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CONNECTED MAPS AND ESSENTIALLY CONNECTED SPACES

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ABSTRACT. The paper discusses some consequences of weak monotonicity for connected maps in relation to essential connectedness of a space. The first main result gives conditions under which the image by a connected map of an essentially connected space is essentially connected. The second is that, for a connected mapping of a connected, l.c. space to a WLOTS-wise and essentially connected space, w-monotonicity implies monotonicity. The remainder of the paper discusses continuity properties of connected, w-monotone mappings with WLOTS-wise and essentially connected range.

1. **Introduction**. The properties of non-continuous mappings of topological spaces have been studied by several authors (see for example, [2], [3], [4], [7], [8]). Our main concern here is with some consequences of weak monotonicity for connected maps in relation to essential connectedness of a space. Our first result, giving conditions under which the image of an essentially connected space is essentially connected, is related to results of L. Friedler [3], and of J. A. Guthrie and H. E. Stone [5].

Of particular interest for connected maps are conditions under which the inverse of a map preserves connected sets. In this connection, K. M. Garg [4] has shown that a connected, real-valued function on a connected, locally connected space is monotone if it is weakly monotone. We extend this result to mappings with WLOTS-wise and essentially connected range.

Also of interest for connected mappings are conditions when the mappings are continuous. The remainder of this paper is devoted to obtaining continuity properties of connected, w-monotone functions when the range is a WLOTS-wise and essentially connected space.

2. **Definitions and notation**. For (X, τ) a topological space, we denote by $\mathcal{K}(\tau)$ the collection of τ-connected subsets of X. The closure of a set A will be denoted by cl (A) and its boundary by fr (A). Singleton sets $\{x\}$ will usually be written without braces. Following [4], a (not necessarily continuous) mapping f: $(X, \tau) \rightarrow (Y, \sigma)$ is called *connected* if $f(K) \in \mathcal{K}(\sigma)$ whenever $K \in \mathcal{K}(\tau)$; *monotone* if $f^{-1}(K) \in \mathcal{K}(\tau)$ whenever

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$K \in \mathcal{K}(\sigma)$; and *weakly monotone* (w-monotone) if $f^{-1}(y) \in \mathcal{K}(\tau)$ for each $y \in Y$.

Using the terminology of [5], a connected space (X, τ) is called *essentially connected* if for every connected expansion $\sigma \supset \tau$, $\mathcal{K}(\sigma) = \mathcal{K}(\tau)$, and is called *irreducibly connected* if the intersection of an arbitrary collection of connected sets is connected. For points x and x' in an irreducibly connected space X, the smallest connected set in X containing x and x' will be called a *segment* and be denoted by $[x, x']$. A space is said to be *LOTS-wise (resp. WLOTS-wise) connected* if each pair of points in the space is contained in a connected, (weakly) linearly orderable subspace.

3. **Preliminary Lemmas**. We now state some lemmas needed for the proofs of the main theorems.

LEMMA 1. [7]. *Let $f: X \to Y$ be connected, where Y is T_1. If $B \subset Y$ is closed, then the components of $f^{-1}(B)$ are closed.*

LEMMA 2. *Let $f: X \to Y$ be connected and w-monotone, where X is l.c. (locally connected), and Y is T_1 and connected. For each $y \in Y$, let $\mathcal{C}$ denote the collection of components of $Y - y$. Then the following hold:*

(a) $f^{-1}(C)$ is open for each $C \in \mathcal{C}$.

If $\mathcal{C}$ is a closure-preserving collection, then for each $C \in \mathcal{C}$:

(b) C is open,

(c) $f^{-1}(\mathrm{cl}(C)) = f^{-1}(C) \cup f^{-1}(y)$ is closed,

(d) $\mathrm{cl}(f^{-1}(C)) \subset f^{-1}(\mathrm{cl}(C))$.

PROOF. (*a*) Let $x \in f^{-1}(C)$. By Lemma 1, $f^{-1}(y)$ is closed, hence there exists a connected neighborhood U of x which is disjoint from $f^{-1}(y)$. Since $f(U)$ is connected, $U \subset f^{-1}(C)$. Thus, $f^{-1}(C)$ is open.

(*b*) For each component C, $\mathrm{cl}(C) \subset C \cup y$. Since $\mathcal{C}$ is closure-preserving, we have $C = Y - \cup\{\mathrm{cl}(C') : C' \in \mathcal{C}, C' \neq C\} = Y - \mathrm{cl}(\cup\{C' : C' \in \mathcal{C}, C' \neq C\})$ is open.

(*c*) If C is closed, then Y is not connected, therefore $\mathrm{cl}(C) = C \cup y$. Hence $f^{-1}(\mathrm{cl}(C)) = f^{-1}(C) \cup f^{-1}(y)$, which is closed by part (*a*). Part (*d*) follows from (*c*).

LEMMA 3 [3]. *If $f: X \to Y$ is onto, open and w-monotone, then f is monotone.*

If (X, τ) is a topological space and $A \subset X$, then $\tau(A)$ will denote the smallest topology containing $\tau \cup \{A\}$ [5]; the members of $\tau(A)$ are of the form $U \cup (V \cap A)$, where $U, V \in \tau$.

LEMMA 4. *Let f be an open and w-monotone mapping of (X, τ) onto (Y, σ). If $A \subset Y$, then*

$$f: (X, \tau(f^{-1}(A))) \to (Y, \sigma(A))$$

is also open and w-monotone, hence monotone.

PROOF. If $W \in \tau(f^{-1}(A))$, then $W = U \cup (V \cap f^{-1}(A))$ for some $U, V \in \tau$. Since f is (τ, σ)-open, then $f(W) = f(U) \cup (f(V) \cap A)$ is in $\sigma(A)$. Thus f is open as a map from $\tau(f^{-1}(A))$ to $\sigma(A)$.

Suppose $f^{-1}(y)$ is not $\tau(f^{-1}(A))$-connected for some $y \in Y$. Let $W = U \cup (V \cap f^{-1}(A))$ and $W' = U' \cup (V' \cap f^{-1}(A))$ separate $f^{-1}(y)$, where $U, U', V, V' \in \tau$.

If $y \notin A$, then U and U' would separate $f^{-1}(y)$, which contradicts its τ-connectedness. Hence $y \in A$, from which it follows that $U \cup U'$ and $V \cup V'$ separate $f^{-1}(y)$, which is again a contradiction.

LEMMA 5 [6]. *If $\{C\}$ is a family of arbitrary subsets of a l.c. space, then*

$$\mathrm{fr}(\cup\{C\}) \subset \mathrm{cl}(\cup\{\mathrm{fr}(C)\}).$$

4. **Preservation of essential connectedness**. It is shown in [3] that if an open, connected and w-monotone mapping f of a maximally connected space X has a T_1 image Y, then Y is also maximally connected. On the other hand, it is proved in [5] that if X is essentially connected and f (continuous), mapping X onto Y, is hereditarily quotient and w-monotone, then Y is also essentially connected. The theorem below gives a result parallel to the second of these and a slight improvement of the first (removing the T_1 condition.)

THEOREM 1. *Let f: $(X, \tau) \to (Y, \sigma)$ be onto, open, connected and w-monotone. If (X, τ) is essentially connected, then (Y, σ) is also. If (X, τ) is maximally connected, so also is (Y, σ).*

PROOF. To prove the first assertion, let σ' be a connected expansion of σ, and suppose $K \in \mathcal{K}(\sigma) - \mathcal{K}(\sigma')$. Let $G, H \in \sigma'$ separate K and put $\tau' = [\tau(f^{-1}(G))](f^{-1}(H))$. By Lemma 4, f is (τ', σ')-monotone. Therefore τ' is a connected expansion of τ. Thus $\mathcal{K}(\tau') = \mathcal{K}(\tau)$.

Since f is (τ, σ)-monotone, $f^{-1}(K) \in \mathcal{K}(\tau)$. But $f^{-1}(K)$ is separated by $f^{-1}(G)$ and $f^{-1}(H)$, hence not in $\mathcal{K}(\tau')$, a contradiction. Thus (Y, σ) is essentially connected.

To verify the second, let σ' be a connected expansion of σ. Let $V \in \sigma' - \sigma$ and put $U = f^{-1}(V)$. According to Lemma 4, f is $(\tau(U), \sigma(V))$-monotone and open. Hence X is $\tau(U)$-connected. Since τ is maximally connected, $U \in \tau$, from which $V \in \sigma$, and σ is therefore maximally connected.

5. **Monotonicity of connected maps**. We shall need the following definition. A *branch* of a connected set K in a space Y is a component of $Y - K$ which clusters in K. The previously mentioned theorem of K. Garg generalizes to:

THEOREM 2. *Let (X, τ) be connected and l.c., and let f: $(X, \tau) \to (Y, \sigma)$ be connected, where (Y, σ) is WLOTS-wise and essentially connected. Then f is monotone if, and only if, it is w-monotone.*

PROOF. By Theorem 8 of [5], a WLOTS-wise connected space which is essentially connected is irreducibly connected. Let $[y, y']$ be a segment in Y, and let $\mathcal{B}$ denote the collection of branches of $[y, y']$. It is shown in [1] that $[y, y']$ is closed, that each component of $Y - [y, y']$ is a branch of $[y, y']$, and that each branch $B \in \mathcal{B}$ has a unique cluster point $z_B \in [y, y']$. According to Theorem 9 of [5], $\mathcal{B}$ is a closure-preserving family. Hence if $B \in \mathcal{B}$, then B is open and $\text{cl}(B) = B \cup z_B$. From Lemma 2($a$) it follows that $f^{-1}([y, y'])$ is closed.

Assume now that $f^{-1}([y, y'])$ is not connected. Let $f^{-1}([y, y']) = M \cup N$, where M and N are disjoint and closed. For each $B \in \mathcal{B}, f^{-1}(z_B)$ being connected, is contained in either M or N. Let $\mathcal{M} = \{B: z_B \in f(M)\}$ and $\mathcal{N} = \{B: z_B \in f(N)\}$. Since $f(M) \cup f(N) = [y, y']$ and $f(M) \cap f(N) = \emptyset$, then $\mathcal{M} \cup \mathcal{N} = \mathcal{B}$ and $\mathcal{M} \cap \mathcal{N} = \emptyset$. Put $P = \cup\{f^{-1}(B): B \in \mathcal{M}\}$ and $Q = \cup\{f^{-1}(B): B \in \mathcal{N}\}$. By Lemma 2($a$), P and Q are open. Furthermore, $P \cup Q = X - f^{-1}([y, y'])$ and $P \cap Q = \emptyset$.

By Lemma 5 we have $\text{fr}(P) \subset \text{cl}[\cup\{\text{fr}(f^{-1}(B)): B \in \mathcal{M}\}]$. Since by Lemma 2, $\text{fr}(f^{-1}(B)) \subset f^{-1}(z_B)$ for each $B \in \mathcal{B}$, we have $\text{fr}(P) \subset \text{cl}[\cup\{f^{-1}(z_B): B \in \mathcal{M}\}] \subset M$. Thus $\text{cl}(P) \cap N = \emptyset$. Similarly $\text{cl}(Q) \cap M = \emptyset$. It thus follows that $X = (P \cup M) \cup (Q \cup N)$ is a separation of X, contradicting its connectedness.

To complete the proof of sufficiency we need only note that for any connected set $K \subset Y$, $K = \cup\{[y, y']: y \in K\}$ for any fixed $y \in K$. The necessity is clear.

6. **Continuity conditions**. We now examine continuity properties of connected mappings into WLOTS-wise and essentially connected spaces.

THEOREM 3. *Let f be a connected, w-monotone mapping of a connected, l.c. space X onto a WLOTS-wise and essentially connected space Y.*

(a) *If f is continuous, then Y is LOTS-wise connected.*

(b) *If Y is LOTS-wise connected and $Y - y$ has a finite number of components for each $y \in Y$, then f is continuous.*

PROOF. (a) We wish to show that each segment $[y, y']$ in Y is a LOTS. If not, there exists a point $z \in [y, y']$ and a neighborhood V of z such that $W = V \cap [y, y']$ contains no relatively open order-interval in $[y, y']$ containing z.

Let U be a connected open set containing $f^{-1}(z)$ such that $f(U) \subset V$. Since $f(U)$ is connected, then $I = f(U) \cap [y, y']$ is a (possibly degenerate) interval in $[y, y']$. Since z is in the boundary of I, we must have $I \subset [y, z]$ or $I \subset [z, y']$, say $I \subset [y, z]$. We then have $f^{-1}(z) \subset U \cap f^{-1}([y, y']) \subset f^{-1}([y, z])$, from which, $f^{-1}([y, z]) = f^{-1}([y, z])) \cup (U \cap f^{-1}([y, y']))$ is both open and closed in $f^{-1}([y, y'])$ if $y \neq z$. But this contradicts the connectedness of $f^{-1}([y, y'])$. Thus $z = y$, in which case $f^{-1}(z)$ is both open and closed in $f^{-1}([y, y'])$, again contradicting its connectedness. Hence Y is LOTS-wise connected.

(b) Let $y \in Y$ and let V be a neighborhood of y. If $C_1, \ldots, C_n$ are the components of $Y - y$, let D_k be the component of $V \cap (C_k \cup y)$ which contains y. Since the branch points of a segment are discrete and closed ([5], Theorem 10), we can find points $y_k \in D_k$ such that (y, y_k) contains no branch points. If E_k is the component of $Y - y_k$ which contains $[y, y_k)$, then $\cup_k\{[y, y_k]\} = \cap_k\{E_k\}$ is a neighborhood of y and $f^{-1}(\cap_k\{E_k\})$ is open.

COROLLARY 1. *If $f: X \rightarrow Y$ is onto, connected and w-monotone, X is connected and l.c., and Y is a WLOTS, then f is continuous if, and only if, Y is a LOTS.*

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COROLLARY 2 [4]. *If $f: X \to R$ is connected and w-monotone, where X is connected and l.c., then f is continuous.*

THEOREM 4. *Let $f: X \to Y$ be connected and w-monotone, where X is connected and l.c., and Y is LOTS-wise and essentially connected. If the open segment $(y, y') \subset Y$ has no branch points and $f^{-1}(y)$ is l.c., then $f|f^{-1}([y, y'))$ is continuous.*

PROOF. Let C be the component of $Y - y'$ which contains y. Then $f^{-1}((y, y'))$ is a component of $f^{-1}(C - y)$. Since $f^{-1}(y)$ is closed and l.c., and $f^{-1}(C - y)$ is l.c., then $f^{-1}(y) \cup f^{-1}((y, y')) = f^{-1}([y, y'))$ is l.c. By corollary 1, $f|f^{-1}([y, y'))$ is continuous.

THEOREM 5. *If f is a one-one, continuous mapping of a connected l.c. space X onto a WLOTS-wise and essentially connected space Y, then X is irreducibly and essentially connected.*

PROOF. Since f is one-one, connected and monotone, it follows that $K \subset X$ is connected if, and only if, $f(K)$ is connected. Hence X is irreducibly connected. According to Theorem 9 of [5], to show that X is essentially connected, it suffices to verify that the branches of each segment form a closure-preserving family.

Thus let $\mathcal{B}$ denote the collection of branches of the segment $[x, x']$ in X. If $y = f(x)$ and $y' = f(x')$, then $f([x, x']) = [y, y']$. If $\mathcal{C}$ denotes the collection of branches of $[y, y']$, then $f(\mathcal{B}) = \mathcal{C}$.

Since f is one-one, it follows that $\mathrm{cl}(B) = f^{-1}(\mathrm{cl}(f(B)))$ for each $B \in \mathcal{B}$. Since f is continuous and $\mathcal{C}$ is closure-preserving, we have for any $\mathcal{B}' \subset \mathcal{B}$, $\cup\{\mathrm{cl}(B): B \in \mathcal{B}'\} \supset \mathrm{cl}[\cup\{B: B \in \mathcal{B}'\}]$ and thus $\mathcal{B}$ is closure-preserving.

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