natural examples, the fundamental difficulties associated with this problem are revealed; nevertheless remarkably strong general local and global theorems are subsequently obtained, most of which are published here for the first time. This is followed by a short chapter on the (global) theory of closed extremals and periodic variational problems, which represents an extension of the work of Poincaré and Hadamard. The final chapter is concerned with the problem of Lagrange which results from the imposition of constraints. Firstly, it is shown that the usual formulation of such problems is entirely unsatisfactory owing to the fact that constraints often render the variation of admissible curves of comparison impossible; thus a new formulation is devised, again in terms of the method of equivalent integrals, which is treated once more by means of an appropriate canonical formalism. Secondly, a description of the author's intricate and conceptually difficult theory of what he calls the 'class' of a problem of Lagrange is given : this is concerned (roughly) with the determination of the dimensionality of distinguished fields of extremals of such problems.

Although 33 years have elapsed since the first appearance of this book, there is no doubt that there is no single modern work which can claim to have superseded this masterpiece of Carathéodory. A great deal of new work has resulted from the book, but even so the latter has not yet been fully exploited. It is to be expected that the new, competent translation under review will serve to stimulate further activities in this direction.

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Linear and quasi-linear elliptic equations, by O. Ladyzhenskaya and N. Ural'tseva. Academic Press, New York, 1968. xviii + 495 pages. U.S. $24.

The 19th problem of Hilbert asked whether every solution of an elliptic equation with smooth (e.g. analytic) coefficients was itself smooth (analytic); the 20th problem was to prove that variational problems (and their associated Euler equations) always have solutions, provided that "solution" is understood in a sufficiently general sense. (In particular, "solution" should presumably have the same meaning for both problems.) These two related problems have determined much of the subsequent research in partial differential equations. For the case of linear equations of arbitrary order, with continuous coefficients, questions concerning existence and regularity of generalized solutions were completely resolved by 1960 (at the latest); for a good survey see Partial Differential Equations, by L. Bers, F. John and M. Schechter, Interscience, New York (1964). For equations with discontinuous coefficients, and especially for non-linear equations, we have only partial solutions. Second order equations in two space variables can be treated, for example, by the techniques of quasi-conformal mappings; cf. Generalized Analytic Functions, by I. N. Vekua, Pergamon Press, London (1962).

The book under review is concerned with linear and quasi-linear equations of second order in n variables, having discontinuous coefficients. As is well known, the shape of the theory must depend strongly on whether the equation is in "divergence" form

$$Lu = \Sigma D_i(a_{ij}(x, u, u_x)D_i u) + \ldots = f$$

or "non-divergence" form

$$Lu = \Sigma a_{ij}(x, u, u_x) D_i D_j u + \ldots = f .$$

Hence there are essentially four separate (but related) theories discussed in this book. (There are also chapters on elliptic systems, and on variational problems per se.)

Let $W_2^k(\Omega)$ denote the Sobolev space of real-valued functions on Ω having square-summable generalized derivatives of all orders $\leq k$. Call a subspace W of $W_2^1(\Omega)$ "admissible" for a given class of equations, if the uniqueness theorem for the Dirichlet problem is valid in W "in the small," i.e., if solutions in W of $Lu(x) = f(x)$, $x \in \Omega'$; $u(x) = 0$, $x \in \partial\Omega'$, are unique for small sets $\Omega' \subset \Omega$ The principal results of the book consist of the determination of necessary and sufficient conditions (in terms of integrability properties of the coefficients) in order that an arbitrary admissible solution to a given class of equations possess some additional regularity property, such as boundedness or Holder continuity. Necessary conditions are derived (in the Introduction) by simple considerations; the main part of the book is devoted to proving their sufficiency. It is assumed throughout the book that Ω is a bounded set in n-space, and that the equations studied are uniformly elliptic on Ω . Uniform ellipticity is defined as usual for linear equations, whereas for quasi-linear equations, e.g. in divergence form, the definition is as follows:

$$\nu(|u|)(1+|p|)^{m-2} \, |\xi|^2 \leq \Sigma \frac{\partial a_i(x, u, p)}{\partial p_j} \xi_i \xi_j$$

$$\leq \mu(|u|)(1+|p|)^{m-2} \, |\xi|^2$$

for all real n-vectors ξ, where $m > 1$ is a fixed constant, and where ν and μ are constants depending only on $|u|$.

The authors have attempted to give a self-contained and complete solution to the problems posed; many of the results are published here for the first time. Granted a few basic results of functional analysis, the arguments are elementary. Most of them are also difficult - Chapter 3, for example, gives a complete derivation of the famous a priori estimates of Schauder, which though frequently referred to, are seldom proved in textbooks.

No applications are given, neither to physical nor mathematical topics. For far-reaching mathematical applications of related results, we refer to Multiple Integrals in the Calculus of Variations, by C.B. Morrey, Grundlehren der Math. Wiss. Vol. 130, Springer, Berlin (1966).

The translation from the 1964 Russian edition was edited by L. Ehrenpreis, and is completely adequate.

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