively. Formally, these reduction formulas may also be derived from (8) and (9) by writing x/λ for x and taking the limit of each side as $\lambda\to\infty$.

Formula (12) is due to Burchnall and Chaundy [2, p. 124], while (13) was given recently by the author [6, p. 53].

Finally, we let $c_n=1$, $n\geq 0$, and the reduction formulas (4) and (5) lead at once to the familiar results ([3, p. 185, Equation (2); p. 186, Equation (3)])

$$(14)\qquad {}_0F_1[-; \nu; x]\, {}_0F_1[-; \sigma; x) = {}_2F_3\left[\begin{matrix}\frac{1}{2}(\nu+\sigma-1), \frac{1}{2}(\nu+\sigma); \\ \nu, \sigma, \nu+\sigma-1\quad ;\end{matrix} 4x\right]$$

and

$$(15)\qquad {}_0F_1[-; \nu; x]\, {}_0F_1[-; \nu; -x] = {}_0F_3[-; \nu, \tfrac{1}{2}\nu, \tfrac{1}{2}(\nu+1); -\tfrac{1}{4}x^2].$$

In terms of Bessel functions these last identities (14) and (15) are well known.

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Added in Proof. It would seem appropriate to record here two variations of the series transformations (4) and (5) given by

$$(16)\qquad \sum_{m,n=0}^{\infty} c_{m+n}(\nu)_m(\sigma)_n \frac{x^{m+n}}{m!\,n!} = \sum_{n=0}^{\infty} c_n(\nu+\sigma)_n \frac{x^n}{n!}$$

and

$$(17)\qquad \sum_{m,n=0}^{\infty} (-1)^n c_{m+n}(\nu)_m(\nu)_n \frac{x^{m+n}}{m!\,n!} = \sum_{n=0}^{\infty} c_{2n}(\nu)_n \frac{x^{2n}}{n!},$$

provided that the series involved converge absolutely. Formula (16) is a generalization, for instance, of the known result ([3, p. 239, Equation (11)]) involving Appell's function F_1 defined by [op. cit., p. 224]

$$(18)\quad F_1[\alpha, \beta, \beta'; \gamma; x, y] = \sum_{m,n=0}^{\infty} \frac{(\alpha)_{m+n}(\beta)_m(\beta')_n}{(\gamma)_{m+n}} \frac{x^m}{m!}\frac{y^n}{n!}, \qquad \max\{|x|, |y|\} < 1.$$

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